Nonlinear Dynamical Systems and Control for Large-Scale, Hybrid, and Network Systems

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Nonlinear Dynamical Systems and Control for Large-Scale, Hybrid, and Network Systems

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To my parents and my wife

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Table of Contents

List of Figures

Summary

Modern complex engineering systems involve multiple modes of operation placing stringent demands on controller design and implementation of increasing complexity. Such systems typically possesses a multiechelon hierarchical hybrid control architecture characterized by continuous-time dynamics at the lower levels of the hierarchy and logic decision-making units at the higher levels of the hierarchy. The ability of developing a hierarchical nonlinear integrated hybrid control-system design methodology for robust, high performance controllers satisfying multiple design criteria and real-world hardware constraints is imperative in light of the increasingly complex nature of modern controlled dynamical systems involving hierarchical embedded subsystems. In this research, we concentrate on developing novel control schemes as well as stability results for large-scale, hybrid, and network systems. Specifically, we consider the following research topics in this dissertation:

In analyzing large-scale systems, it is often desirable to treat the overall system as a collection of interconnected subsystems. Solution properties of the large-scale system are then deduced from the solution properties of the individual subsystems and the nature of the system interconnections. In this research, we develop an analysis framework for discrete-time large-scale dynamical systems based on *vector dissipativity* notions. Specifically, using vector storage functions and vector supply rates, dissipativity properties of the discrete-time composite large-scale system are shown to be determined from the dissipativity properties of the subsystems and their interconnections. In particular, extended Kalman-Yakubovich-Popov conditions, in terms of the subsystem dynamics and interconnection constraints, characterizing vector dissipativeness via vector system storage functions are derived. Finally, these results are used to develop feedback interconnection stability results for discrete-time large-scale nonlinear dynamical systems using vector Lyapunov functions.

Next, we develop thermodynamic models for discrete-time, large-scale dynamical sys-

tems. Specifically, using compartmental dynamical system theory, we develop energy flow models possessing energy conservation, energy equipartition, temperature equipartition, and entropy nonconservation principles for discrete-time, large-scale dynamical systems. Furthermore, we introduce a *new* and dual notion to entropy, namely, *ectropy*, as a measure of the tendency of a dynamical system to do useful work and grow more organized, and show that conservation of energy in an isolated thermodynamic system necessarily leads to nonconservation of ectropy and entropy. In addition, using the system ectropy as a Lyapunov function candidate we show that our discrete-time, large-scale thermodynamic energy flow model has convergent trajectories to Lyapunov stable equilibria determined by the system initial subsystem energies.

Modern complex large-scale impulsive systems involve multiple modes of operation placing stringent demands on controller analysis of increasing complexity. In analyzing these large-scale systems, it is often desirable to treat the overall impulsive system as a collection of interconnected impulsive subsystems. Solution properties of the large-scale impulsive system are then deduced from the solution properties of the individual impulsive subsystems and the nature of the impulsive system interconnections. In this research, we develop vector dissipativity theory for large-scale impulsive dynamical systems. Specifically, using vector storage functions and vector hybrid supply rates, dissipativity properties of the composite large-scale impulsive system are shown to be determined from the dissipativity properties of the impulsive subsystems and their interconnections. Furthermore, extended Kalman-Yakubovich-Popov conditions, in terms of the impulsive subsystem dynamics and interconnection constraints, characterizing vector dissipativeness via vector system storage functions are derived. Finally, these results are used to develop feedback interconnection stability results for large-scale impulsive dynamical systems using vector Lyapunov functions.

A novel class of dynamic, energy-based hybrid controllers is proposed as a means for achieving enhanced energy dissipation in lossless dynamical systems. These dynamic controllers combine a logical switching architecture with continuous dynamics to guarantee that the system plant energy is strictly decreasing across switchings. The general framework leads to closed-loop systems described by impulsive differential equations. In addition, we construct hybrid dynamic controllers that guarantee that the closed-loop system is consistent with basic thermodynamic principles. In particular, the existence of an entropy function for the closed-loop system is established that satisfies a hybrid Clausius-type inequality. Special cases of energy-based and entropy-based hybrid controllers involving state-dependent switching are described. Moreover, we extend this novel class of fixed-order, energy-based hybrid controllers to nonsmooth Euler-Lagrange, hybrid port-controlled Hamiltonian, and lossless impulsive dynamical systems.

In the analysis of complex, large-scale dynamical systems it is often essential to decompose the overall dynamical system into a collection interacting subsystems. Because of implementation constraints, cost, and reliability considerations, a decentralized controller architecture is often required for controlling large-scale interconnected dynamical systems. In this research, a novel class of fixed-order, energy-based hybrid decentralized controllers is proposed as a means for achieving enhanced energy dissipation in large-scale lossless and dissipative dynamical systems. These dynamic decentralized controllers combine a logical switching architecture with continuous dynamics to guarantee that the system plant energy is strictly decreasing across switchings. The general framework leads to hybrid closed-loop systems described by impulsive differential equations. In addition, we construct hybrid dynamic controllers that guarantee that each subsystem-subcontroller pair of the hybrid closed-loop system is consistent with basic thermodynamic principles. Special cases of energy-based hybrid controllers involving state-dependent switching are described, and an illustrative combustion control example is given to demonstrate the efficacy of the proposed approach.

Finite-time stability involves dynamical systems whose trajectories converge to an equilibrium state in finite time. Since finite-time convergence implies nonuniqueness of system solutions in reverse time, such systems possess non-Lipschitzian dynamics. Sufficient conditions for finite-time stability have been developed in the literature using Hölder continuous

Lyapunov functions. In this research, we develop a general framework for finite-time stability analysis based on vector Lyapunov functions. Specifically, we construct a vector comparison system whose solution is finite-time stable and relate this finite-time stability property to the stability properties of a nonlinear dynamical system using a vector comparison principle. Furthermore, we design a universal decentralized finite-time stabilizer for large-scale dynamical systems that is robust against full modeling uncertainty.

Next, we turn our attention to finite-time stability, semistability, and network systems. Semistability is the property whereby the solutions of a dynamical system converge to Lyapunov stable equilibrium points determined by the system initial conditions. In this research, we merge the theories of semistability and finite-time stability to develop a rigorous framework for finite-time semistability. In particular, finite-time semistability for a continuum of equilibria of continuous autonomous systems is established. Continuity of the settling-time function as well as Lyapunov and converse Lyapunov theorems for semistability are also developed. In addition, necessary and sufficient conditions for finite-time semistability of homogeneous systems are addressed by exploiting the fact that a homogeneous system is finite-time semistable if and only if it is semistable and has a negative degree of homogeneity. Unlike previous work on homogeneous systems, our results involve homogeneity with respect to semistable dynamics, and require us to adopt a geometric description of homogeneity. Finally, we use these results to develop a general framework for designing semistable protocols in dynamical networks for achieving coordination tasks in finite time.

Using our results on semistability, we develop a thermodynamic framework for addressing consensus problems for nonlinear multiagent dynamical systems with fixed and switching topologies. Specifically, we present distributed nonlinear static and dynamic controller architectures for multiagent coordination. The proposed controller architectures are predicated on system thermodynamic notions resulting in controller architectures involving the exchange of information between agents that guarantee that the closed-loop dynamical network is consistent with basic thermodynamic principles. In addition, we extend the theory of semistability to discontinuous time-invariant and time-varying dynamical systems. In particular, Lyapunov-based tests for semistability, weak semistability, as well as uniform semistability for autonomous and nonautonomous differential inclusions are established. Using these results we develop a framework for designing semistable protocols in dynamical networks with switching topologies.

Even though many consensus protocol algorithms have been developed over the last several years in the literature, robustness properties of these algorithms involving nonlinear dynamics have been largely ignored. Robustness here refers to sensitivity of the control algorithm achieving semistability and consensus in the face of model uncertainty. In this research, we examine the robustness of several control algorithms for network consensus protocols with information model uncertainty of a specified structure. In particular, we develop sufficient conditions for robust stability of control protocol functions involving higher-order perturbation terms that scale in a consistent fashion with respect to a scaling operation on an underlying space with the additional property that the protocol functions can be written as a sum of functions, each homogeneous with respect to a fixed scaling operation, that retain system semistability and consensus.

Next, we focus on optimality notions for the network consensus problem. Specifically, we develop $\mathcal{H}_2$ semistability theory for linear dynamical systems. Using this theory, we design $\mathcal{H}_2$ optimal semistable controllers for linear dynamical systems. Unlike the standard $\mathcal{H}_2$ optimal control problem, a complicating feature of the $\mathcal{H}_2$ optimal semistable stabilization problem is that the closed-loop Lyapunov equation guaranteeing semistability can admit multiple solutions. An interesting feature of the proposed approach, however, is that a least squares solution over all possible semistabilizing solutions corresponds to the $\mathcal{H}_2$ optimal solution. It is shown that this least squares solution can be characterized by a linear matrix inequality minimization problem.

Finally, we develop a thermodynamic framework for addressing consensus problems for

Eulerian swarm models. Specifically, we present a distributed boundary controller architecture involving the exchange of information between uniformly distributed swarms over an n-dimensional (not necessarily Euclidian) space that guarantee that the closed-loop system is consistent with basic thermodynamic principles. In addition, we establish the existence of a unique continuously differentiable entropy functional for all equilibrium and nonequilibrium states of our thermodynamically consistent dynamical system. Information consensus and semistability are shown using the well-known Sobolev embedding theorems and the notion of generalized (or weak) solutions. Finally, since the closed-loop system is guaranteed to satisfy basic thermodynamic principles, robustness to individual agent failures and unplanned individual agent behavior is automatically guaranteed.

Chapter 1

Introduction

Due to advances in embedded computational resources over the last several years, a considerable research effort has been devoted to the control of networks and control over networks [3,65,72,80,135,139,152,155,160,166,185,187,194,205,207,227,230,231]. Network systems involve distributed decision-making for coordination of networks of dynamic agents involving information flow enabling enhanced operational effectiveness via cooperative control in autonomous systems. These dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles (UAV's) and autonomous underwater vehicles (AUV's) for combat, surveillance, and reconnaissance [239]; distributed reconfigurable sensor networks for managing power levels of wireless networks [60]; air and ground transportation systems for air traffic control and payload transport and traffic management [226]; swarms of air and space vehicle formations for command and control between heterogeneous air and space vehicles [72, 231]; and congestion control in communication networks for routing the flow of information through a network [194].

To enable the applications for these multiagent aerospace systems, cooperative control tasks such as formation control, rendezvous, flocking, cyclic pursuit, cohesion, separation, alignment, and consensus need to be developed [123,124,135,158,166,185,187,225]. To realize these tasks, individual agents need to share information of the system objectives as well as the dynamical network. In particular, in many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. Information consensus over dynamic information-exchange topologies guarantees agreement between agents for a given coordination task. Distributed consensus algorithms involve neighbor-to-neighbor interaction between agents wherein agents update their information state based on the information states of the neighboring agents. A unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in a dynamical network is the existence of a continuum of equilibria representing a state of consensus. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial state as well.

In systems possessing a continuum of equilibria, *semistability*, and not asymptotic stability is the relevant notion of stability [31, 32]. Semistability is the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. Semistability thus implies Lyapunov stability, and is implied by asymptotic stability. From a practical viewpoint, it is not sufficient to only guarantee that a network converges to a state of consensus since steady state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequently, semistable.

Modern complex aerospace dynamical systems and multiagent systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication network constraints. The sheer size (i.e., dimensionality) and complexity of these large-scale dynamical systems often necessitates a decentralized architecture for analyzing and controlling these systems. Specifically, in the control-system design of complex large-scale interconnected dynamical systems it is often desirable to treat the overall system as a collection of interconnected subsystems. The behavior of the composite (i.e., large-scale) system can then be predicted from the behaviors of the individual subsystems and their interconnections. The need for decentralized control design of large-scale systems is a direct consequence of the physical size and complexity of the dynamical model. In particular, computational complexity may be too large for model analysis while severe constraints on communication links between system sensors, actuators, and processors may render centralized control architectures impractical. Moreover, even when communication constraints do not exist, decentralized processing may be more economical.

The complexity of modern controlled large-scale dynamical systems is further exacerbated by the use of hierarchical embedded control subsystems within the feedback control system, that is, abstract decision-making units performing logical checks that identity system mode operation and specify the continuous-variable subcontroller to be activated. Such systems typically possess a multiechelon hierarchical hybrid decentralized control architecture characterized by continuous-time dynamics at the lower levels of the hierarchy and discrete-time dynamics at the higher levels of the hierarchy. The lower-level units directly interact with the dynamical system to be controlled while the higher-level units receive information from the lower-level units as inputs and provide (possibly discrete) output commands which serve to coordinate and reconcile the (sometimes competing) actions of the lower-level units. The hierarchical controller organization reduces processor cost and controller complexity by breaking up the processing task into relatively small pieces and decomposing the fast and slow control functions. Typically, the higher-level units perform logical checks that determine system mode operation, while the lower-level units execute continuous-variable commands for a given system mode of operation. Due to their multiechelon hierarchical structure, hybrid dynamical systems are capable of simultaneously exhibiting continuous-time dynamics, discrete-time dynamics, logic commands, discrete events, and resetting events. Such systems include dynamical switching systems [38, 153, 201], nonsmooth impact systems [37, 40], biological systems [147], sampled-data systems [110], discrete-event systems [198], intelligent vehicle/highway systems [163], constrained mechanical systems [37], and flight control systems [229], to cite but a few examples. The mathematical descriptions of many of these systems can be characterized by impulsive differential equations [14, 15, 127, 147, 215]. Impulsive dynamical systems will be discussed in Chapters 4–6 and can be viewed as a subclass of hybrid systems.

Since implementation constraints, cost, and reliability considerations often require decentralized controller architectures for controlling large-scale interconnected systems, decentralized control has received considerable attention in the literature [21, 27, 50, 51, 64, 128–131, 137, 156, 159, 192, 204, 214, 219, 222]. A straightforward decentralized control design technique is that of *sequential optimization* [21, 64, 137], wherein a sequential centralized subcontroller design procedure is applied to an augmented closed-loop plant composed of the actual plant and the remaining subcontrollers. Clearly, a key difficulty with decentralized control predicated on sequential optimization is that of dimensionality. An alternative approach to sequential optimization for decentralized control is based on *subsystem decomposition* with centralized design procedures applied to the individual subsystems of the large-scale system [50, 51, 128–131, 156, 159, 192, 204, 214, 219]. Decomposition techniques exploit subsystem interconnection data and in many cases, such as in the presence of very high system dimensionality, is absolutely essential for designing decentralized controllers.

Alternatively, to enable the autonomous operation for multiagent aerospace systems, the development of functional algorithms for agent coordination and control is needed. In particular, control algorithms need to address agent interactions, cooperative and non-cooperative control, task assignments, and resource allocations. To realize these tasks, appropriate sensory and cognitive capabilities such as adaptation, learning, decision-making, and agreement (or consensus) on the agent and multiagent levels are required. The common approach for addressing the autonomous operation of multiagent systems is using distributed control algorithms involving neighbor-to-neighbor interaction between agents wherein agents update their information state based on the information states of the neighboring agents. Since most multiagent network systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication networks, these systems are characterized by high-dimensional, large-scale interconnected dynamical systems. To develop distributed methods for control and coordination of autonomous multiagent systems, many researchers have looked to autonomous *swarm* systems appearing in nature for inspiration [152, 154, 176, 197, 207, 230].

In light of the above, it seems both natural and appropriate to postulate the following paradigm for nonlinear analysis and control law design of large-scale interconnected dynamical systems and multiagent systems: Develop a unified network system framework for hybrid hierarchical nonlinear large-scale interconnected dynamical systems and multiagent systems in the face of a specified level of modeling uncertainty. This dissertation provides a rigorous foundation for developing a unified network system analysis and synthesis framework for large-scale aerospace systems possessing hybrid, hierarchical, and feedback structures. Correspondingly, the main goal of this research is to make progress towards the development of analysis and hierarchical hybrid nonlinear control law tools for nonlinear large-scale interconnected dynamical systems and multiagent systems which support this paradigm. The results in this dissertation provide the basis for control-system partitioning/embedding and develops concepts of energy-based and information-based thermodynamic hybrid stabilization for complex, large-scale dynamical systems.

This dissertation focuses on large-scale interconnected dynamical systems, energy-based decentralized control, maximum entropy stabilization, and distributed hybrid control for multiagent systems. Research topics include decentralized control design for interconnected dynamical systems, hierarchical control vector Lyapunov function architectures, maximum entropy decentralized hybrid control, finite-time stabilization, distributed nonlinear control algorithms for achieving consensus, flocking, and cyclic pursuit in multiagent systems, nonlinear consensus protocols for networks of dynamic agents with directed and undirected information flow, switching network topologies, system time-delays, and distributed boundary control for Eulerian swarm models.

Chapters 2–7 address the problem of decentralized control design for large-scale interconnected dynamical systems. Since the sheer size and complexity of large-scale aerospace systems often necessitates a hierarchical decentralized architecture for analyzing and controlling these systems, here we develop several fundamental results on control vector Lyapunov function theory, thermodynamic modeling of large-scale systems, hybrid decentralized control, and finite-time control. Specifically, since large-scale aerospace systems are inherently nonlinear with multiple modes of operation, plant nonlinearities as well as high-level, abstract protocol layers for multi-modal control must be accounted for in the control-system design process. These systems typically possess a hierarchical hybrid structure characterized by continuous-time dynamics at the lower-levels of the hierarchy and discrete-time dynamics at the higher-levels of the hierarchy. Chapter 6 addresses the problem of energy-based hybrid maximum entropy decentralized control for large-scale dynamical systems. Specifically, we address three research areas involving energy-based hybrid control; namely, impulsive control systems to address systems that combine logical and continuous processes, energy-based hybrid decentralized control that affects a one-way energy transfer between the plant and each decentralized controller thereby efficiently removing energy from the physical system, and thermodynamic stabilization guaranteeing that the energy of the closed-loop large-scale dynamical system is always flowing from regions of higher to lower energies in accordance with the second law of thermodynamics.

Although the theory of distributed control for linear networks has been addressed in the literature, nonlinear protocols for network systems remain relatively undeveloped. Key issues such as robustness, disturbance rejection, switching network topologies, message transmission and processing delays, and information asynchrony between agents have been largely ignored for nonlinear networks. In Chapters 8–13, 15, and 16, we develop a unified framework for addressing consensus, flocking, and cyclic pursuit problems for multiagent nonlinear dynamical systems. Specifically, we develop continuous and discontinuous distributed controller architectures for multiagent coordination. The proposed controller architectures are predicated on system thermodynamic notions resulting in thermodynamically consistent continuous and discontinuous controller architectures involving the exchange of information between agents that guarantee that the closed-loop dynamical network is consistent with basic thermodynamic principles. Robustness, finite-time coordination, system time-delays, and dynamic system topologies are also explored.

Finally, in Chapter 14, we develop a thermodynamic framework for addressing consensus problems for Eulerian swarm models. Specifically, we develop distributed boundary controller architectures involving the exchange of information between uniformly distributed swarms over an n-dimensional (not necessarily Euclidian) space that guarantee that the closed-loop system is consistent with basic thermodynamic principles. Since the closed-loop system satisfies basic thermodynamic principles, robustness to individual agent failures and unplanned individual agent behavior are automatically guaranteed.

Chapter 2

Vector Dissipativity Theory for Large-Scale Nonlinear Dynamical Systems

2.1. Introduction

Modern complex dynamical systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication network constraints. The sheer size (i.e., dimensionality) and complexity of these large-scale dynamical systems often necessitates a hierarchical decentralized architecture for analyzing and controlling these systems. Specifically, in the analysis and control-system design of complex large-scale dynamical systems it is often desirable to treat the overall system as a collection of interconnected subsystems. The behavior of the aggregate or composite (i.e., large-scale) system can then be predicted from the behaviors of the individual subsystems and their interconnections. The need for decentralized analysis and control design of large-scale systems is a direct consequence of the physical size and complexity of the dynamical model. In particular, computational complexity may be too large for model analysis while severe constraints on communication links between system sensors, actuators, and processors may render centralized control architectures impractical.

An approach to analyzing large-scale dynamical systems was introduced by the pioneering work of Šiljak [50] and involves the notion of *connective stability*. In particular, the large-scale dynamical system is decomposed into a collection of subsystems with local dynamics and uncertain interactions. Then, each subsystem is considered independently so that the stability of each subsystem is combined with the interconnection constraints to obtain a *vector Lyapunov function* for the composite large-scale dynamical system guaranteeing connective stability for the overall system.

Vector Lyapunov functions were first introduced by Bellman [17] and Matrosov [171] and further developed by Lakshmikantham *et al.* [148], with [50,51,86,162,168,169,174] exploiting their utility for analyzing large-scale systems. The use of vector Lyapunov functions in large-scale system analysis offers a very flexible framework since each component of the vector Lyapunov function can satisfy less rigid requirements as compared to a single scalar Lyapunov function. Moreover, in large-scale systems several Lyapunov functions arise naturally from the stability properties of each subsystem. An alternative approach to vector Lyapunov functions for analyzing large-scale dynamical systems is an input-output approach wherein stability criteria are derived by assuming that each subsystem is either finite gain, passive, or conic [7, 150, 151, 232].

Since most physical processes evolve naturally in continuous-time, it is not surprising that the bulk of large-scale dynamical system theory has been developed for continuous-time systems. Nevertheless, it is the overwhelming trend to implement controllers digitally. Hence, in this chapter we extend the notions of dissipativity theory [236, 237] to develop *vector dissipativity* notions for large-scale nonlinear discrete-time dynamical systems; a notion not previously considered in the literature. In particular, we introduce a generalized definition of dissipativity for large-scale nonlinear discrete-time dynamical systems in terms of a *vector inequality* involving a *vector supply rate*, a *vector storage function*, and a nonnegative, semistable dissipation matrix. Generalized notions of vector available storage and vector required supply are also defined and shown to be element-by-element ordered, nonnegative, and finite. On the subsystem level, the proposed approach provides a discrete energy flow balance in terms of the stored subsystem energy, the supplied subsystem energy, the subsystem energy gained from all other subsystems independent of the subsystem coupling strengths, and the subsystem energy dissipated.

For large-scale discrete-time dynamical systems decomposed into interconnected subsystems, dissipativity of the composite system is shown to be determined from the dissipativity properties of the individual subsystems and the nature of the interconnections. In particular, we develop extended Kalman-Yakubovich-Popov conditions, in terms of the local subsystem dynamics and the interconnection constraints, for characterizing vector dissipativeness via vector storage functions for large-scale discrete-time dynamical systems. Finally, using the concepts of vector dissipativity and vector storage functions as candidate vector Lyapunov functions, we develop feedback interconnection stability results of large-scale discrete-time nonlinear dynamical systems. General stability criteria are given for Lyapunov and asymptotic stability of feedback interconnections of large-scale discrete-time dynamical systems. In the case of vector quadratic supply rates involving net subsystem powers and input-output subsystem energies, these results provide a positivity and small gain theorem for large-scale discrete-time systems predicated on vector Lyapunov functions.

2.2. Notation and Mathematical Preliminaries

In this section we introduce notation, several definitions, and some key results needed for analyzing discrete-time large-scale nonlinear dynamical systems. Let $\mathbb{R}$ denote the set of real numbers, $\overline{\mathbb{Z}}_+$ denote the set of nonnegative integers, $\mathbb{R}^n$ denote the set of $n \times 1$ column vectors, $\mathbb{S}^n$ denote the set of $n \times n$ symmetric matrices, $\mathbb{N}^n$ (respectively, $\mathbb{P}^n$) denote the the set of $n \times n$ nonnegative (respectively, positive) definite matrices, $(\cdot)^{\mathrm{T}}$ denote transpose, and let I_n or I denote the $n \times n$ identity matrix. For $v \in \mathbb{R}^q$ we write $v \geq\geq 0$ (respectively, $v >> 0$) to indicate that every component of v is nonnegative (respectively, positive). In this case, we say that v is *nonnegative* or *positive*, respectively. Let $\overline{\mathbb{R}}_+^q$ and $\mathbb{R}_+^q$ denote the nonnegative and positive orthants of $\mathbb{R}^q$; that is, if $v \in \mathbb{R}^q$, then $v \in \overline{\mathbb{R}}_+^q$ and $v \in \mathbb{R}_+^q$ are equivalent, respectively, to $v \geq\geq 0$ and $v >> 0$. Finally, we write $\|\cdot\|$ for the Euclidean vector norm, $\operatorname{spec}(M)$ for the spectrum of the square matrix M, $\rho(M)$ for the spectral radius of the square matrix M, $\Delta V(x(k))$ for $V(x(k+1)) - V(x(k))$, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball centered at α with radius ε, and $M \geq 0$ (respectively, $M > 0$) to denote the fact that the Hermitian matrix M is nonnegative (respectively, positive) definite. The following definition introduces the notion of nonnegative matrices.

Definition 2.1 [19, 26, 97]. Let $W \in \mathbb{R}^{q\times q}$. W is *nonnegative*[1] (respectively, *positive*) if $W_{(i,j)} \geq 0$ (respectively, $W_{(i,j)} > 0$), $i, j = 1, \ldots, q$.

The following definition introduces the notion of class $\mathcal{W}$ functions involving nondecreasing functions.

Definition 2.2. A function $w = [w_1, ..., w_q]^{\mathrm{T}} : \mathbb{R}^q \to \mathbb{R}^q$ is of *class* $\mathcal{W}$ if $w_i(r') \leq w_i(r'')$, $i = 1, ..., q$, for all $r', r'' \in \mathbb{R}^q$ such that $r'_j \leq r''_j$, $j = 1, ..., q$, where r_j denotes the jth component of r.

Note that if $w(r) = Wr$, where $W \in \mathbb{R}^{q\times q}$, then the function $w(\cdot)$ is of class $\mathcal{W}$ if and only if W is nonnegative. The following definition introduces the notion of nonnegative functions [95].

Definition 2.3. Let $w = [w_1, \cdots, w_q]^{\mathrm{T}} : \mathcal{V} \to \mathbb{R}^q$, where $\mathcal{V}$ is an open subset of $\mathbb{R}^q$ that contains $\overline{\mathbb{R}}_+^q$. Then w is *nonnegative* if $w(r) \geq\geq 0$ for all $r \in \overline{\mathbb{R}}_+^q$.

Note that if $w : \mathbb{R}^q \to \mathbb{R}^q$ is such that $w(\cdot) \in \mathcal{W}$ and $w(0) \geq\geq 0$, then w is nonnegative. Note that, if $w(r) = Wr$, then $w(\cdot)$ is nonnegative if and only if $W \in \mathbb{R}^{q\times q}$ is nonnegative.

Proposition 2.1 [95]. Suppose $\overline{\mathbb{R}}_+^q \subset \mathcal{V}$. Then $\overline{\mathbb{R}}_+^q$ is an invariant set with respect to

$$r(k+1) = w(r(k)), \quad r(0) = r_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{2.1}$$

if and only if $w : \mathcal{V} \to \mathbb{R}^q$ is nonnegative.

The following lemma is needed for developing several of the results in later sections. For the statement of this lemma the following definition is required.

[1] In this dissertation it is important to distinguish between a square nonnegative (respectively, positive) matrix and a nonnegative-definite (respectively, positive-definite) matrix.

Definition 2.4. The equilibrium solution $r(k) \equiv r_{\mathrm{e}}$ of (2.1) is *Lyapunov stable* if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $r_0 \in \mathcal{B}_\delta(r_{\mathrm{e}}) \cap \overline{\mathbb{R}}_+^q$, then $r(k) \in \mathcal{B}_\varepsilon(r_{\mathrm{e}}) \cap \overline{\mathbb{R}}_+^q$, $k \in \overline{\mathbb{Z}}_+$. The equilibrium solution $r(k) \equiv r_{\mathrm{e}}$ of (2.1) is *semistable* if it is Lyapunov stable and there exists $\delta > 0$ such that if $r_0 \in \mathcal{B}_\delta(r_{\mathrm{e}}) \cap \overline{\mathbb{R}}_+^q$, then $\lim_{k\to\infty} r(k)$ exists and converges to a Lyapunov stable equilibrium point. The equilibrium solution $r(k) \equiv r_{\mathrm{e}}$ of (2.1) is *asymptotically stable* if it is Lyapunov stable and there exists $\delta > 0$ such that if $r_0 \in \mathcal{B}_\delta(r_{\mathrm{e}}) \cap \overline{\mathbb{R}}_+^q$, then $\lim_{k\to\infty} r(k) = r_{\mathrm{e}}$. Finally, the equilibrium solution $r(k) \equiv r_{\mathrm{e}}$ of (2.1) is *globally asymptotically stable* if the previous statement holds for all $r_0 \in \overline{\mathbb{R}}_+^q$.

Recall that a matrix $W \in \mathbb{R}^{q\times q}$ is (discrete-time) *semistable* if and only if $\lim_{k\to\infty} W^k$ exists [95] while W is *asymptotically stable* if and only if $\lim_{k\to\infty} W^k = 0$.

Lemma 2.1. Suppose $W \in \mathbb{R}^{q\times q}$ is nonsingular and nonnegative. If W is semistable (respectively, asymptotically stable), then there exist a scalar $\alpha \geq 1$ (respectively, $\alpha > 1$) and a nonnegative vector $p \in \overline{\mathbb{R}}_+^q$, $p \neq 0$, (respectively, positive vector $p \in \mathbb{R}_+^q$) such that

$$W^{-\mathrm{T}} p = \alpha p. \tag{2.2}$$

Proof. Since W is semistable, it follows from Theorem 3.3 of [95] that $|\lambda| < 1$ or $\lambda = 1$ and $\lambda = 1$ is semisimple, where $\lambda \in \mathrm{spec}(W)$. Since $W^{\mathrm{T}} \geq\geq 0$, it follows from the Perron-Frobenius theorem [19] that $\rho(W) \in \mathrm{spec}(W)$, and hence, there exists $p \geq\geq 0$, $p \neq 0$, such that $W^{\mathrm{T}} p = \rho(W) p$. In addition, since W is nonsingular, $\rho(W) > 0$. Hence, $W^{\mathrm{T}} p = \alpha^{-1} p$, where $\alpha \triangleq 1/\rho(W)$, which proves that there exist $p \geq\geq 0$, $p \neq 0$, and $\alpha \geq 1$ such that (2.2) holds. In the case where W is asymptotically stable, the result is a direct consequence of the Perron-Frobenius theorem. □

Next, we present a stability result for discrete-time large-scale nonlinear dynamical systems using vector Lyapunov functions. In particular, we consider discrete-time nonlinear dynamical systems of the form

$$x(k+1) = F(x(k)), \quad x(k_0) = x_0, \quad k \geq k_0, \tag{2.3}$$

where $F : \mathcal{D} \to \mathbb{R}^n$ is continuous on $\mathcal{D}$, $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$, and $F(0) = 0$. Here, we assume that (2.3) characterizes a discrete-time, large-scale nonlinear dynamical system composed of q interconnected subsystems such that, for all $i = 1, ..., q$, each element of $F(x)$ is given by $F_i(x) = f_i(x_i) + \mathcal{I}_i(x)$, where $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ defines the vector field of each isolated subsystem of (2.3), $\mathcal{I}_i : \mathcal{D} \to \mathbb{R}^{n_i}$ defines the structure of interconnection dynamics of the ith subsystem with all other subsystems, $x_i \in \mathbb{R}^{n_i}$, $f_i(0) = 0$, $\mathcal{I}_i(0) = 0$, and $\sum_{i=1}^{q} n_i = n$. For the discrete-time, large-scale nonlinear dynamical system (2.3) we note that the subsystem states $x_i(k)$, $k \geq k_0$, for all $i = 1, ..., q$, belong to $\mathbb{R}^{n_i}$ as long as $x(k) \triangleq [x_1^{\mathrm{T}}(k), ..., x_q^{\mathrm{T}}(k)]^{\mathrm{T}} \in \mathcal{D}$, $k \geq k_0$. The next theorem presents a stability result for (2.3) via vector Lyapunov functions by relating the stability properties of a *comparison system* to the stability properties of the discrete-time, large-scale nonlinear dynamical system.

Theorem 2.1 [148]. Consider the discrete-time, large-scale nonlinear dynamical system given by (2.3). Suppose there exist a continuous vector function $V : \mathcal{D} \to \overline{\mathbb{R}}_+^q$ and a positive vector $p \in \mathbb{R}_+^q$ such that $V(0) = 0$, the scalar function $v : \mathcal{D} \to \overline{\mathbb{R}}_+$ defined by $v(x) = p^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, is such that $v(0) = 0$, $v(x) > 0$, $x \neq 0$, and

$$V(F(x)) \leq\leq w(V(x)), \quad x \in \mathcal{D}, \tag{2.4}$$

where $w : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ is a class $\mathcal{W}$ function such that $w(0) = 0$. Then the stability properties of the zero solution $r(k) \equiv 0$ to

$$r(k+1) = w(r(k)), \quad r(k_0) = r_0, \quad k \geq k_0, \tag{2.5}$$

imply the corresponding stability properties of the zero solution $x(k) \equiv 0$ to (2.3). That is, if the zero solution $r(k) \equiv 0$ to (2.5) is Lyapunov (respectively, asymptotically) stable, then the zero solution $x(k) \equiv 0$ to (2.3) is Lyapunov (respectively, asymptotically) stable. If, in addition, $\mathcal{D} = \mathbb{R}^n$ and $V(x) \to \infty$ as $\|x\| \to \infty$, then global asymptotic stability of the zero solution $r(k) \equiv 0$ to (2.5) implies global asymptotic stability of the zero solution $x(k) \equiv 0$ to (2.3).

If $V : \mathcal{D} \to \overline{\mathbb{R}}_+^q$ satisfies the conditions of Theorem 2.1 we say that $V(x)$, $x \in \mathcal{D}$, is a *vector Lyapunov function* for the discrete-time large-scale nonlinear dynamical system (2.3). Finally, we recall the notions of dissipativity [53] and geometric dissipativity [92, 95] for discrete-time nonlinear dynamical systems $\mathcal{G}$ of the form

$$x(k+1) = f(x(k)) + G(x(k))u(k), \quad x(k_0) = x_0, \quad k \geq k_0, \tag{2.6}$$

$$y(k) = h(x(k)) + J(x(k))u(k), \tag{2.7}$$

where $x \in \mathcal{D} \subseteq \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, $y \in \mathcal{Y} \subseteq \mathbb{R}^l$, $f : \mathcal{D} \to \mathbb{R}^n$ and satisfies $f(0) = 0$, $G : \mathcal{D} \to \mathbb{R}^{n\times m}$, $h : \mathcal{D} \to \mathbb{R}^l$ and satisfies $h(0) = 0$, and $J : \mathcal{D} \to \mathbb{R}^{l\times m}$. For the discrete-time nonlinear dynamical system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $u(\cdot)$ satisfies sufficient regularity conditions such that (2.6) has a unique solution forward in time. Note that since all input-output pairs $u \in \mathcal{U}$, $y \in \mathcal{Y}$, of the discrete-time nonlinear dynamical system $\mathcal{G}$ are defined on $\overline{\mathbb{Z}}_+$, the *supply rate* [236] satisfying $s(0, 0) = 0$ is locally summable for all input-output pairs satisfying (2.6) and (2.7), that is, for all input-output pairs $u \in \mathcal{U}$, $y \in \mathcal{Y}$ satisfying (2.6) and (2.7), $s(\cdot, \cdot)$ satisfies $\sum_{k=k_1}^{k_2} |s(u(k), y(k))| < \infty$, $k_1, k_2 \in \overline{\mathbb{Z}}_+$.

Definition 2.5 [53, 92]. The discrete-time nonlinear dynamical system $\mathcal{G}$ given by (2.6) and (2.7) is *geometrically dissipative* (respectively, *dissipative*) with respect to the supply rate $s(u, y)$ if there exist a continuous nonnegative-definite function $v_{\mathrm{s}} : \mathbb{R}^n \to \overline{\mathbb{R}}_+$, called a *storage function*, and a scalar $\rho > 1$ (respectively, $\rho = 1$) such that $v_{\mathrm{s}}(0) = 0$ and the *dissipation inequality*

$$\rho^{k_2} v_{\mathrm{s}}(x(k_2)) \leq \rho^{k_1} v_{\mathrm{s}}(x(k_1)) + \sum_{i=k_1}^{k_2-1} \rho^{i+1} s(u(i), y(i)), \quad k_2 \geq k_1, \tag{2.8}$$

is satisfied for all $k_2 \geq k_1 \geq k_0$, where $x(k), k \geq k_0$, is the solution to (2.6) with $u \in \mathcal{U}$. The discrete-time nonlinear dynamical system $\mathcal{G}$ given by (2.6) and (2.7) is *lossless with respect to the supply rate* $s(u, y)$ if the dissipation inequality is satisfied as an equality with $\rho = 1$ for all $k_2 \geq k_1 \geq k_0$.

An equivalent statement for dissipativity of the dynamical system (2.6) and (2.7) is

$$\Delta v_{\mathrm{s}}(x(k)) \leq s(u(k), y(k)), \quad k \geq k_0, \quad u \in \mathcal{U}, \quad y \in \mathcal{Y}. \tag{2.9}$$

Alternatively, an equivalent statement for geometric dissipativity of the dynamical system (2.6) and (2.7) is

$$\rho v_{\mathrm{s}}(x(k+1)) - v_{\mathrm{s}}(x(k)) \leq \rho s(u(k), y(k)), \quad k \geq k_0, \quad u \in \mathcal{U}, \quad y \in \mathcal{Y}. \tag{2.10}$$

2.3. Vector Dissipativity Theory for Discrete-Time Large-Scale Nonlinear Dynamical Systems

In this section, we extend the notion of dissipative dynamical systems to develop the generalized notion of vector dissipativity for discrete-time large-scale nonlinear dynamical systems. We begin by considering discrete-time nonlinear dynamical systems $\mathcal{G}$ of the form

$$x(k+1) = F(x(k), u(k)), \quad x(k_0) = x_0, \quad k \geq k_0, \tag{2.11}$$

$$y(k) = H(x(k), u(k)), \tag{2.12}$$

where $x \in \mathcal{D} \subseteq \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, $y \in \mathcal{Y} \subseteq \mathbb{R}^l$, $F : \mathcal{D} \times \mathcal{U} \to \mathbb{R}^n$, $H : \mathcal{D} \times \mathcal{U} \to \mathcal{Y}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, and $F(0, 0) = 0$. Here, we assume that $\mathcal{G}$ represents a discrete-time large-scale dynamical system composed of q interconnected controlled subsystems $\mathcal{G}_i$ such that, for all $i = 1, ..., q$,

$$F_i(x, u_i) = f_i(x_i) + \mathcal{I}_i(x) + G_i(x_i)u_i, \tag{2.13}$$

$$H_i(x_i, u_i) = h_i(x_i) + J_i(x_i)u_i, \tag{2.14}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathcal{U}_i \subseteq \mathbb{R}^{m_i}$, $y_i \triangleq H_i(x_i, u_i) \in \mathcal{Y}_i \subseteq \mathbb{R}^{l_i}$, (u_i, y_i) is the input-output pair for the ith subsystem, $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ and $\mathcal{I}_i : \mathcal{D} \to \mathbb{R}^{n_i}$ are continuous and satisfy $f_i(0) = 0$ and $\mathcal{I}_i(0) = 0$, $G_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i \times m_i}$ is continuous, $h_i : \mathbb{R}^{n_i} \to \mathbb{R}^{l_i}$ and satisfies $h_i(0) = 0$, $J_i : \mathbb{R}^{n_i} \to \mathbb{R}^{l_i \times m_i}$, $\sum_{i=1}^{q} n_i = n$, $\sum_{i=1}^{q} m_i = m$, and $\sum_{i=1}^{q} l_i = l$. Furthermore, for the system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied. We define the composite input and composite output for the discrete-time large-scale system $\mathcal{G}$ as $u \triangleq [u_1^{\mathrm{T}}, ..., u_q^{\mathrm{T}}]^{\mathrm{T}}$ and $y \triangleq [y_1^{\mathrm{T}}, ..., y_q^{\mathrm{T}}]^{\mathrm{T}}$, respectively. Note that in this case the set $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_q$ contains the set of input values and $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_q$ contains the set of output values.

Definition 2.6. For the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12) a vector function $S = [s_1, ..., s_q]^{\mathrm{T}} : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}^q$ such that $S(u, y) \triangleq [s_1(u_1, y_1), ..., s_q(u_q, y_q)]^{\mathrm{T}}$ and $S(0, 0) = 0$ is called a *vector supply rate*.

Note that since all input-output pairs $(u_i, y_i) \in \mathcal{U}_i \times \mathcal{Y}_i$, $i = 1, ..., q$, satisfying (2.11) and (2.12) are defined on $\overline{\mathbb{Z}}_+$, $s_i(\cdot, \cdot)$ satisfies $\sum_{k=k_1}^{k_2} |s_i(u_i(k), y_i(k))| < \infty$, $k_1, k_2 \in \overline{\mathbb{Z}}_+$.

Definition 2.7. The discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12) is *vector dissipative* (respectively, *geometrically vector dissipative*) *with respect to the vector supply rate* $S(u, y)$ if there exist a continuous, nonnegative definite vector function $V_{\mathrm{s}} = [v_{\mathrm{s}1}, ..., v_{\mathrm{s}q}]^{\mathrm{T}} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$, called a *vector storage function*, and a nonsingular nonnegative *dissipation matrix* $W \in \mathbb{R}^{q \times q}$ such that $V_{\mathrm{s}}(0) = 0$, W is semistable (respectively, asymptotically stable), and the *vector dissipation inequality*

$$V_{\mathrm{s}}(x(k)) \leq\leq W^{k-k_0} V_{\mathrm{s}}(x(k_0)) + \sum_{i=k_0}^{k-1} W^{k-1-i} S(u(i), y(i)), \quad k \geq k_0, \tag{2.15}$$

is satisfied, where $x(k)$, $k \geq k_0$, is the solution to (2.11) with $u \in \mathcal{U}$. The discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12) is *vector lossless with respect to the vector supply rate* $S(u, y)$ if the vector dissipation inequality is satisfied as an equality with W semistable.

Note that if the subsystems $\mathcal{G}_i$ of $\mathcal{G}$ are *disconnected*, that is, $\mathcal{I}_i(x) \equiv 0$ for all $i = 1, ..., q$, and $W \in \mathbb{R}^{q \times q}$ is diagonal, positive definite, and semistable, then it follows from Definition 2.7 that each of isolated subsystems $\mathcal{G}_i$ is dissipative or geometrically dissipative in the sense of Definition 2.5. A similar remark holds in the case where $q = 1$. Next, define the *vector available storage* of the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ by

$$V_{\mathrm{a}}(x_0) \triangleq \sup_{K \geq k_0,\, u(\cdot)} \left[-\sum_{k=k_0}^{K-1} W^{-(k+1-k_0)} S(u(k), y(k)) \right], \tag{2.16}$$

where $x(k)$, $k \geq k_0$, is the solution to (2.11) with $x(k_0) = x_0$ and admissible inputs $u \in \mathcal{U}$. The supremum in (2.16) is taken componentwise which implies that for different elements of $V_{\mathrm{a}}(\cdot)$ the supremum is calculated separately. Note, that $V_{\mathrm{a}}(x_0) \geq\geq 0$, $x_0 \in \mathcal{D}$, since $V_{\mathrm{a}}(x_0)$ is the supremum over a set of vectors containing the zero vector ($K = k_0$). To state the main results of this section the following definition is required.

Definition 2.8 [95]. The discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12) is *completely reachable* if for all $x_0 \in \mathcal{D} \subseteq \mathbb{R}^n$, there exist a $k_{\mathrm{i}} < k_0$ and a square summable input $u(\cdot)$ defined on $[k_{\mathrm{i}}, k_0]$ such that the state $x(k)$, $k \geq k_{\mathrm{i}}$, can be driven from $x(k_{\mathrm{i}}) = 0$ to $x(k_0) = x_0$. A discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ is *zero-state observable* if $u(k) \equiv 0$ and $y(k) \equiv 0$ imply $x(k) \equiv 0$.

Theorem 2.2. Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12), and assume that $\mathcal{G}$ is completely reachable. Let $W \in \mathbb{R}^{q \times q}$ be nonsingular, nonnegative, and semistable (respectively, asymptotically stable). Then

$$\sum_{k=k_0}^{K-1} W^{-(k+1-k_0)} S(u(k), y(k)) \geq\geq 0, \quad K \geq k_0, \quad u \in \mathcal{U}, \tag{2.17}$$

for $x(k_0) = 0$ if and only if $V_{\mathrm{a}}(0) = 0$ and $V_{\mathrm{a}}(x)$ is finite for all $x \in \mathcal{D}$. Moreover, if (2.17) holds, then $V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$ and hence $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u, y)$.

Proof. Suppose $V_{\rm a}(0)=0$ and $V_{\rm a}(x)$, $x\in\mathcal{D}$, is finite. Then

$$0=V_{\rm a}(0)=\sup_{K\geq k_0,\,u(\cdot)}\left[-\sum_{k=k_0}^{K-1}W^{-(k+1-k_0)}S(u(k),y(k))\right], \tag{2.18}$$

which implies (2.17).

Next, suppose (2.17) holds. Then for $x(k_0)=0$,

$$\sup_{K\geq k_0,\,u(\cdot)}\left[-\sum_{k=k_0}^{K-1}W^{-(k+1-k_0)}S(u(k),y(k))\right]\leq\leq 0, \tag{2.19}$$

which implies that $V_{\rm a}(0)\leq\leq 0$. However, since $V_{\rm a}(x_0)\geq\geq 0$, $x_0\in\mathcal{D}$, it follows that $V_{\rm a}(0)=0$. Moreover, since $\mathcal{G}$ is completely reachable it follows that for every $x_0\in\mathcal{D}$ there exists $\hat{k}>k_0$ and an admissible input $u(\cdot)$ defined on $[k_0,\hat{k}]$ such that $x(\hat{k})=x_0$. Now, since (2.17) holds for $x(k_0)=0$ it follows that for all admissible $u(\cdot)\in\mathcal{U}$,

$$\sum_{k=k_0}^{K-1}W^{-(k+1-k_0)}S(u(k),y(k))\geq\geq 0,\quad K\geq\hat{k}, \tag{2.20}$$

or, equivalently, multiplying (2.20) by the nonnegative matrix $W^{\hat{k}-k_0}$, $\hat{k}>k_0$, yields

$$-\sum_{k=\hat{k}}^{K-1}W^{-(k+1-\hat{k})}S(u(k),y(k))\leq\leq\sum_{k=k_0}^{\hat{k}-1}W^{-(k+1-\hat{k})}S(u(k),y(k))\leq\leq Q(x_0)<<\infty,$$
$$K\geq\hat{k},\quad u\in\mathcal{U}, \tag{2.21}$$

where $Q:\mathcal{D}\to\mathbb{R}^q$. Hence,

$$V_{\rm a}(x_0)=\sup_{K\geq\hat{k},\,u(\cdot)}\left[-\sum_{k=\hat{k}}^{K-1}W^{-(k+1-\hat{k})}S(u(k),y(k))\right]\leq\leq Q(x_0)<<\infty,\quad x_0\in\mathcal{D}, \tag{2.22}$$

which implies that $V_{\rm a}(x_0)$, $x_0\in\mathcal{D}$, is finite.

Finally, since (2.17) implies that $V_{\rm a}(0)=0$ and $V_{\rm a}(x)$, $x\in\mathcal{D}$, is finite it follows from the definition of the vector available storage that

$$-V_{\rm a}(x_0)\leq\leq\sum_{k=k_0}^{K-1}W^{-(k+1-k_0)}S(u(k),y(k))=\sum_{k=k_0}^{k_{\rm f}-1}W^{-(k+1-k_0)}S(u(k),y(k))$$
$$+\sum_{k=k_{\rm f}}^{K-1}W^{-(k+1-k_0)}S(u(k),y(k)),\quad K\geq k_0. \tag{2.23}$$

Now, multiplying (2.23) by the nonnegative matrix $W^{k_{\mathrm{f}}-k_0}$, $k_{\mathrm{f}} > k_0$, it follows that

$$\begin{aligned} W^{k_{\mathrm{f}}-k_0}V_{\mathrm{a}}(x_0) + \sum_{k=k_0}^{k_{\mathrm{f}}-1} W^{-(k+1-k_{\mathrm{f}})}S(u(k), y(k)) \geq\geq & \sup_{K\geq k_{\mathrm{f}},\, u(\cdot)} \left[-\sum_{k=k_{\mathrm{f}}}^{K-1} W^{-(k+1-k_{\mathrm{f}})}S(u(k), y(k)) \right] \\ = & \; V_{\mathrm{a}}(x(k_{\mathrm{f}})), \end{aligned} \tag{2.24}$$

which implies that $V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, is a vector storage function and hence $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u, y)$.

□

It follows from Lemma 2.1 that if $W \in \mathbb{R}^{q\times q}$ is nonsingular, nonnegative, and semistable (respectively, asymptotically stable), then there exist a scalar $\alpha \geq 1$ (respectively, $\alpha > 1$) and a nonnegative vector $p \in \overline{\mathbb{R}}_+^q$, $p \neq 0$, (respectively, $p \in \mathbb{R}_+^q$) such that (2.2) holds. In this case,

$$p^{\mathrm{T}}W^{-k} = \alpha p^{\mathrm{T}}W^{-(k-1)} = \cdots = \alpha^k p^{\mathrm{T}}, \quad k \in \overline{\mathbb{Z}}_+. \tag{2.25}$$

Using (2.25), we define the (scalar) *available storage* for the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ by

$$\begin{aligned} v_{\mathrm{a}}(x_0) &\triangleq \sup_{K\geq k_0,\, u(\cdot)} \left[-\sum_{k=k_0}^{K-1} p^{\mathrm{T}}W^{-(k+1-k_0)}S(u(k), y(k)) \right] \\ &= \sup_{K\geq k_0,\, u(\cdot)} \left[-\sum_{k=k_0}^{K-1} \alpha^{k+1-k_0}s(u(k), y(k)) \right], \end{aligned} \tag{2.26}$$

where $s : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}$ defined as $s(u, y) \triangleq p^{\mathrm{T}}S(u, y)$ is the (scalar) supply rate for the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$. Clearly, $v_{\mathrm{a}}(x) \geq 0$ for all $x \in \mathcal{D}$. As in standard dissipativity theory, the available storage $v_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, denotes the maximum amount of (scaled) energy that can be extracted from the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ at any instant K.

The following theorem relates vector storage functions and vector supply rates to scalar storage functions and scalar supply rates of discrete-time large-scale dynamical systems.

Theorem 2.3. Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12). Suppose $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}^q$ and with vector storage function $V_{\rm s} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$. Then there exists $p \in \overline{\mathbb{R}}_+^q$, $p \neq 0$, (respectively, $p \in \mathbb{R}_+^q$) such that $\mathcal{G}$ is dissipative (respectively, geometrically dissipative) with respect to the scalar supply rate $s(u, y) = p^{\rm T}S(u, y)$ and with storage function $v_{\rm s}(x) \triangleq p^{\rm T}V_{\rm s}(x)$, $x \in \mathcal{D}$. Moreover, in this case $v_{\rm a}(x)$, $x \in \mathcal{D}$, is a storage function for $\mathcal{G}$ and

$$0 \leq v_{\rm a}(x) \leq v_{\rm s}(x), \quad x \in \mathcal{D}. \tag{2.27}$$

Proof. Suppose $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u, y)$. Then there exist a nonsingular, nonnegative, and semistable (respectively, asymptotically stable) dissipation matrix W and a vector storage function $V_{\rm s} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$ such that the dissipation inequality (2.15) holds. Furthermore, it follows from Lemma 2.1 that there exist $\alpha \geq 1$ (respectively, $\alpha > 1$) and a nonzero vector $p \in \overline{\mathbb{R}}_+^q$ (respectively, $p \in \mathbb{R}_+^q$) satisfying (2.2). Hence, premultiplying (2.15) by $p^{\rm T}$ and using (2.25) it follows that

$$v_{\rm s}(x(k)) \leq \alpha^{-(k-k_0)}v_{\rm s}(x(k_0)) + \sum_{i=k_0}^{k-1} \alpha^{-(k-1-i)}s(u(i), y(i)), \quad k \geq k_0, \quad u \in \mathcal{U}, \tag{2.28}$$

where $v_{\rm s}(x) = p^{\rm T}V_{\rm s}(x)$, $x \in \mathcal{D}$, which implies dissipativity (respectively, geometric dissipativity) of $\mathcal{G}$ with respect to the supply rate $s(u, y)$ and with storage function $v_{\rm s}(x)$, $x \in \mathcal{D}$. Moreover, since $v_{\rm s}(0) = 0$, it follows from (2.28) that for $x(k_0) = 0$,

$$\sum_{i=k_0}^{k-1} \alpha^{i+1-k_0}s(u(i), y(i)) \geq 0, \quad k \geq k_0, \quad u \in \mathcal{U}, \tag{2.29}$$

which, using (2.26), implies that $v_{\rm a}(0) = 0$. Now, it can be easily shown that $v_{\rm a}(x)$, $x \in \mathcal{D}$, satisfies (2.28), and hence the available storage defined by (2.26) is a storage function for $\mathcal{G}$.

Finally, it follows from (2.28) that

$$v_{\rm s}(x(k_0)) \geq \alpha^{k-k_0}v_{\rm s}(x(k)) - \sum_{i=k_0}^{k-1} \alpha^{i+1-k_0}s(u(i), y(i))$$

$$\geq -\sum_{i=k_0}^{k-1} \alpha^{i+1-k_0} s(u(i), y(i)), \quad k \geq k_0, \quad u \in \mathcal{U}, \tag{2.30}$$

which implies

$$v_{\mathrm{s}}(x(k_0)) \geq \sup_{k \geq k_0,\, u(\cdot)} \left[-\sum_{i=k_0}^{k-1} \alpha^{i+1-k_0} s(u(i), y(i)) \right] = v_{\mathrm{a}}(x(k_0)), \tag{2.31}$$

and hence, (2.27) holds. □

Remark 2.1. It follows from Theorem 2.2 that if (2.17) holds for $x(k_0) = 0$, then the vector available storage $V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$. In this case, it follows from Theorem 2.3 that there exists $p \in \overline{\mathbb{R}}_+^q$, $p \neq 0$, such that $v_{\mathrm{s}}(x) \triangleq p^{\mathrm{T}} V_{\mathrm{a}}(x)$ is a storage function for $\mathcal{G}$ that satisfies (2.28), and hence by (2.27), $v_{\mathrm{a}}(x) \leq p^{\mathrm{T}} V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$.

Remark 2.2. It is important to note that it follows from Theorem 2.3 that if $\mathcal{G}$ is vector dissipative, then $\mathcal{G}$ can either be (scalar) dissipative or (scalar) geometrically dissipative.

The following theorem provides sufficient conditions guaranteeing that all scalar storage functions defined in terms of vector storage functions, that is, $v_{\mathrm{s}}(x) = p^{\mathrm{T}} V_{\mathrm{s}}(x)$, of a given vector dissipative discrete-time large-scale nonlinear dynamical system are positive definite.

Theorem 2.4. Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12), and assume that $\mathcal{G}$ is zero-state observable. Furthermore, assume that $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u, y)$ and there exist $\alpha \geq 1$ and $p \in \mathbb{R}_+^q$ such that (2.2) holds. In addition, assume that there exist functions $\kappa_i : \mathcal{Y}_i \to \mathcal{U}_i$ such that $\kappa_i(0) = 0$ and $s_i(\kappa_i(y_i), y_i) < 0$, $y_i \neq 0$, for all $i = 1, ..., q$. Then for all vector storage functions $V_{\mathrm{s}} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$ the storage function $v_{\mathrm{s}}(x) \triangleq p^{\mathrm{T}} V_{\mathrm{s}}(x)$, $x \in \mathcal{D}$, is positive definite, that is, $v_{\mathrm{s}}(0) = 0$ and $v_{\mathrm{s}}(x) > 0$, $x \in \mathcal{D}$, $x \neq 0$.

Proof. It follows from Theorem 2.3 that $v_{\rm a}(x)$, $x \in \mathcal{D}$, is a storage function for $\mathcal{G}$ that satisfies (2.28). Next, suppose, *ad absurdum*, there exists $x \in \mathcal{D}$ such that $v_{\rm a}(x) = 0$, $x \neq 0$. Then it follows from the definition of $v_{\rm a}(x)$, $x \in \mathcal{D}$, that for $x(k_0) = x$,

$$\sum_{k=k_0}^{K-1} \alpha^{k+1-k_0} s(u(k), y(k)) \geq 0, \quad K \geq k_0, \quad u \in \mathcal{U}. \tag{2.32}$$

However, for $u_i = k_i(y_i)$ we have $s_i(\kappa_i(y_i), y_i) < 0$, $y_i \neq 0$, for all $i = 1, ..., q$ and since $p >> 0$ it follows that $y_i(k) = 0$, $k \geq k_0$, $i = 1, ..., q$, which further implies that $u_i(k) = 0$, $k \geq k_0$, $i = 1, ..., q$. Since $\mathcal{G}$ is zero-state observable it follows that $x = 0$ and hence $v_{\rm a}(x) = 0$ if and only if $x = 0$. The result now follows from (2.27). Finally, for the geometrically vector dissipative case it follows from Lemma 2.1 that $p >> 0$ with the rest of the proof being identical as above. $\square$

Next, we introduce the concept of *vector required supply* of a discrete-time large-scale nonlinear dynamical system. Specifically, define the vector required supply of the discrete-time large-scale dynamical system $\mathcal{G}$ by

$$V_{\rm r}(x_0) \triangleq \inf_{K \geq -k_0+1,\, u(\cdot)} \sum_{k=-K}^{k_0-1} W^{-(k+1-k_0)} S(u(k), y(k)), \tag{2.33}$$

where $x(k)$, $k \geq -K$, is the solution to (2.11) with $x(-K) = 0$ and $x(k_0) = x_0$. Note that since, with $x(k_0) = 0$, the infimum in (2.33) is the zero vector it follows that $V_{\rm r}(0) = 0$. Moreover, since $\mathcal{G}$ is completely reachable it follows that $V_{\rm r}(x) << \infty$, $x \in \mathcal{D}$. Using the notion of the vector required supply we present necessary and sufficient conditions for dissipativity of a large-scale dynamical system with respect to a vector supply rate.

Theorem 2.5. Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12), and assume that $\mathcal{G}$ is completely reachable. Then $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u, y)$ if and only if

$$0 \leq\leq V_{\rm r}(x) << \infty, \quad x \in \mathcal{D}. \tag{2.34}$$

Moreover, if (2.34) holds, then $V_{\mathrm{r}}(x)$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$. Finally, if the vector available storage $V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$, then

$$0 \leq\leq V_{\mathrm{a}}(x) \leq\leq V_{\mathrm{r}}(x) << \infty, \quad x \in \mathcal{D}. \tag{2.35}$$

Proof. Suppose (2.34) holds and let $x(k)$, $k \in \overline{\mathbb{Z}}_+$, satisfy (2.11) with admissible inputs $u(k) \in \mathcal{U}$, $k \in \overline{\mathbb{Z}}_+$, and $x(k_0) = x_0$. Then it follows from the definition of $V_{\mathrm{r}}(\cdot)$ that for $-K \leq k_{\mathrm{f}} \leq k_0 - 1$ and $u(\cdot) \in \mathcal{U}$,

$$\begin{aligned} V_{\mathrm{r}}(x_0) &\leq\leq \sum_{k=-K}^{k_0-1} W^{-(k+1-k_0)} S(u(k), y(k)) \\ &= \sum_{k=-K}^{k_{\mathrm{f}}-1} W^{-(k+1-k_0)} S(u(k), y(k)) + \sum_{k=k_{\mathrm{f}}}^{k_0-1} W^{-(k+1-k_0)} S(u(k), y(k)), \end{aligned} \tag{2.36}$$

and hence,

$$\begin{aligned} V_{\mathrm{r}}(x_0) \leq\leq\ & W^{k_0-k_{\mathrm{f}}} \inf_{K \geq -k_{\mathrm{f}}+1,\, u(\cdot)} \left[\sum_{k=-K}^{k_{\mathrm{f}}-1} W^{-(k+1-k_{\mathrm{f}})} S(u(k), y(k)) \right] \\ & + \sum_{k=k_{\mathrm{f}}}^{k_0-1} W^{-(k+1-k_0)} S(u(k), y(k)) \\ =\ & W^{k_0-k_{\mathrm{f}}} V_{\mathrm{r}}(x(k_{\mathrm{f}})) + \sum_{k=k_{\mathrm{f}}}^{k_0-1} W^{k_0-1-k} S(u(k), y(k)), \end{aligned} \tag{2.37}$$

which shows that $V_{\mathrm{r}}(x)$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$ and, hence, $\mathcal{G}$ is vector dissipative with respect to the vector supply rate $S(u, y)$.

Conversely, suppose that $\mathcal{G}$ is vector dissipative with respect to the vector supply rate $S(u, y)$. Then there exists a nonnegative vector storage function $V_{\mathrm{s}}(x)$, $x \in \mathcal{D}$, such that $V_{\mathrm{s}}(0) = 0$. Since $\mathcal{G}$ is completely reachable it follows that for $x(k_0) = x_0$ there exist $K > -k_0$ and $u(k)$, $k \in [-K, k_0]$, such that $x(-K) = 0$. Hence, it follows from the vector dissipation inequality (2.15) that

$$0 \leq\leq V_{\mathrm{s}}(x(k_0)) \leq\leq W^{k_0+K} V_{\mathrm{s}}(x(-K)) + \sum_{k=-K}^{k_0-1} W^{k_0-1-k} S(u(k), y(k)), \tag{2.38}$$

which implies that for all $K \geq -k_0 + 1$ and $u \in \mathcal{U}$,

$$0 \leq\leq \sum_{k=-K}^{k_0-1} W^{-(k+1-k_0)} S(u(k), y(k)) \tag{2.39}$$

or, equivalently,

$$0 \leq\leq \inf_{K \geq -k_0+1,\, u(\cdot)} \sum_{k=-K}^{k_0-1} W^{-(k+1-k_0)} S(u(k), y(k)) = V_{\mathrm{r}}(x_0). \tag{2.40}$$

Since, by complete reachability $V_{\mathrm{r}}(x) << \infty$, $x \in \mathcal{D}$, it follows that (2.34) holds.

Finally, suppose that $V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, is a vector storage function. Then for $x(-K) = 0$, $x(k_0) = x_0$, and $u \in \mathcal{U}$, it follows that

$$V_{\mathrm{a}}(x(k_0)) \leq\leq W^{k_0+K} V_{\mathrm{a}}(x(-K)) + \sum_{k=-K}^{k_0-1} W^{k_0-1-k} S(u(k), y(k)), \tag{2.41}$$

which implies that

$$0 \leq\leq V_{\mathrm{a}}(x(k_0)) \leq\leq \inf_{K \geq -k_0+1,\, u(\cdot)} \sum_{k=-K}^{k_0-1} W^{-(k+1-k_0)} S(u(k), y(k)) = V_{\mathrm{r}}(x(k_0)), \quad x \in \mathcal{D}. \tag{2.42}$$

Since $x(k_0) = x_0 \in \mathcal{D}$ is arbitrary and, by complete reachability, $V_{\mathrm{r}}(x) << \infty$, $x \in \mathcal{D}$, (2.42) implies (2.35). □

The next result is a direct consequence of Theorems 2.2 and 2.5.

Proposition 2.2. Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12). Let $M = \mathrm{diag}\,[\mu_1, ..., \mu_q]$ be such that $0 \leq \mu_i \leq 1$, $i = 1, ..., q$. If $V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, and $V_{\mathrm{r}}(x)$, $x \in \mathcal{D}$, are vector storage functions for $\mathcal{G}$, then

$$V_{\mathrm{s}}(x) = M V_{\mathrm{a}}(x) + (I_q - M) V_{\mathrm{r}}(x), \quad x \in \mathcal{D}, \tag{2.43}$$

is a vector storage function for $\mathcal{G}$.

Proof. First note that $M \geq\geq 0$ and $I_q - M \geq\geq 0$ if and only if $M = \operatorname{diag}[\mu_1, ..., \mu_q]$ and $\mu_i \in [0, 1]$, $i = 1, ..., q$. Now, the result is a direct consequence of the vector dissipation inequality (2.15) by noting that if $V_{\mathrm{a}}(x)$ and $V_{\mathrm{r}}(x)$ satisfy (2.15), then $V_{\mathrm{s}}(x)$ satisfies (2.15).

□

Next, recall that if $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative), then there exist $p \in \overline{\mathbb{R}}_+^q$, $p \neq 0$, and $\alpha \geq 1$ (respectively, $p \in \mathbb{R}_+^q$ and $\alpha > 1$) such that (2.2) and (2.25) hold. Now, define the (scalar) *required supply* for the large-scale nonlinear dynamical system $\mathcal{G}$ by

$$\begin{aligned} v_{\mathrm{r}}(x_0) &\triangleq \inf_{K \geq -k_0+1,\, u(\cdot)} \sum_{k=-K}^{k_0-1} p^{\mathrm{T}} W^{-(k+1-k_0)} S(u(k), y(k)) \\ &= \inf_{K \geq -k_0+1,\, u(\cdot)} \sum_{k=-K}^{k_0-1} \alpha^{k+1-k_0} s(u(k), y(k)), \quad x_0 \in \mathcal{D}, \end{aligned} \tag{2.44}$$

where $s(u, y) = p^{\mathrm{T}} S(u, y)$ and $x(k)$, $k \geq -K$, is the solution to (2.11) with $x(-K) = 0$ and $x(k_0) = x_0$. It follows from (2.44) that the required supply of a discrete-time large-scale nonlinear dynamical system is the minimum amount of generalized energy which can be delivered to the discrete-time large-scale system in order to transfer it from an initial state $x(-K) = 0$ to a given state $x(k_0) = x_0$. Using the same arguments as in case of the vector required supply, it follows that $v_{\mathrm{r}}(0) = 0$ and $v_{\mathrm{r}}(x) < \infty$, $x \in \mathcal{D}$.

Next, using the notion of required supply, we show that all storage functions of the form $v_{\mathrm{s}}(x) = p^{\mathrm{T}} V_{\mathrm{s}}(x)$, where $p \in \overline{\mathbb{R}}_+^q$, $p \neq 0$, are bounded from above by the required supply and bounded from below by the available storage. Hence, a dissipative discrete-time large-scale nonlinear dynamical system can only deliver to its surroundings a fraction of all of its stored subsystem energies and can only store a fraction of the work done to all of its subsystems.

Corollary 2.1. Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12). Assume that $\mathcal{G}$ is vector dissipative with respect to a vector supply rate $S(u, y)$ and with vector storage function $V_{\mathrm{s}} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$. Then $v_{\mathrm{r}}(x)$, $x \in \mathcal{D}$, is a storage function for $\mathcal{G}$. Moreover, if $v_{\mathrm{s}}(x) \triangleq p^{\mathrm{T}} V_{\mathrm{s}}(x)$, $x \in \mathcal{D}$, where $p \in \overline{\mathbb{R}}_{+}^{q}$, $p \neq 0$, then

$$0 \leq v_{\mathrm{a}}(x) \leq v_{\mathrm{s}}(x) \leq v_{\mathrm{r}}(x) < \infty, \quad x \in \mathcal{D}. \tag{2.45}$$

Proof. It follows from Theorem 2.3 that if $\mathcal{G}$ is vector dissipative with respect to the vector supply rate $S(u, y)$ and with a vector storage function $V_{\mathrm{s}} : \mathcal{D} \to \overline{\mathbb{R}}_{+}^{q}$, then there exists $p \in \overline{\mathbb{R}}_{+}^{q}$, $p \neq 0$, such that $\mathcal{G}$ is dissipative with respect to the supply rate $s(u, y) = p^{\mathrm{T}} S(u, y)$ and with storage function $v_{\mathrm{s}}(x) = p^{\mathrm{T}} V_{\mathrm{s}}(x)$, $x \in \mathcal{D}$. Hence, it follows from (2.28), with $x(-K) = 0$ and $x(k_0) = x_0$, that

$$\sum_{k=-K}^{k_0-1} \alpha^{k+1-k_0} s(u(k), y(k)) \geq 0, \quad K \geq -k_0, \quad u \in \mathcal{U}, \tag{2.46}$$

which implies that $v_{\mathrm{r}}(x_0) \geq 0$, $x_0 \in \mathcal{D}$. Furthermore, it is easy to see from the definition of a required supply that $v_{\mathrm{r}}(x)$, $x \in \mathcal{D}$, satisfies the dissipation inequality (2.28). Hence, $v_{\mathrm{r}}(x)$, $x \in \mathcal{D}$, is a storage function for $\mathcal{G}$. Moreover, it follows from the dissipation inequality (2.28), with $x(-K) = 0$, $x(k_0) = x_0$, and $u \in \mathcal{U}$, that

$$\alpha^{k_0} v_{\mathrm{s}}(x(k_0)) \leq \alpha^{-K} v_{\mathrm{s}}(x(-K)) + \sum_{k=-K}^{k_0-1} \alpha^{k+1} s(u(k), y(k)) = \sum_{k=-K}^{k_0-1} \alpha^{k+1} s(u(k), y(k)), \tag{2.47}$$

which implies that

$$v_{\mathrm{s}}(x(k_0)) \leq \inf_{K \geq -k_0+1,\, u(\cdot)} \sum_{k=-K}^{k_0-1} \alpha^{k+1-k_0} s(u(k), y(k)) = v_{\mathrm{r}}(x(k_0)). \tag{2.48}$$

Finally, it follows from Theorem 2.3 that $v_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, is a storage function for $\mathcal{G}$ and hence, using (2.27) and (2.48), (2.45) holds. $\square$

Remark 2.3. It follows from Theorem 2.5 that if $\mathcal{G}$ is vector dissipative with respect to the vector supply rate $S(u, y)$, then $V_{\mathrm{r}}(x)$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$ and, by Theorem 2.3, there exists $p \in \overline{\mathbb{R}}_{+}^{q}$, $p \neq 0$, such that $v_{\mathrm{s}}(x) \triangleq p^{\mathrm{T}} V_{\mathrm{r}}(x)$, $x \in \mathcal{D}$, is a storage function for $\mathcal{G}$ satisfying (2.28). Hence, it follows from Corollary 2.1 that $p^{\mathrm{T}} V_{\mathrm{r}}(x) \leq v_{\mathrm{r}}(x)$, $x \in \mathcal{D}$.

The next result relates vector (respectively, scalar) available storage and vector (respectively, scalar) required supply for vector lossless discrete-time large-scale dynamical systems.

Theorem 2.6. Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12). Assume that $\mathcal{G}$ is completely reachable to and from the origin. If $\mathcal{G}$ is vector lossless with respect to the vector supply rate $S(u,y)$ and $V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, is a vector storage function, then $V_{\mathrm{a}}(x) = V_{\mathrm{r}}(x)$, $x \in \mathcal{D}$. Moreover, if $V_{\mathrm{s}}(x)$, $x \in \mathcal{D}$, is a vector storage function, then all (scalar) storage functions of the form $v_{\mathrm{s}}(x) = p^{\mathrm{T}}V_{\mathrm{s}}(x)$, $x \in \mathcal{D}$, where $p \in \overline{\mathbb{R}}_{+}^{q}$, $p \neq 0$, are given by

$$v_{\mathrm{s}}(x_0) = v_{\mathrm{a}}(x_0) = v_{\mathrm{r}}(x_0) = -\sum_{k=k_0}^{K-1} \alpha^{k+1-k_0} s(u(k),y(k)) = \sum_{k=-K}^{k_0-1} \alpha^{k+1-k_0} s(u(k),y(k)), \tag{2.49}$$

where $x(k)$, $k \geq k_0$, is the solution to (2.11) with $u \in \mathcal{U}$, $x(-K) = 0$, $x(K) = 0$, $x(k_0) = x_0 \in \mathcal{D}$, and $s(u,y) = p^{\mathrm{T}}S(u,y)$.

Proof. Suppose $\mathcal{G}$ is vector lossless with respect to the vector supply rate $S(u,y)$. Since $\mathcal{G}$ is completely reachable to and from the origin it follows that for every $x_0 = x(k_0) \in \mathcal{D}$ there exist $K_+ > k_0$, $-K_- < k_0$, and $u(k) \in \mathcal{U}$, $k \in [-K_-, K_+]$, such that $x(-K_-) = 0$, $x(K_+) = 0$, and $x(k_0) = x_0$. Now, it follows from the dissipation inequality (2.15) which is satisfied as an equality that

$$0 = \sum_{k=-K_-}^{K_+-1} W^{K_+-1-k} S(u(k),y(k)), \tag{2.50}$$

or, equivalently,

$$\begin{aligned} 0 &= \sum_{k=-K_-}^{K_+-1} W^{-(k+1-k_0)} S(u(k),y(k)) \\ &= \sum_{k=-K_-}^{k_0-1} W^{-(k+1-k_0)} S(u(k),y(k)) + \sum_{k=k_0}^{K_+-1} W^{-(k+1-k_0)} S(u(k),y(k)) \\ &\geq\geq \inf_{K \geq -k_0+1,\, u(\cdot)} \sum_{k=-K}^{k_0-1} W^{-(k+1-k_0)} S(u(k),y(k)) + \inf_{K \geq k_0,\, u(\cdot)} \sum_{k=k_0}^{K-1} W^{-(k+1-k_0)} S(u(k),y(k)) \\ &= V_{\mathrm{r}}(x_0) - V_{\mathrm{a}}(x_0), \end{aligned} \tag{2.51}$$

which implies that $V_{\rm r}(x_0) \leq\leq V_{\rm a}(x_0)$, $x_0 \in \mathcal{D}$. However, it follows from Theorem 2.5 that if $\mathcal{G}$ is vector dissipative and $V_{\rm a}(x)$, $x \in \mathcal{D}$, is a vector storage function, then $V_{\rm a}(x) \leq\leq V_{\rm r}(x)$, $x \in \mathcal{D}$, which along with (2.51) implies that $V_{\rm a}(x) = V_{\rm r}(x)$, $x \in \mathcal{D}$. Furthermore, since $\mathcal{G}$ is vector lossless there exist a nonzero vector $p \in \overline{\mathbb{R}}_+^q$ and a scalar $\alpha \geq 0$ satisfying (2.2).

Now, it follows from (2.50) that

$$\begin{aligned}
0 &= \sum_{k=-K_-}^{K_+-1} p^{\rm T} W^{-(k+1-k_0)} S(u(k), y(k)) = \sum_{k=-K_-}^{K_+-1} \alpha^{k+1-k_0} s(u(k), y(k)) \\
&= \sum_{k=-K_-}^{k_0-1} \alpha^{k+1-k_0} s(u(k), y(k)) + \sum_{k=k_0}^{K_+-1} \alpha^{k+1-k_0} s(u(k), y(k)) \\
&\geq \inf_{K \geq -k_0+1,\, u(\cdot)} \sum_{k=-K}^{k_0-1} \alpha^{k+1-k_0} s(u(k), y(k)) + \inf_{K \geq k_0,\, u(\cdot)} \sum_{k=k_0}^{K-1} \alpha^{k+1-k_0} s(u(k), y(k)) \\
&= v_{\rm r}(x_0) - v_{\rm a}(x_0), \quad x_0 \in \mathcal{D}, \qquad (2.52)
\end{aligned}$$

which along with (2.45) implies that for any (scalar) storage function of the form $v_{\rm s}(x) = p^{\rm T} V_{\rm s}(x)$, $x \in \mathcal{D}$, the equality $v_{\rm a}(x) = v_{\rm s}(x) = v_{\rm r}(x)$, $x \in \mathcal{D}$, holds. Moreover, since $\mathcal{G}$ is vector lossless the inequalities (2.28) and (2.47) are satisfied as equalities and

$$v_{\rm s}(x_0) = -\sum_{k=k_0}^{K-1} \alpha^{k+1-k_0} s(u(k), y(k)) = \sum_{k=-K}^{k_0-1} \alpha^{k+1-k_0} s(u(k), y(k)), \qquad (2.53)$$

where $x(k)$, $k \geq k_0$, is the solution to (2.11) with $u \in \mathcal{U}$, $x(-K) = 0$, $x(K) = 0$, and $x(k_0) = x_0 \in \mathcal{D}$. $\square$

The next proposition presents a characterization for vector dissipativity of discrete-time large-scale nonlinear dynamical systems.

Proposition 2.3. Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12) and assume $V_{\rm s} = [v_{{\rm s}1}, ..., v_{{\rm s}q}]^{\rm T} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$ is a continuous vector storage function for $\mathcal{G}$. Then $\mathcal{G}$ is vector dissipative with respect to the vector supply rate

$S(u, y)$ if and only if

$$V_{\rm s}(x(k+1)) \leq\leq WV_{\rm s}(x(k)) + S(u(k), y(k)), \quad k \geq k_0, \quad u \in \mathcal{U}. \tag{2.54}$$

Proof. The proof is immediate from (2.15) and, hence, is omitted. $\square$

As a special case of vector dissipativity theory we can analyze the stability of discrete-time large-scale nonlinear dynamical systems. Specifically, assume that the discrete-time large-scale dynamical system $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u, y)$ and with a continuous vector storage function $V_{\rm s} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$. Moreover, assume that the conditions of Theorem 2.4 are satisfied. Then it follows from Proposition 2.3, with $u(k) \equiv 0$ and $y(k) \equiv 0$, that

$$V_{\rm s}(x(k+1)) \leq\leq WV_{\rm s}(x(k)), \quad k \geq k_0, \tag{2.55}$$

where $x(k)$, $k \geq k_0$, is a solution to (2.11) with $x(k_0) = x_0$ and $u(k) \equiv 0$. Now, it follows from Theorem 2.1, with $w(r) = Wr$, that the zero solution $x(k) \equiv 0$ to (2.11), with $u(k) \equiv 0$, is Lyapunov (respectively, asymptotically) stable.

More generally, the problem of control system design for discrete-time large-scale nonlinear dynamical systems can be addressed within the framework of vector dissipativity theory. In particular, suppose that there exists a continuous vector function $V_{\rm s} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$ such that $V_{\rm s}(0) = 0$ and

$$V_{\rm s}(x(k+1)) \leq\leq \mathcal{F}(V_{\rm s}(x(k)), u(k)), \quad k \geq k_0, \quad u \in \mathcal{U}, \tag{2.56}$$

where $\mathcal{F} : \overline{\mathbb{R}}_+^q \times \mathbb{R}^m \to \mathbb{R}^q$ and $\mathcal{F}(0, 0) = 0$. Then the control system design problem for a discrete-time large-scale dynamical system reduces to constructing an *energy* feedback control law $\phi : \overline{\mathbb{R}}_+^q \to \mathcal{U}$ of the form

$$u = \phi(V_{\rm s}(x)) \triangleq [\phi_1^{\rm T}(V_{\rm s}(x)), ..., \phi_q^{\rm T}(V_{\rm s}(x))]^{\rm T}, \quad x \in \mathcal{D}, \tag{2.57}$$

where $\phi_i : \overline{\mathbb{R}}_+^q \to \mathcal{U}_i$, $\phi_i(0) = 0$, $i = 1, ..., q$, such that the zero solution $r(k) \equiv 0$ to the comparison system

$$r(k+1) = w(r(k)), \quad r(k_0) = V_{\rm s}(x(k_0)), \quad k \geq k_0, \tag{2.58}$$

is rendered asymptotically stable, where $w(r) \triangleq \mathcal{F}(r, \phi(r))$ is of class $\mathcal{W}$. In this case, if there exists $p \in \mathbb{R}_+^q$ such that $v_{\rm s}(x) \triangleq p^{\rm T} V_{\rm s}(x)$, $x \in \mathcal{D}$, is positive definite, then it follows from Theorem 2.1 that the zero solution $x(k) \equiv 0$ to (2.11), with u given by (2.57), is asymptotically stable.

As can be seen from the above discussion, using an energy feedback control architecture and exploiting the comparison system within the control design for discrete-time large-scale nonlinear dynamical systems can significantly reduce the dimensionality of a control synthesis problem in terms of a number of states that need to be stabilized. It should be noted, however, that for stability analysis of discrete-time large-scale dynamical systems the comparison system need not be linear as implied by (2.55). A discrete-time nonlinear comparison system would still guarantee stability of a discrete-time large-scale dynamical system provided that the conditions of Theorem 2.1 are satisfied.

2.4. Extended Kalman-Yakubovich-Popov Conditions for Discrete-Time Large-Scale Nonlinear Dynamical Systems

In this section, we show that vector dissipativeness (respectively, geometric vector dissipativeness) of a discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ of the form (2.11) and (2.12) can be characterized in terms of the local subsystem functions $f_i(\cdot)$, $G_i(\cdot)$, $h_i(\cdot)$, and $J_i(\cdot)$, along with the interconnection structures $\mathcal{I}_i(\cdot)$ for $i = 1, ..., q$. For the results in this section we consider the special case of dissipative systems with quadratic vector supply rates and set $\mathcal{D} = \mathbb{R}^n$, $\mathcal{U}_i = \mathbb{R}^{m_i}$, and $\mathcal{Y}_i = \mathbb{R}^{l_i}$. Specifically, let $R_i \in \mathbb{S}^{m_i}$, $S_i \in \mathbb{R}^{l_i \times m_i}$, and $Q_i \in \mathbb{S}^{l_i}$ be given and assume $S(u, y)$ is such that $s_i(u_i, y_i) = y_i^{\rm T} Q_i y_i + 2y_i^{\rm T} S_i u_i + u_i^{\rm T} R_i u_i$, $i = 1, ..., q$. For the statement of the next result recall that $x = [x_1^{\rm T}, ..., x_q^{\rm T}]^{\rm T}$, $u = [u_1^{\rm T}, ..., u_q^{\rm T}]^{\rm T}$,

$y = [y_1^{\mathrm{T}}, ..., y_q^{\mathrm{T}}]^{\mathrm{T}}$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^{l_i}$, $i = 1, ..., q$, $\sum_{i=1}^q n_i = n$, $\sum_{i=1}^q m_i = m$, and $\sum_{i=1}^q l_i = l$. Furthermore, for (2.11) and (2.12) define $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \to \mathbb{R}^l$, and $J : \mathbb{R}^n \to \mathbb{R}^{l\times m}$ by $\mathcal{F}(x) \triangleq [\mathcal{F}_1^{\mathrm{T}}(x), ..., \mathcal{F}_q^{\mathrm{T}}(x)]^{\mathrm{T}}$, where $\mathcal{F}_i(x) \triangleq f_i(x_i) + \mathcal{I}_i(x)$, $i = 1, ..., q$, $G(x) \triangleq \mathrm{diag}[G_1(x_1), ..., G_q(x_q)]$, $h(x) \triangleq [h_1^{\mathrm{T}}(x_1), ..., h_q^{\mathrm{T}}(x_q)]^{\mathrm{T}}$, and $J(x) \triangleq \mathrm{diag}[J_1(x_1), ..., J_q(x_q)]$. In addition, for all $i = 1, ..., q$, define $\hat{R}_i \in \mathbb{S}^m$, $\hat{S}_i \in \mathbb{R}^{l\times m}$, and $\hat{Q}_i \in \mathbb{S}^l$ such that each of these matrices consists of zero blocks except, respectively, for the matrix blocks $R_i \in \mathbb{S}^{m_i}$, $S_i \in \mathbb{R}^{l_i\times m_i}$, and $Q_i \in \mathbb{S}^{l_i}$ on (i, i) position. Finally, we introduce a more general definition of vector dissipativity involving an underlying nonlinear comparsion system.

Definition 2.9. The discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12) is *vector dissipative* (respectively, *geometrically vector dissipative*) *with respect to the vector supply rate* $S(u, y)$ if there exist a continuous, nonnegative definite vector function $V_{\mathrm{s}} = [v_{\mathrm{s}1}, ..., v_{\mathrm{s}q}]^{\mathrm{T}} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$, called a *vector storage function*, and a class $\mathcal{W}$ function $w : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ such that $V_{\mathrm{s}}(0) = 0$, $w(0) = 0$, the zero solution $r(k) \equiv 0$ to the comparison system

$$r(k+1) = w(r(k)), \quad r(k_0) = r_0, \quad k \geq k_0, \tag{2.59}$$

is Lyapunov (respectively, asymptotically) stable, and the *vector dissipation inequality*

$$V_{\mathrm{s}}(x(k+1)) \leq\leq w(V_{\mathrm{s}}(x(k))) + S(u(k), y(k)), \quad k \geq k_0, \tag{2.60}$$

is satisfied, where $x(k)$, $k \geq k_0$, is the solution to (2.11) with $u \in \mathcal{U}$. The discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12) is *vector lossless with respect to the vector supply rate* $S(u, y)$ if the vector dissipation inequality is satisfied as an equality with the zero solution $r(k) \equiv 0$ to (2.59) being Lyapunov stable.

Remark 2.4. If in Definition 2.9 the function $w : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ is such that $w(r) = Wr$, where $W \in \mathbb{R}^{q\times q}$, then W is nonnegative and Definition 2.9 collapses to Definition 2.7.

Theorem 2.7. Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12). Let $R_i \in \mathbb{S}^{m_i}$, $S_i \in \mathbb{R}^{l_i \times m_i}$, and $Q_i \in \mathbb{S}^{l_i}$, $i = 1, ..., q$. If there exist functions $V_{\rm s} = [v_{\rm s1}, ..., v_{\rm sq}]^{\rm T} : \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$, $P_{1i} : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, $P_{2i} : \mathbb{R}^n \to \mathbb{N}^m$, $w = [w_1, ..., w_q]^{\rm T} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$, $\ell_i : \mathbb{R}^n \to \mathbb{R}^{s_i}$, and $\mathcal{Z}_i : \mathbb{R}^n \to \mathbb{R}^{s_i \times m}$, such that $v_{{\rm s}i}(\cdot)$ is continuous, $v_{{\rm s}i}(0) = 0$, $i = 1, ..., q$, $w \in \mathcal{W}$, $w(0) = 0$,

$$v_{{\rm s}i}(\mathcal{F}(x) + G(x)u) = v_{{\rm s}i}(\mathcal{F}(x)) + P_{1i}(x)u + u^{\rm T}P_{2i}(x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \tag{2.61}$$

the zero solution $r(k) \equiv 0$ to (2.59) is Lyapunov (respectively, asymptotically) stable, and, for all $x \in \mathbb{R}^n$ and $i = 1, ..., q$,

$$0 = v_{{\rm s}i}(\mathcal{F}(x)) - h^{\rm T}(x)\hat{Q}_i h(x) - w_i(V_{\rm s}(x)) + \ell_i^{\rm T}(x)\ell_i(x), \tag{2.62}$$

$$0 = \tfrac{1}{2}P_{1i}(x) - h^{\rm T}(x)(\hat{S}_i + \hat{Q}_i J(x)) + \ell_i^{\rm T}(x)\mathcal{Z}_i(x), \tag{2.63}$$

$$0 = \hat{R}_i + J^{\rm T}(x)\hat{S}_i + \hat{S}_i^{\rm T}J(x) + J^{\rm T}(x)\hat{Q}_i J(x) - P_{2i}(x) - \mathcal{Z}_i^{\rm T}(x)\mathcal{Z}_i(x), \tag{2.64}$$

then $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector quadratic supply rate $S(u, y)$, where $s_i(u_i, y_i) = u_i^{\rm T}R_iu_i + 2y_i^{\rm T}S_iu_i + y_i^{\rm T}Q_iy_i$, $i = 1, ..., q$.

Proof. Suppose that there exist functions $v_{{\rm s}i} : \mathbb{R}^n \to \overline{\mathbb{R}}_+$, $\ell_i : \mathbb{R}^n \to \mathbb{R}^{s_i}$, $\mathcal{Z}_i : \mathbb{R}^n \to \mathbb{R}^{s_i\times m}$, $w : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$, $P_{1i} : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, $P_{2i} : \mathbb{R}^n \to \mathbb{N}^m$, such that $v_{{\rm s}i}(\cdot)$ is continuous and nonnegative-definite, $v_{{\rm s}i}(0) = 0$, $i = 1, ..., q$, $w(0) = 0$, $w \in \mathcal{W}$, the zero solution $r(k) \equiv 0$ to (2.59) is Lyapunov (respectively, asymptotically) stable, and (2.61)–(2.64) are satisfied. Then for any $u \in \mathcal{U}$ and $x \in \mathbb{R}^n$, $i = 1, ..., q$, it follows from (2.61)–(2.64) that

$$\begin{aligned}
s_i(u_i, y_i) =\ & u^{\rm T}\hat{R}_iu + 2y^{\rm T}\hat{S}_iu + y^{\rm T}\hat{Q}_iy \\
=\ & h^{\rm T}(x)\hat{Q}_ih(x) + 2h^{\rm T}(x)(\hat{S}_i + \hat{Q}_iJ(x))u \\
& +u^{\rm T}(J^{\rm T}(x)\hat{Q}_iJ(x) + J^{\rm T}(x)\hat{S}_i + \hat{S}_i^{\rm T}J(x) + \hat{R}_i)u \\
=\ & v_{{\rm s}i}(\mathcal{F}(x)) - w_i(V_{\rm s}(x)) + P_{1i}(x)u + \ell_i^{\rm T}(x)\ell_i(x) + 2\ell_i^{\rm T}(x)\mathcal{Z}_i(x)u \\
& +u^{\rm T}P_{2i}(x)u + u^{\rm T}\mathcal{Z}_i^{\rm T}(x)\mathcal{Z}_i(x)u \\
=\ & v_{{\rm s}i}(\mathcal{F}(x) + G(x)u) + [\ell_i(x) + \mathcal{Z}_i(x)u]^{\rm T}[\ell_i(x) + \mathcal{Z}_i(x)u] - w_i(V_{\rm s}(x)) \\
\geq\ & v_{{\rm s}i}(\mathcal{F}(x) + G(x)u) - w_i(V_{\rm s}(x)),
\end{aligned} \tag{2.65}$$

where $x(k)$, $k \geq k_0$, satisfies (2.11). Now, the result follows from (2.65) with vector storage function $V_{\mathrm{s}}(x) = [v_{\mathrm{s}1}(x), ..., v_{\mathrm{s}q}(x)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$. $\square$

Using (2.62)–(2.64) it follows that for $k \geq k_0$ and $i = 1, ..., q$,

$$s_i(u_i(k), y_i(k)) + [w_i(V_{\mathrm{s}}(x(k))) - v_{\mathrm{s}i}(x(k))] = \Delta v_{\mathrm{s}i}(x(k)) + [\ell_i(x(k)) + \mathcal{Z}_i(x(k))u(k)]^{\mathrm{T}}[\ell_i(x(k)) + \mathcal{Z}_i(x(k))u(k)], \tag{2.66}$$

where $V_{\mathrm{s}}(x) = [v_{\mathrm{s}1}(x), ..., v_{\mathrm{s}q}(x)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$, which can be interpreted as a *generalized energy* balance equation for the ith subsystem of $\mathcal{G}$ where $\Delta v_{\mathrm{s}i}(x(k))$ is the change in energy between consecutive discrete times, the two discrete terms on the left are, respectively, the external supplied energy to the ith subsystem and the energy gained by the ith subsystem from the net energy flow between all subsystems due to subsystem coupling, and the second discrete term on the right corresponds to the dissipated energy from the ith subsystem.

Remark 2.5. Note that if $\mathcal{G}$ with $u(k) \equiv 0$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector quadratic supply rate where $Q_i \leq 0$, $i = 1, ..., q$, then it follows from the vector dissipation inequality that

$$V_{\mathrm{s}}(x(k+1)) \leq\leq w(V_{\mathrm{s}}(x(k))) + S(0, y(k)) \leq\leq w(V_{\mathrm{s}}(x(k))), \quad k \geq k_0, \tag{2.67}$$

where $S(0, y) = [s_1(0, y_1), ..., s_q(0, , y_q)]^{\mathrm{T}}$, $s_i(0, y_i(k)) = y_i^{\mathrm{T}}(k)Q_i y_i(k) \leq 0$, $k \geq k_0$, $i = 1, ..., q$, and $x(k)$, $k \geq k_0$, is the solution to (2.11) with $u(k) \equiv 0$. If, in addition, there exists $p \in \mathbb{R}_+^q$ such that $p^{\mathrm{T}}V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$, is positive definite, then it follows from Theorem 2.1 that the undisturbed ($u(k) \equiv 0$) large-scale nonlinear dynamical system (2.11) is Lyapunov (respectively, asymptotically) stable.

Next, we extend the notions of passivity and nonexpansivity to vector passivity and vector nonexpansivity.

Definition 2.10. The discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12) with $m_i = l_i$, $i = 1, ..., q$, is *vector passive* (respectively, *geometrically vector* *passive*) if it is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u, y)$, where $s_i(u_i, y_i) = 2y_i^{\mathrm{T}} u_i$, $i = 1, ..., q$.

Definition 2.11. The discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12) is *vector nonexpansive* (respectively, *geometrically vector nonexpansive*) if it is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u, y)$, where $s_i(u_i, y_i) = \gamma_i^2 u_i^{\mathrm{T}} u_i - y_i^{\mathrm{T}} y_i$, $i = 1, ..., q$, and $\gamma_i > 0$, $i = 1, ..., q$, are given.

Remark 2.6. Note that a mixed vector passive-nonexpansive formulation of $\mathcal{G}$ can also be considered. Specifically, one can consider discrete-time large-scale nonlinear dynamical systems $\mathcal{G}$ which are vector dissipative with respect to vector supply rate $S(u, y)$, where $s_i(u_i, y_i) = 2y_i^{\mathrm{T}} u_i$, $i \in \mathbb{Z}_{\mathrm{p}}$, $s_j(u_j, y_j) = \gamma_j^2 u_j^{\mathrm{T}} u_j - y_j^{\mathrm{T}} y_j$, $\gamma_j > 0$, $j \in \mathbb{Z}_{\mathrm{ne}}$, and $\mathbb{Z}_{\mathrm{p}} \cup \mathbb{Z}_{\mathrm{ne}} = \{1, ..., q\}$. Furthermore, vector supply rates for vector input strict passivity, vector output strict passivity, and vector input-output strict passivity generalizing the passivity notions given in [118] can also be considered. However, for simplicity of exposition we do not do so here.

The next result presents constructive sufficient conditions guaranteeing vector dissipativity of $\mathcal{G}$ with respect to a vector quadratic supply rate for the case where the vector storage function $V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$, is component decoupled; that is, $V_{\mathrm{s}}(x) = [v_{\mathrm{s}1}(x_1), ..., v_{\mathrm{s}q}(x_q)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$.

Theorem 2.8. Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12). Assume that there exist functions $V_{\mathrm{s}} = [v_{\mathrm{s}1}, ..., v_{\mathrm{s}q}]^{\mathrm{T}} : \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$, $P_{1i} : \mathbb{R}^n \to \mathbb{R}^{1\times m_i}$, $P_{2i} : \mathbb{R}^n \to \mathbb{N}^{m_i}$, $w = [w_1, ..., w_q]^{\mathrm{T}} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$, $\ell_i : \mathbb{R}^n \to \mathbb{R}^{s_i}$, $\mathcal{Z}_i : \mathbb{R}^n \to \mathbb{R}^{s_i\times m_i}$ such that $v_{\mathrm{s}i}(\cdot)$ is continuous, $v_{\mathrm{s}i}(0) = 0$, $i = 1, ..., q$, $w \in \mathcal{W}$, $w(0) = 0$, the zero solution $r(k) \equiv 0$ to (2.59) is Lyapunov (respectively, asymptotically) stable, and, for all $x \in \mathbb{R}^n$ and $i = 1, ..., q$,

$$0 \leq v_{\mathrm{s}i}(\mathcal{F}_i(x)) - v_{\mathrm{s}i}(\mathcal{F}_i(x) + G_i(x_i)u_i) + P_{1i}(x)u_i + u_i^{\mathrm{T}} P_{2i}(x)u_i, \tag{2.68}$$

$$0 \geq v_{\mathrm{s}i}(\mathcal{F}_i(x)) - h_i^{\mathrm{T}}(x_i)Q_i h_i(x_i) - w_i(V_{\mathrm{s}}(x)) + \ell_i^{\mathrm{T}}(x_i)\ell_i(x_i), \tag{2.69}$$

$$0 = \tfrac{1}{2}P_{1i}(x) - h_i^{\mathrm{T}}(x_i)(S_i + Q_i J_i(x_i)) + \ell_i^{\mathrm{T}}(x_i)\mathcal{Z}_i(x_i), \tag{2.70}$$

$$0 \leq R_i + J_i^{\mathrm{T}}(x_i)S_i + S_i^{\mathrm{T}} J_i(x_i) + J_i^{\mathrm{T}}(x_i)Q_i J_i(x_i) - P_{2i}(x) - \mathcal{Z}_i^{\mathrm{T}}(x_i)\mathcal{Z}_i(x_i). \tag{2.71}$$

Then $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u, y)$, where $s_i(u_i, y_i) = u_i^{\mathrm{T}} R_i u_i + 2y_i^{\mathrm{T}} S_i u_i + y_i^{\mathrm{T}} Q_i y_i$, $i = 1, ..., q$.

Proof. For any admissible input $u = [u_1^{\mathrm{T}}, ..., u_q^{\mathrm{T}}]^{\mathrm{T}}$ such that $u_i \in \mathbb{R}^{m_i}$, $k \in \overline{\mathbb{Z}}_+$, and $i = 1, ..., q$, it follows from (2.68)–(2.71) that

$$\begin{aligned}
s_i(u_i(k), y_i(k)) &= u_i^{\mathrm{T}}(k)R_i u_i(k) + 2y_i^{\mathrm{T}}(k)S_i u_i(k) + y_i^{\mathrm{T}}(k)Q_i y_i(k) \\
&= h_i^{\mathrm{T}}(x_i(k))Q_i h_i(x_i(k)) + 2h_i^{\mathrm{T}}(x_i(k))(S_i + Q_i J_i(x_i(k)))u_i(k) \\
&\quad + u_i^{\mathrm{T}}(k)(J_i^{\mathrm{T}}(x_i(k))Q_i J_i(x_i(k)) + J_i^{\mathrm{T}}(x_i(k))S_i \\
&\quad + S_i^{\mathrm{T}} J_i(x_i(k)) + R_i)u_i(k) \\
&\geq v_{\mathrm{s}i}(\mathcal{F}_i(x(k))) + P_{1i}(x(k))u_i(k) + \ell_i^{\mathrm{T}}(x_i(k))\ell_i(x_i(k)) \\
&\quad + 2\ell_i^{\mathrm{T}}(x_i(k))\mathcal{Z}_i(x_i(k))u_i(k) + u_i^{\mathrm{T}}(k)P_{2i}(x(k))u_i(k) \\
&\quad + u_i^{\mathrm{T}}(k)\mathcal{Z}_i^{\mathrm{T}}(x_i(k))\mathcal{Z}_i(x_i(k))u_i(k) - w_i(V_{\mathrm{s}}(x(k))) \\
&\geq v_{\mathrm{s}i}(x_i(k+1)) + [\ell_i(x_i(k)) + \mathcal{Z}_i(x_i(k))u_i(k)]^{\mathrm{T}}[\ell_i(x_i(k)) \\
&\quad + \mathcal{Z}_i(x_i(k))u_i(k)] - w_i(V_{\mathrm{s}}(x(k))) \\
&\geq v_{\mathrm{s}i}(x_i(k+1)) - w_i(V_{\mathrm{s}}(x(k))),
\end{aligned} \tag{2.72}$$

where $x(k)$, $k \geq k_0$, satisfies (2.11). Now, the result follows from (2.72) with vector storage function $V_{\mathrm{s}}(x) = [v_{\mathrm{s}1}(x_1), ..., v_{\mathrm{s}q}(x_q)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$. □

Finally, we provide necessary and sufficient conditions for the case where the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ is vector lossless with respect to a vector quadratic supply rate.

Theorem 2.9. Consider the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ given by (2.11) and (2.12). Let $R_i \in \mathbb{S}^{m_i}$, $S_i \in \mathbb{R}^{l_i \times m_i}$, and $Q_i \in \mathbb{S}^{l_i}$, $i = 1, ..., q$. Then $\mathcal{G}$ is vector lossless with respect to the vector quadratic supply rate $S(u, y)$, where $s_i(u_i, y_i) = u_i^{\mathrm{T}} R_i u_i + 2y_i^{\mathrm{T}} S_i u_i + y_i^{\mathrm{T}} Q_i y_i$, $i = 1, ..., q$, if and only if there exist functions $V_{\mathrm{s}} = [v_{\mathrm{s}1}, ..., v_{\mathrm{s}q}]^{\mathrm{T}} : \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$, $P_{1i} : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, $P_{2i} : \mathbb{R}^n \to \mathbb{N}^m$, $w = [w_1, ..., w_q]^{\mathrm{T}} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ such that $v_{\mathrm{s}i}(\cdot)$ is continuous, $v_{\mathrm{s}i}(0) = 0$, $i = 1, ..., q$, $w \in \mathcal{W}$, $w(0) = 0$, the zero solution $r(k) \equiv 0$ to (2.59) is Lyapunov stable, and, for all $x \in \mathbb{R}^n$, $i = 1, ..., q$, (2.61) holds and

$$0 = v_{\mathrm{s}i}(\mathcal{F}(x)) - h^{\mathrm{T}}(x)\hat{Q}_i h(x) - w_i(V_{\mathrm{s}}(x)), \tag{2.73}$$

$$0 = \tfrac{1}{2}P_{1i}(x) - h^{\mathrm{T}}(x)(\hat{S}_i + \hat{Q}_i J(x)), \tag{2.74}$$

$$0 = \hat{R}_i + J^{\mathrm{T}}(x)\hat{S}_i + \hat{S}_i^{\mathrm{T}} J(x) + J^{\mathrm{T}}(x)\hat{Q}_i J(x) - P_{2i}(x). \tag{2.75}$$

Proof. Sufficiency follows as in the proof of Theorem 2.7. To show necessity, suppose that $\mathcal{G}$ is lossless with respect to the vector quadratic supply rate $S(u, y)$. Then, there exist continuous functions $V_{\mathrm{s}} = [v_{\mathrm{s}1}, ..., v_{\mathrm{s}q}]^{\mathrm{T}} : \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$ and $w = [w_1, ..., w_q]^{\mathrm{T}} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ such that $V_{\mathrm{s}}(0) = 0$, the zero solution $r(k) \equiv 0$ to (2.59) is Lyapunov stable and

$$\begin{aligned} v_{\mathrm{s}i}(\mathcal{F}(x) + G(x)u) =&\ w_i(V_{\mathrm{s}}(x)) + s_i(u_i, y_i) \\ =&\ w_i(V_{\mathrm{s}}(x)) + u^{\mathrm{T}}\hat{R}_i u + 2y^{\mathrm{T}}\hat{S}_i u + y^{\mathrm{T}}\hat{Q}_i y \\ =&\ w_i(V_{\mathrm{s}}(x)) + h^{\mathrm{T}}(x)\hat{Q}_i h(x) + 2h^{\mathrm{T}}(x)(\hat{Q}_i J(x) + \hat{S}_i)u \\ &\ +u^{\mathrm{T}}(\hat{R}_i + \hat{S}_i^{\mathrm{T}} J(x) + J^{\mathrm{T}}(x)\hat{S}_i + J^{\mathrm{T}}(x)\hat{Q}_i J(x))u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m. \end{aligned} \tag{2.76}$$

Since the right-hand-side of (2.76) is quadratic in u it follows that $v_{\mathrm{s}i}(\mathcal{F}(x) + G(x)u)$ is quadratic in u, and hence, there exist $P_{1i} : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ and $P_{2i} : \mathbb{R}^n \to \mathbb{N}^m$ such that

$$v_{\mathrm{s}i}(\mathcal{F}(x) + G(x)u) = v_{\mathrm{s}i}(\mathcal{F}(x)) + P_{1i}(x)u + u^{\mathrm{T}}P_{2i}(x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m. \tag{2.77}$$

Now, using (2.77) and equating coefficients of equal powers in (2.76) yields (2.73)–(2.75). □

2.5. Specialization to Discrete-Time Large-Scale Linear Dynamical Systems

In this section, we specialize the results of Section 2.4 to the case of discrete-time large-scale linear dynamical systems. Specifically, we assume that $w \in \mathcal{W}$ is linear so that $w(r) = Wr$, where $W \in \mathbb{R}^{q\times q}$ is nonnegative, and consider the discrete-time large-scale linear dynamical system $\mathcal{G}$ given by

$$x(k+1) = Ax(k) + Bu(k), \quad x(k_0) = x_0, \quad k \geq k_0, \tag{2.78}$$

$$y(k) = Cx(k) + Du(k), \tag{2.79}$$

where $A \in \mathbb{R}^{n\times n}$ and A is partitioned as $A \triangleq [A_{ij}], i, j = 1, ..., q$, $A_{ij} \in \mathbb{R}^{n_i\times n_j}$, $\sum_{i=1}^{q} n_i = n$, $B = \text{block–diag}[B_1, ..., B_q]$, $C = \text{block–diag}[C_1, ..., C_q]$, $D = \text{block–diag}[D_1, ..., D_q]$, $B_i \in \mathbb{R}^{n_i\times m_i}$, $C_i \in \mathbb{R}^{l_i\times n_i}$, $D_i \in \mathbb{R}^{l_i\times m_i}$, and $i = 1, ..., q$.

Theorem 2.10. Consider the discrete-time large-scale linear dynamical system $\mathcal{G}$ given by (2.78) and (2.79). Let $R_i \in \mathbb{S}^{m_i}$, $S_i \in \mathbb{R}^{l_i\times m_i}$, $Q_i \in \mathbb{S}^{l_i}$, $i = 1, ..., q$. Then $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u, y)$, where $s_i(u_i, y_i) = u_i^{\mathrm{T}} R_i u_i + 2y_i^{\mathrm{T}} S_i u_i + y_i^{\mathrm{T}} Q_i y_i$, $i = 1, ..., q$, and with a three-times continuously differentiable vector storage function if and only if there exist $W \in \mathbb{R}^{q\times q}$, $P_i \in \mathbb{N}^n$, $L_i \in \mathbb{R}^{s_i\times n}$, and $Z_i \in \mathbb{R}^{s_i\times m}$, $i = 1, ..., q$, such that W is nonnegative and semistable (respectively, asymptotically stable), and, for all $i = 1, ..., q$,

$$0 = A^{\mathrm{T}} P_i A - C^{\mathrm{T}} \hat{Q}_i C - \sum_{j=1}^{q} W_{(i,j)} P_j + L_i^{\mathrm{T}} L_i, \tag{2.80}$$

$$0 = A^{\mathrm{T}} P_i B - C^{\mathrm{T}}(\hat{S}_i + \hat{Q}_i D) + L_i^{\mathrm{T}} Z_i, \tag{2.81}$$

$$0 = \hat{R}_i + D^{\mathrm{T}} \hat{S}_i + \hat{S}_i^{\mathrm{T}} D + D^{\mathrm{T}} \hat{Q}_i D - B^{\mathrm{T}} P_i B - Z_i^{\mathrm{T}} Z_i. \tag{2.82}$$

Proof. Sufficiency follows from Theorem 2.7 with $\mathcal{F}(x) = Ax$, $G(x) = B$, $h(x) = Cx$, $J(x) = D$, $P_{1i}(x) = 2x^{\mathrm{T}} A^{\mathrm{T}} P_i B$, $P_{2i}(x) = B^{\mathrm{T}} P_i B$, $w(r) = Wr$, $\ell_i(x) = L_i x$, $\mathcal{Z}_i(x) = Z_i$, and $v_{\mathrm{s}i}(x) = x^{\mathrm{T}} P_i x$, $i = 1, ..., q$. To show necessity, suppose $\mathcal{G}$ is vector dissipative with respect to the vector supply rate $S(u,y)$, where $s_i(u_i,y_i) = u_i^{\mathrm{T}} R_i u_i + 2y_i^{\mathrm{T}} S_i u_i + y_i^{\mathrm{T}} Q_i y_i$, $i = 1, ..., q$. Then, with $w(r) = Wr$, there exists $V_{\mathrm{s}} : \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$ such that W is nonnegative and semistable (respectively, asymptotically stable), $V_{\mathrm{s}}(x) \triangleq [v_{\mathrm{s}1}(x), ..., v_{\mathrm{s}q}(x)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$, $V_{\mathrm{s}}(0) = 0$, and for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$,

$$V_{\mathrm{s}}(Ax + Bu) - WV_{\mathrm{s}}(x) \leq\leq S(u,y). \tag{2.83}$$

Next, it follows from (2.83) that there exists a three-times continuously differentiable vector function $d = [d_1, ..., d_q]^{\mathrm{T}} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q$ such that $d(x,u) \geq\geq 0$, $d(0,0) = 0$, and

$$0 = V_{\mathrm{s}}(Ax + Bu) - WV_{\mathrm{s}}(x) - S(u, Cx + Du) + d(x,u). \tag{2.84}$$

Now, expanding $v_{\mathrm{s}i}(\cdot)$ and $d_i(\cdot,\cdot)$ via Taylor series expansion about $x = 0$, $u = 0$, and using the fact that $v_{\mathrm{s}i}(\cdot)$ and $d_i(\cdot,\cdot)$ are nonnegative and $v_{\mathrm{s}i}(0) = 0, d_i(0,0) = 0$, $i = 1, ..., q$, it follows that there exist $P_i \in \mathbb{N}^n$, $L_i \in \mathbb{R}^{s_i \times n}$, $Z_i \in \mathbb{R}^{s_i \times m}$, $i = 1, ..., q$, such that

$$v_{\mathrm{s}i}(x) = x^{\mathrm{T}} P_i x + v_{\mathrm{sr}i}(x), \tag{2.85}$$

$$d_i(x,u) = (L_i x + Z_i u)^{\mathrm{T}} (L_i x + Z_i u) + d_{\mathrm{r}i}(x,u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad i = 1, ..., q, \tag{2.86}$$

where $v_{\mathrm{sr}i} : \mathbb{R}^n \to \mathbb{R}$ and $d_{\mathrm{r}i} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ contain the higher-order terms of $v_{\mathrm{s}i}(\cdot)$, $d_i(\cdot,\cdot)$, respectively. Using the above expressions, (2.84) can be written componentwise as

$$\begin{aligned} 0 = {} & (Ax + Bu)^{\mathrm{T}} P_i (Ax + Bu) - \sum_{j=1}^{q} W_{(i,j)} x^{\mathrm{T}} P_j x - (x^{\mathrm{T}} C^{\mathrm{T}} \hat{Q}_i Cx + 2x^{\mathrm{T}} C^{\mathrm{T}} \hat{Q}_i Du \\ & + u^{\mathrm{T}} D^{\mathrm{T}} \hat{Q}_i Du + 2x^{\mathrm{T}} C^{\mathrm{T}} \hat{S}_i u + 2u^{\mathrm{T}} D^{\mathrm{T}} \hat{S}_i u + u^{\mathrm{T}} \hat{R}_i u) \\ & + (L_i x + Z_i u)^{\mathrm{T}} (L_i x + Z_i u) + \delta(x,u), \end{aligned} \tag{2.87}$$

where $\delta(x,u)$ is such that

$$\lim_{\|x\|^2 + \|u\|^2 \to 0} \frac{|\delta(x,u)|}{\|x\|^2 + \|u\|^2} = 0. \tag{2.88}$$

Finally, viewing (2.87) as the componentwise Taylor series expansion of (2.84) about $x = 0$ and $u = 0$ it follows that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,

$$\begin{aligned} 0 = \; & x^{\mathrm{T}}(A^{\mathrm{T}} P_i A - \sum_{j=1}^{q} W_{(i,j)} P_j - C^{\mathrm{T}} \hat{Q}_i C + L_i^{\mathrm{T}} L_i)x \\ & +2x^{\mathrm{T}}(A^{\mathrm{T}} P_i B - C^{\mathrm{T}} \hat{S}_i - C^{\mathrm{T}} \hat{Q}_i D + L_i^{\mathrm{T}} Z_i)u \\ & +u^{\mathrm{T}}(Z_i^{\mathrm{T}} Z_i - D^{\mathrm{T}} \hat{Q}_i D - D^{\mathrm{T}} \hat{S}_i - \hat{S}_i^{\mathrm{T}} D - \hat{R}_i + B^{\mathrm{T}} P_i B)u, \quad i = 1, ..., q. \end{aligned} \tag{2.89}$$

Now, equating coefficients of equal powers in (2.89) yields (2.80)–(2.82). $\square$

Remark 2.7. Note that (2.80)–(2.82) are equivalent to

$$\begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{B}_i^{\mathrm{T}} & \mathcal{C}_i \end{bmatrix} = - \begin{bmatrix} L_i^{\mathrm{T}} \\ Z_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} L_i & Z_i \end{bmatrix} \le 0, \quad i = 1, ..., q, \tag{2.90}$$

where, for all $i = 1, ..., q$,

$$\mathcal{A}_i = A^{\mathrm{T}} P_i A - C^{\mathrm{T}} \hat{Q}_i C - \sum_{j=1}^{q} W_{(i,j)} P_j, \tag{2.91}$$

$$\mathcal{B}_i = A^{\mathrm{T}} P_i B - C^{\mathrm{T}} (\hat{S}_i + \hat{Q}_i D), \tag{2.92}$$

$$\mathcal{C}_i = -(\hat{R}_i + D^{\mathrm{T}} \hat{S}_i + \hat{S}_i^{\mathrm{T}} D + D^{\mathrm{T}} \hat{Q}_i D - B^{\mathrm{T}} P_i B). \tag{2.93}$$

Hence, vector dissipativity of discrete-time large-scale linear dynamical systems with respect to vector quadratic supply rates can be characterized via (cascade) linear matrix inequalities (LMIs) [36]. A similar remark holds for Theorem 2.11 below.

The next result presents sufficient conditions guaranteeing vector dissipativity of $\mathcal{G}$ with respect to a vector quadratic supply rate in case where the vector storage function is component decoupled.

Theorem 2.11. Consider the discrete-time large-scale linear dynamical system $\mathcal{G}$ given by (2.78) and (2.79). Let $R_i \in \mathbb{S}^{m_i}$, $S_i \in \mathbb{R}^{l_i \times m_i}$, $Q_i \in \mathbb{S}^{l_i}$, $i = 1, ..., q$, be given. Assume there exist matrices $W \in \mathbb{R}^{q \times q}$, $P_i \in \mathbb{N}^{n_i}$, $L_{ii} \in \mathbb{R}^{s_{ii} \times n_i}$, $Z_{ii} \in \mathbb{R}^{s_{ii} \times m_i}$, $i = 1, ..., q$, $L_{ij} \in \mathbb{R}^{s_{ij} \times n_i}$, and

$Z_{ij} \in \mathbb{R}^{s_{ij} \times n_j}$, $i, j = 1, ..., q$, $i \neq j$, such that W is nonnegative and semistable (respectively, asymptotically stable), and, for all $i = 1, ..., q$,

$$0 \geq A_{ii}^{\mathrm{T}} P_i A_{ii} - C_i^{\mathrm{T}} Q_i C_i - W_{(i,i)} P_i + L_{ii}^{\mathrm{T}} L_{ii} + \sum_{j=1,\, j \neq i}^{q} L_{ij}^{\mathrm{T}} L_{ij}, \tag{2.94}$$

$$0 = A_{ii}^{\mathrm{T}} P_i B_i - C_i^{\mathrm{T}} S_i - C_i^{\mathrm{T}} Q_i D_i + L_{ii}^{\mathrm{T}} Z_{ii}, \tag{2.95}$$

$$0 \leq R_i + D_i^{\mathrm{T}} S_i + S_i^{\mathrm{T}} D_i + D_i^{\mathrm{T}} Q_i D_i - B_i^{\mathrm{T}} P_i B_i - Z_{ii}^{\mathrm{T}} Z_{ii}, \tag{2.96}$$

and for $j = 1, ..., q, l = 1, ..., q$, $j \neq i$, $l \neq i$, $l \neq j$,

$$0 = A_{ij}^{\mathrm{T}} P_i B_i, \tag{2.97}$$

$$0 = A_{ij}^{\mathrm{T}} P_i A_{il}, \tag{2.98}$$

$$0 = A_{ii}^{\mathrm{T}} P_i A_{ij} + L_{ij}^{\mathrm{T}} Z_{ij}, \tag{2.99}$$

$$0 \leq W_{(i,j)} P_j - Z_{ij}^{\mathrm{T}} Z_{ij} - A_{ij}^{\mathrm{T}} P_i A_{ij}. \tag{2.100}$$

Then $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u, y) \triangleq [s_1(u_1, y_1), ..., s_q(u_q, y_q)]^{\mathrm{T}}$, where $s_i(u_i, y_i) = u_i^{\mathrm{T}} R_i u_i + 2 y_i^{\mathrm{T}} S_i u_i + y_i^{\mathrm{T}} Q_i y_i$, $i = 1, ..., q$.

Proof. Since $P_i \in \mathbb{N}^{n_i}$, the function $v_{\mathrm{s}i}(x_i) \triangleq x_i^{\mathrm{T}} P_i x_i$, $x_i \in \mathbb{R}^{n_i}$, is nonnegative definite and $v_{\mathrm{s}i}(0) = 0$. Moreover, since $v_{\mathrm{s}i}(\cdot)$ is continuous it follows from (2.94)–(2.100) that for all $u_i \in \mathbb{R}^{m_i}$, $i = 1, ..., q$, and $k \geq k_0$,

$$\begin{aligned}
v_{\mathrm{s}i}(x_i(k+1)) = & \left[\sum_{j=1}^{q} A_{ij} x_j(k) + B_i u_i(k)\right]^{\mathrm{T}} P_i \left[\sum_{j=1}^{q} A_{ij} x_j(k) + B_i u_i(k)\right] \\
\leq & \; x_i^{\mathrm{T}}(k) \left[W_{(i,i)} P_i + C_i^{\mathrm{T}} Q_i C_i - L_{ii}^{\mathrm{T}} L_{ii} - \sum_{j=1,\, j \neq i}^{q} L_{ij}^{\mathrm{T}} L_{ij}\right] x_i(k) \\
& - \sum_{j=1,\, j \neq i}^{q} 2 x_i^{\mathrm{T}}(k) L_{ij}^{\mathrm{T}} Z_{ij} x_j(k) + 2 x_i^{\mathrm{T}}(k) C_i^{\mathrm{T}} S_i u_i(k) + 2 x_i^{\mathrm{T}}(k) C_i^{\mathrm{T}} Q_i D_i u_i(k) \\
& - 2 x_i^{\mathrm{T}}(k) L_{ii}^{\mathrm{T}} Z_{ii} u_i(k) + \sum_{j=1,\, j \neq i}^{q} x_j^{\mathrm{T}}(k) [W_{(i,j)} P_j - Z_{ij}^{\mathrm{T}} Z_{ij}] x_j(k) \\
& + u_i^{\mathrm{T}}(k) R_i u_i(k) + 2 u_i^{\mathrm{T}}(k) D_i^{\mathrm{T}} S_i u_i(k) + u_i^{\mathrm{T}}(k) D_i^{\mathrm{T}} Q_i D_i u_i(k)
\end{aligned}$$

$$
\begin{aligned}
&\quad -u_i^{\mathrm{T}}(k)Z_{ii}^{\mathrm{T}}Z_{ii}u_i(k)\\
&= \sum_{j=1}^{q} W_{(i,j)}v_{\mathrm{s}j}(x_j(k)) + u_i^{\mathrm{T}}(k)R_iu_i(k) + 2y_i^{\mathrm{T}}(k)S_iu_i(k) + y_i^{\mathrm{T}}(k)Q_iy_i(k)\\
&\quad -[L_{ii}x_i(k) + Z_{ii}u_i(k)]^{\mathrm{T}}[L_{ii}x_i(k) + Z_{ii}u_i(k)]\\
&\quad -\sum_{j=1,\, j\neq i}^{q} (L_{ij}x_i(k) + Z_{ij}x_j(k))^{\mathrm{T}}(L_{ij}x_i(k) + Z_{ij}x_j(k))\\
&\leq s_i(u_i(k), y_i(k)) + \sum_{j=1}^{q} W_{(i,j)}v_{\mathrm{s}j}(x_j(k)),
\end{aligned}
\tag{2.101}
$$

or, equivalently, in vector form

$$
V_{\mathrm{s}}(x(k+1)) \leq\leq WV_{\mathrm{s}}(x(k)) + S(u,y), \quad u \in \mathcal{U}, \quad k \geq k_0, \tag{2.102}
$$

where $V_{\mathrm{s}}(x) \triangleq [v_{\mathrm{s}1}(x_1), ..., v_{\mathrm{s}q}(x_q)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$. Now, it follows from Proposition 2.3 that $\mathcal{G}$ is vector dissipative (respectively, geometrically vector dissipative) with respect to the vector supply rate $S(u,y)$ and with vector storage function $V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$. □

2.6. Stability of Feedback Interconnections of Discrete-Time Large-Scale Nonlinear Dynamical Systems

In this section, we consider stability of feedback interconnections of discrete-time large-scale nonlinear dynamical systems. Specifically, for the discrete-time large-scale dynamical system $\mathcal{G}$ given by (2.11) and (2.12) we consider either a dynamic or static discrete-time large-scale feedback system $\mathcal{G}_{\mathrm{c}}$. Then by appropriately combining vector storage functions for each system we show stability of the feedback interconnection. We begin by considering the discrete-time large-scale nonlinear dynamical system (2.11) and (2.12) with the large-scale feedback system $\mathcal{G}_{\mathrm{c}}$ given by

$$
x_{\mathrm{c}}(k+1) = F_{\mathrm{c}}(x_{\mathrm{c}}(k), u_{\mathrm{c}}(k)), \quad x_{\mathrm{c}}(k_0) = x_{\mathrm{c}0}, \quad k \geq k_0, \tag{2.103}
$$

$$
y_{\mathrm{c}}(k) = H_{\mathrm{c}}(x_{\mathrm{c}}(k), u_{\mathrm{c}}(k)), \tag{2.104}
$$

where $F_{\rm c}:\mathbb{R}^{n_{\rm c}}\times\mathcal{U}_{\rm c}\to\mathbb{R}^{n_{\rm c}}$, $H_{\rm c}:\mathbb{R}^{n_{\rm c}}\times\mathcal{U}_{\rm c}\to\mathcal{Y}_{\rm c}$, $F_{\rm c}\triangleq[F_{{\rm c}1}^{\rm T},...,F_{{\rm c}q}^{\rm T}]^{\rm T}$, $H_{\rm c}\triangleq[H_{{\rm c}1}^{\rm T},...,H_{{\rm c}q}^{\rm T}]^{\rm T}$, $\mathcal{U}_{\rm c}\subseteq\mathbb{R}^{l}$, $\mathcal{Y}_{\rm c}\subseteq\mathbb{R}^{m}$. Moreover, for all $i=1,...,q$, we assume that

$$F_{{\rm c}i}(x_{\rm c},u_{{\rm c}i}) = f_{{\rm c}i}(x_{{\rm c}i})+\mathcal{I}_{{\rm c}i}(x_{\rm c})+G_{{\rm c}i}(x_{{\rm c}i})u_{{\rm c}i}, \tag{2.105}$$

$$H_{{\rm c}i}(x_{{\rm c}i},u_{{\rm c}i}) = h_{{\rm c}i}(x_{{\rm c}i})+J_{{\rm c}i}(x_{{\rm c}i})u_{{\rm c}i}, \tag{2.106}$$

where $u_{{\rm c}i}\in\mathcal{U}_{{\rm c}i}\subseteq\mathbb{R}^{l_i}$, $y_{{\rm c}i}\triangleq H_{{\rm c}i}(x_{{\rm c}i},u_{{\rm c}i})\in\mathcal{Y}_i\subseteq\mathbb{R}^{m_i}$, $(u_{{\rm c}i},y_{{\rm c}i})$ is the input-output pair for the ith subsystem of $\mathcal{G}_{\rm c}$, $f_{{\rm c}i}:\mathbb{R}^{n_{{\rm c}i}}\to\mathbb{R}^{n_{{\rm c}i}}$ and $\mathcal{I}_{{\rm c}i}:\mathbb{R}^{n_{\rm c}}\to\mathbb{R}^{n_{{\rm c}i}}$ satisfy $f_{{\rm c}i}(0)=0$ and $\mathcal{I}_{{\rm c}i}(0)=0$, $G_{{\rm c}i}:\mathbb{R}^{n_{{\rm c}i}}\to\mathbb{R}^{n_{{\rm c}i}\times l_i}$, $h_{{\rm c}i}:\mathbb{R}^{n_{{\rm c}i}}\to\mathbb{R}^{m_i}$ and satisfies $h_{{\rm c}i}(0)=0$, $J_{{\rm c}i}:\mathbb{R}^{n_{{\rm c}i}}\to\mathbb{R}^{m_i\times l_i}$, and $\sum_{i=1}^{q}n_{{\rm c}i}=n_{\rm c}$.

We define the composite input and composite output for the system $\mathcal{G}_{\rm c}$ as $u_{\rm c}\triangleq[u_{{\rm c}1}^{\rm T},\ldots,u_{{\rm c}q}^{\rm T}]^{\rm T}$ and $y_{\rm c}\triangleq[y_{{\rm c}1}^{\rm T},...,y_{{\rm c}q}^{\rm T}]^{\rm T}$, respectively. In this case, $\mathcal{U}_{\rm c}=\mathcal{U}_{{\rm c}1}\times\cdots\times\mathcal{U}_{{\rm c}q}$ and $\mathcal{Y}_{\rm c}=\mathcal{Y}_{{\rm c}1}\times\cdots\times\mathcal{Y}_{{\rm c}q}$. Note that with the feedback interconnection given by Figure 2.1, $u_{\rm c}=y$ and $y_{\rm c}=-u$. We assume that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is well posed, that is, $\det(I_{m_i}+J_{{\rm c}i}(x_{{\rm c}i})J_i(x_i))\neq 0$ for all $x_i\in\mathbb{R}^{n_i}$, $x_{{\rm c}i}\in\mathbb{R}^{n_{{\rm c}i}}$, and $i=1,...,q$. Furthermore, we assume that for the discrete-time large-scale systems $\mathcal{G}$ and $\mathcal{G}_{\rm c}$, the conditions of Theorem 2.4 are satisfied; that is, if $V_{\rm s}(x)$, $x\in\mathbb{R}^n$, and $V_{\rm cs}(x_{\rm c})$, $x_{\rm c}\in\mathbb{R}^{n_{\rm c}}$, are vector storage functions for $\mathcal{G}$ and $\mathcal{G}_{\rm c}$, respectively, then there exist $p\in\mathbb{R}_+^q$ and $p_{\rm c}\in\mathbb{R}_+^q$ such that the functions $v_{\rm s}(x)=p^{\rm T}V_{\rm s}(x)$, $x\in\mathbb{R}^n$, and $v_{\rm cs}(x_{\rm c})=p_{\rm c}^{\rm T}V_{\rm cs}(x_{\rm c})$, $x_{\rm c}\in\mathbb{R}^{n_{\rm c}}$, are positive definite. The following result gives sufficient conditions for Lyapunov and asymptotic stability of the feedback interconnection given by Figure 2.1.

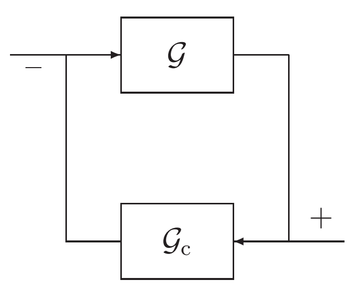

Figure 2.1: Feedback interconnection of large-scale systems $\mathcal{G}$ and $\mathcal{G}_{\rm c}$

Theorem 2.12. Consider the discrete-time large-scale nonlinear dynamical systems $\mathcal{G}$ and $\mathcal{G}_c$ given by (2.11) and (2.12), and (2.103) and (2.104), respectively. Assume that $\mathcal{G}$ and $\mathcal{G}_c$ are vector dissipative with respect to the vector supply rates $S(u,y)$ and $S_c(u_c,y_c)$, and with continuous vector storage functions $V_s(\cdot)$ and $V_{cs}(\cdot)$ and dissipation matrices $W \in \mathbb{R}^{q\times q}$ and $W_c \in \mathbb{R}^{q\times q}$, respectively.

i) If there exists $\Sigma \triangleq \mathrm{diag}[\sigma_1, ..., \sigma_q] > 0$ such that $S(u,y) + \Sigma S_c(u_c,y_c) \leq\leq 0$ and $\tilde{W} \in \mathbb{R}^{q\times q}$ is semistable (respectively, asymptotically stable), where $\tilde{W}_{(i,j)} \triangleq \max\{W_{(i,j)}, (\Sigma W_c \Sigma^{-1})_{(i,j)}\} = \max\{W_{(i,j)}, \frac{\sigma_i}{\sigma_j} W_{c(i,j)}\}$, $i,j = 1,...,q$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is Lyapunov (respectively, asymptotically) stable.

ii) Let $Q_i \in \mathbb{S}^{l_i}$, $S_i \in \mathbb{R}^{l_i\times m_i}$, $R_i \in \mathbb{S}^{m_i}$, $Q_{ci} \in \mathbb{S}^{m_i}$, $S_{ci} \in \mathbb{R}^{m_i\times l_i}$, and $R_{ci} \in \mathbb{S}^{l_i}$, and suppose $S(u,y) = [s_1(u_1,y_1), ..., s_q(u_q,y_q)]^{\mathrm{T}}$ and $S_c(u_c,y_c) = [s_{c1}(u_{c1},y_{c1}), ..., s_q(u_{cq},y_{cq})]^{\mathrm{T}}$, where $s_i(u_i,y_i) = u_i^{\mathrm{T}} R_i u_i + 2y_i^{\mathrm{T}} S_i u_i + y_i^{\mathrm{T}} Q_i y_i$ and $s_{ci}(u_{ci},y_{ci}) = u_{ci}^{\mathrm{T}} R_{ci} u_{ci} + 2y_{ci}^{\mathrm{T}} S_{ci} u_{ci} + y_{ci}^{\mathrm{T}} Q_{ci} y_{ci}$, $i = 1,...,q$. If there exists $\Sigma \triangleq \mathrm{diag}[\sigma_1,...,\sigma_q] > 0$ such that for all $i = 1,...,q$,

$$\tilde{Q}_i \triangleq \begin{bmatrix} Q_i + \sigma_i R_{ci} & -S_i + \sigma_i S_{ci}^{\mathrm{T}} \\ -S_i^{\mathrm{T}} + \sigma_i S_{ci} & R_i + \sigma_i Q_{ci} \end{bmatrix} \leq 0 \tag{2.107}$$

and $\tilde{W} \in \mathbb{R}^{q\times q}$ is semistable (respectively, asymptotically stable), where $\tilde{W}_{(i,j)} \triangleq \max\{W_{(i,j)}, (\Sigma W_c \Sigma^{-1})_{(i,j)}\} = \max\{W_{(i,j)}, \frac{\sigma_i}{\sigma_j} W_{c(i,j)}\}$, $i,j = 1,...,q$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is Lyapunov (respectively, asymptotically) stable.

Proof. *i)* Consider the vector Lyapunov function candidate $V(x,x_c) = V_s(x) + \Sigma V_{cs}(x_c)$, $(x,x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c}$, and note that

$$\begin{aligned} V(x(k+1),x_c(k+1)) &= V_s(x(k+1)) + \Sigma V_{cs}(x_c(k+1)) \\ &\leq\leq S(u(k),y(k)) + \Sigma S_c(u_c(k),y_c(k)) + W V_s(x(k)) + \Sigma W_c V_{cs}(x_c(k)) \\ &\leq\leq W V_s(x(k)) + \Sigma W_c \Sigma^{-1} \Sigma V_{cs}(x_c(k)) \\ &\leq\leq \tilde{W}(V_s(x(k)) + \Sigma V_{cs}(x_c(k))) \\ &= \tilde{W} V(x(k),x_c(k)), \quad (x(k),x_c(k)) \in \mathbb{R}^n \times \mathbb{R}^{n_c}, \quad k \geq k_0. \end{aligned} \tag{2.108}$$

Next, since for $V_{\rm s}(x)$, $x \in \mathbb{R}^n$, and $V_{\rm cs}(x_{\rm c})$, $x_{\rm c} \in \mathbb{R}^{n_{\rm c}}$, there exist, by assumption, $p \in \mathbb{R}^q_+$ and $p_{\rm c} \in \mathbb{R}^q_+$ such that the functions $v_{\rm s}(x) = p^{\rm T}V_{\rm s}(x)$, $x \in \mathbb{R}^n$, and $v_{\rm cs}(x_{\rm c}) = p_{\rm c}^{\rm T}V_{\rm cs}(x_{\rm c})$, $x_{\rm c} \in \mathbb{R}^{n_{\rm c}}$, are positive definite and noting that $v_{\rm cs}(x_{\rm c}) \leq \max_{i=1,\ldots,q}\{p_{{\rm c}i}\}\mathbf{e}^{\rm T}V_{\rm cs}(x_{\rm c})$, where $p_{{\rm c}i}$ is the ith element of $p_{\rm c}$ and $\mathbf{e} = [1, \ldots, 1]^{\rm T}$, it follows that $\mathbf{e}^{\rm T}V_{\rm cs}(x_{\rm c})$, $x_{\rm c} \in \mathbb{R}^{n_{\rm c}}$, is positive definite. Now, since $\min_{i=1,\ldots,q}\{p_i\sigma_i\}\mathbf{e}^{\rm T}V_{\rm cs}(x_{\rm c}) \leq p^{\rm T}\Sigma V_{\rm cs}(x_{\rm c})$, it follows that $p^{\rm T}\Sigma V_{\rm cs}(x_{\rm c})$, $x_{\rm c} \in \mathbb{R}^{n_{\rm c}}$, is positive definite. Hence, the function $v(x, x_{\rm c}) = p^{\rm T}V(x, x_{\rm c})$, $(x, x_{\rm c}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\rm c}}$, is positive definite. Now, the result is a direct consequence of Theorem 2.1.

ii) The proof follows from *i*) by noting that, for all $i = 1, .., q$,

$$s_i(u_i, y_i) + \sigma_i s_{{\rm c}i}(u_{{\rm c}i}, y_{{\rm c}i}) = \begin{bmatrix} y \\ y_{\rm c} \end{bmatrix}^{\rm T} \tilde{Q}_i \begin{bmatrix} y \\ y_{\rm c} \end{bmatrix}, \tag{2.109}$$

and hence, $S(u, y) + \Sigma S_{\rm c}(u_{\rm c}, y_{\rm c}) \leq\leq 0$. □

For the next result note that if the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ is vector dissipative with respect to the vector supply rate $S(u, y)$, where $s_i(u_i, y_i) = 2y_i^{\rm T}u_i$, $i = 1, \ldots, q$, then with $\kappa_i(y_i) = -\kappa_i y_i$, where $\kappa_i > 0$, $i = 1, \ldots, q$, it follows that $s_i(\kappa_i(y_i), y_i) = -\kappa_i y_i^{\rm T}y_i < 0$, $y_i \neq 0$, $i = 1, \ldots, q$. Alternatively, if $\mathcal{G}$ is vector dissipative with respect to the vector supply rate $S(u, y)$, where $s_i(u_i, y_i) = \gamma_i^2 u_i^{\rm T}u_i - y_i^{\rm T}y_i$, where $\gamma_i > 0$, $i = 1, \ldots, q$, then with $\kappa_i(y_i) = 0$, it follows that $s_i(\kappa_i(y_i), y_i) = -y_i^{\rm T}y_i < 0$, $y_i \neq 0$, $i = 1, \ldots, q$. Hence, if $\mathcal{G}$ is zero-state observable and the dissipation matrix W is such that there exist $\alpha \geq 1$ and $p \in \mathbb{R}^q_+$ such that (2.2) holds, then it follows from Theorem 2.4 that (scalar) storage functions of the form $v_{\rm s}(x) = p^{\rm T}V_{\rm s}(x)$, $x \in \mathbb{R}^n$, where $V_{\rm s}(\cdot)$ is a vector storage function for $\mathcal{G}$, are positive definite. If $\mathcal{G}$ is geometrically vector dissipative, then p is positive.

Corollary 2.2. Consider the discrete-time large-scale nonlinear dynamical systems $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ given by (2.11) and (2.12), and (2.103) and (2.104), respectively. Assume that $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are zero-state observable and the dissipation matrices $W \in \mathbb{R}^{q\times q}$ and $W_{\rm c} \in \mathbb{R}^{q\times q}$ are such that there exist, respectively, $\alpha \geq 1$, $p \in \mathbb{R}^q_+$, $\alpha_{\rm c} \geq 1$, and $p_{\rm c} \in \mathbb{R}^q_+$ such that (2.2) is satisfied. Then the following statements hold:

i) If $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are vector passive and $\tilde{W} \in \mathbb{R}^{q\times q}$ is asymptotically stable, where $\tilde{W}_{(i,j)} \triangleq \max\{W_{(i,j)}, W_{{\rm c}(i,j)}\}$, $i, j = 1, ..., q$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is asymptotically stable.

ii) If $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are vector nonexpansive and $\tilde{W} \in \mathbb{R}^{q\times q}$ is asymptotically stable, where $\tilde{W}_{(i,j)} \triangleq \max\{W_{(i,j)}, W_{{\rm c}(i,j)}\}$, $i, j = 1, ..., q$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is asymptotically stable.

Proof. The proof is a direct consequence of Theorem 2.12. Specifically, *i*) follows from Theorem 2.12 with $R_i = 0$, $S_i = I_{m_i}$, $Q_i = 0$, $R_{{\rm c}i} = 0$, $S_{{\rm c}i} = I_{m_i}$, $Q_{{\rm c}i} = 0$, $i = 1, ..., q$, and $\Sigma = I_q$; while *ii*) follows from Theorem 2.12 with $R_i = \gamma_i^2 I_{m_i}$, $S_i = 0$, $Q_i = -I_{l_i}$, $R_{{\rm c}i} = \gamma_{{\rm c}i}^2 I_{l_i}$, $S_{{\rm c}i} = 0$, $Q_{{\rm c}i} = -I_{m_i}$, $i = 1, ..., q$, and $\Sigma = I_q$. $\square$

Chapter 3

Thermodynamic Modeling, Energy Equipartition, and Nonconservation of Entropy for Dynamical Systems

3.1. Introduction

Thermodynamic principles have been repeatedly used in continuous-time dynamical system theory as well as information theory for developing models that capture the exchange of nonnegative quantities (e.g., mass, energy, fluid, etc.) between coupled subsystems [26, 39, 43, 94, 199, 236, 244]. In particular, conservation laws (e.g., mass and energy) are used to capture the exchange of material between coupled macroscopic subsystems known as compartments. Each compartment is assumed to be kinetically homogeneous, that is, any material entering the compartment is instantaneously mixed with the material in the compartment. These models are known as *compartmental* models and are widespread in engineering systems as well as biological and ecological sciences [2,42,81,132,134,216]. Even though the compartmental models developed in the literature are based on the first law of thermodynamics involving conservation of energy principles, they do not tell us whether any particular process can actually occur; that is, they do not address the second law of thermodynamics involving entropy notions in the energy flow between subsystems.

The goal of this chapter is directed toward developing nonlinear discrete-time compartmental models that are consistent with thermodynamic principles. Specifically, since thermodynamic models are concerned with energy flow among subsystems, we develop a nonlinear compartmental dynamical system model that is characterized by energy conservation laws capturing the exchange of energy between coupled macroscopic subsystems. Furthermore, using graph theoretic notions we state three thermodynamic axioms consistent with the zeroth and second laws of thermodynamics that ensure that our large-scale dynamical system model gives rise to a thermodynamically consistent energy flow model. Specifically, using a large-scale dynamical systems theory perspective, we show that our compartmental dynamical system model leads to a precise formulation of the equivalence between work energy and heat in a large-scale dynamical system.

Next, we give a deterministic definition of entropy for a large-scale dynamical system that is consistent with the classical thermodynamic definition of entropy and show that it satisfies a Clausius-type inequality leading to the law of entropy nonconservation. Furthermore, we introduce a *new* and dual notion to entropy, namely, *ectropy*, as a measure of the tendency of a large-scale dynamical system to do useful work and grow more organized, and show that conservation of energy in an isolated thermodynamically consistent system necessarily leads to nonconservation of ectropy and entropy. Then, using the system ectropy as a Lyapunov function candidate we show that our thermodynamically consistent large-scale nonlinear dynamical system model possesses a continuum of equilibria and is *semistable*; that is, it has convergent subsystem energies to Lyapunov stable energy equilibria determined by the large-scale system initial subsystem energies. In addition, we show that the steady-state distribution of the large-scale system energies is uniform leading to system energy equipartitioning corresponding to a minimum ectropy and a maximum entropy equilibrium state. In the case where the subsystem energies are proportional to subsystem temperatures, we show that our dynamical system model leads to temperature equipartition wherein all the system energy is transferred into heat at a uniform temperature. Furthermore, we show that our system-theoretic definition of entropy and the newly proposed notion of ectropy are consistent with Boltzmann's kinetic theory of gases involving an n-body theory of ideal gases divided by diathermal walls.

3.2. Conservation of Energy and the First Law of Thermodynamics

We start this section by introducing notation and a key definition. We write $\mathcal{R}(M)$ and $\mathcal{N}(M)$ for the range space and the null space of a matrix M, respectively, $\mathrm{rank}(M)$ for the rank of the matrix M, $\mathrm{ind}(M)$ for the index of M, that is, $\min\{k \in \overline{\mathbb{Z}}_+ : \mathrm{rank}(M^k) = \mathrm{rank}(M^{k+1})\}$, $M^{\#}$ for the group generalized inverse of M where $\mathrm{ind}(M) \le 1$, and $\Delta E(x(k))$ for $E(x(k+1)) - E(x(k))$. The following definition introduces the notion of Z-, M-, nonnegative, and compartmental matrices.

Definition 3.1 [19, 26, 97]. Let $W \in \mathbb{R}^{q\times q}$. W is a *Z-matrix* if $W_{(i,j)} \le 0$, $i, j = 1, \ldots, q$, $i \ne j$. W is an *M-matrix* (respectively, a *nonsingular M-matrix*) if W is a Z-matrix and all the principal minors of W are nonnegative (respectively, positive). W is *nonnegative* (respectively, *positive*) if $W_{(i,j)} \ge 0$ (respectively, $W_{(i,j)} > 0$), $i, j = 1, \ldots, q$. Finally, W is *compartmental* if W is nonnegative and $\sum_{i=1}^{q} W_{(i,j)} \le 1$, $j = 1, \ldots, q$.

The fundamental and unifying concept in the analysis of complex (large-scale) dynamical systems is the concept of energy. The energy of a state of a dynamical system is the measure of its ability to produce changes (motion) in its own system state as well as changes in the system states of its surroundings. These changes occur as a direct consequence of the energy flow between different subsystems within the dynamical system. Since heat (energy) is a fundamental concept of thermodynamics involving the capacity of hot bodies (more energetic subsystems) to produce work, thermodynamics is a theory of large-scale dynamical systems [104]. As in thermodynamic systems, dynamical systems can exhibit energy (due to friction) that becomes unavailable to do useful work. This in turn contributes to an increase in system entropy; a measure of the tendency of a system to lose the ability to do useful work.

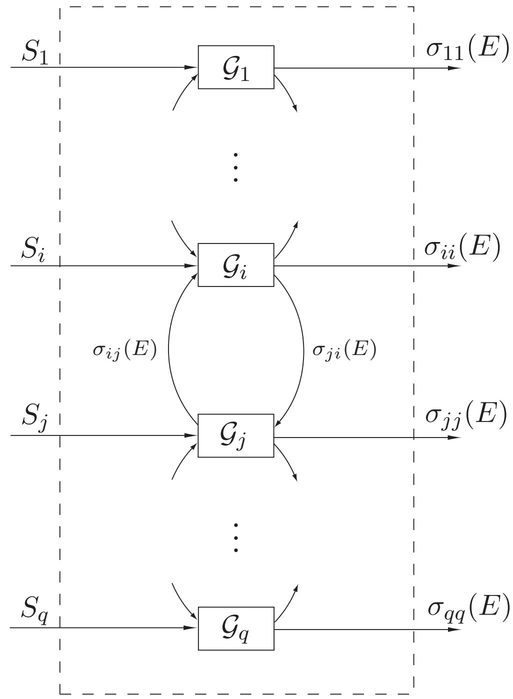

Figure 3.1: Large-scale dynamical system $\mathcal{G}$

To develop discrete-time compartmental models that are consistent with thermodynamic principles, consider the discrete-time large-scale dynamical system $\mathcal{G}$ shown in Figure 3.1 involving q interconnected subsystems. Let $E_i : \overline{\mathbb{Z}}_+ \to \overline{\mathbb{R}}_+$ denote the energy (and hence a nonnegative quantity) of the ith subsystem, let $S_i : \overline{\mathbb{Z}}_+ \to \mathbb{R}$ denote the external energy supplied to (or extracted from) the ith subsystem, let $\sigma_{ij} : \overline{\mathbb{R}}_+^q \to \overline{\mathbb{R}}_+$, $i \neq j$, $i, j = 1, \ldots, q$, denote the exchange of energy from the jth subsystem to the ith subsystem, and let $\sigma_{ii} : \overline{\mathbb{R}}_+^q \to \overline{\mathbb{R}}_+$, $i = 1, \ldots, q$, denote the energy loss from the ith subsystem. An *energy balance* equation for the ith subsystem yields

$$\Delta E_i(k) = \sum_{j=1,\, j\neq i}^{q} [\sigma_{ij}(E(k)) - \sigma_{ji}(E(k))] - \sigma_{ii}(E(k)) + S_i(k), \quad k \geq k_0, \tag{3.1}$$

or, equivalently, in vector form,

$$E(k+1) = w(E(k)) - d(E(k)) + S(k), \quad k \geq k_0, \tag{3.2}$$

where $E(k) = [E_1(k), \ldots, E_q(k)]^{\mathrm{T}}$, $S(k) = [S_1(k), \ldots, S_q(k)]^{\mathrm{T}}$, $d(E(k)) = [\sigma_{11}(E(k)), \ldots, \sigma_{qq}(E(k))]^{\mathrm{T}}$, $k \geq k_0$, and $w = [w_1, \ldots, w_q]^{\mathrm{T}} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ is such that

$$w_i(E) = E_i + \sum_{j=1,\, j\neq i}^{q} [\sigma_{ij}(E) - \sigma_{ji}(E)], \quad E \in \overline{\mathbb{R}}_+^q. \tag{3.3}$$

Equation (3.1) yields a conservation of energy equation and implies that the change of energy stored in the ith subsystem is equal to the external energy supplied to (or extracted from) the ith subsystem plus the energy gained by the ith subsystem from all other subsystems due to subsystem coupling minus the energy dissipated from the ith subsystem. Note that (3.2) or, equivalently, (3.1) is a statement reminiscent of the *first law of thermodynamics* for each of the subsystems, with $E_i(\cdot)$, $S_i(\cdot)$, $\sigma_{ij}(\cdot)$, $i \neq j$, and $\sigma_{ii}(\cdot)$, $i = 1, \ldots, q$, playing the role of the ith subsystem internal energy, energy supplied to (or extracted from) the ith subsystem, the energy exchange between subsystems due to coupling, and the energy dissipated to the environment, respectively.

To further elucidate that (3.2) is essentially the statement of the principle of the conservation of energy let the total energy in the discrete-time large-scale dynamical system $\mathcal{G}$ be given by $U \triangleq \mathbf{e}^{\mathrm{T}}E$, $E \in \overline{\mathbb{R}}_+^q$, where $\mathbf{e}^{\mathrm{T}} \triangleq [1, \ldots, 1]$, and let the energy received by the discrete-time large-scale dynamical system $\mathcal{G}$ (in forms other than work) over the discrete-time interval $\{k_1, \ldots, k_2\}$ be given by $Q \triangleq \sum_{k=k_1}^{k_2} \mathbf{e}^{\mathrm{T}}[S(k) - d(E(k))]$, where $E(k)$, $k \geq k_0$, is the solution to (3.2). Then, premultiplying (3.2) by $\mathbf{e}^{\mathrm{T}}$ and using the fact that $\mathbf{e}^{\mathrm{T}}w(E) \equiv \mathbf{e}^{\mathrm{T}}E$, it follows that

$$\Delta U = Q, \tag{3.4}$$

where $\Delta U \triangleq U(k_2) - U(k_1)$ denotes the variation in the total energy of the discrete-time large-scale dynamical system $\mathcal{G}$ over the discrete-time interval $\{k_1, \ldots, k_2\}$. This is a statement of the first law of thermodynamics for the discrete-time large-scale dynamical system $\mathcal{G}$ and gives a precise formulation of the equivalence between variation in system internal energy and heat.

It is important to note that our discrete-time large-scale dynamical system model does not consider work done by the system on the environment nor work done by the environment on the system. Hence, Q can be interpreted physically as the amount of energy that is received by the system in forms other than work. The extension of addressing work performed by and on the system can be easily handeled by including an additional state equation, coupled to the energy balance equation (3.2), involving volume states for each subsystem [104]. Since this extension does not alter any of the results of the chapter, it is not considered here for simplicity of exposition.

For our large-scale dynamical system model $\mathcal{G}$, we assume that $\sigma_{ij}(E) = 0$, $E \in \overline{\mathbb{R}}_+^q$, whenever $E_j = 0$, $i, j = 1, \ldots, q$. This constraint implies that if the energy of the jth subsystem of $\mathcal{G}$ is zero, then this subsystem cannot supply any energy to its surroundings nor dissipate energy to the environment. Furthermore, for the remainder of this chapter we assume that $E_i \geq \sigma_{ii}(E) - S_i - \sum_{j=1, j\neq i}^{q} [\sigma_{ij}(E) - \sigma_{ji}(E)] = -\Delta E_i$, $E \in \overline{\mathbb{R}}_+^q$, $S \in \mathbb{R}^q$, $i = 1, \ldots, q$. This constraint implies that the energy that can be dissipated, extracted, or exchanged by the ith subsystem cannot exceed the current energy in the subsystem. Note that this assumption implies that $E(k) \geq\geq 0$ for all $k \geq k_0$.

Next, premultiplying (3.2) by $\mathbf{e}^{\mathrm{T}}$ and using the fact that $\mathbf{e}^{\mathrm{T}} w(E) \equiv \mathbf{e}^{\mathrm{T}} E$, it follows that

$$\mathbf{e}^{\mathrm{T}} E(k_1) = \mathbf{e}^{\mathrm{T}} E(k_0) + \sum_{k=k_0}^{k_1-1} \mathbf{e}^{\mathrm{T}} S(k) - \sum_{k=k_0}^{k_1-1} \mathbf{e}^{\mathrm{T}} d(E(k)), \quad k_1 \geq k_0. \tag{3.5}$$

Now, for the discrete-time large-scale dynamical system $\mathcal{G}$ define the input $u(k) \triangleq S(k)$ and the output $y(k) \triangleq d(E(k))$. Hence, it follows from (3.5) that the discrete-time large-scale dynamical system $\mathcal{G}$ is *lossless* [236] with respect to the *energy supply rate* $r(u, y) = \mathbf{e}^{\mathrm{T}} u - \mathbf{e}^{\mathrm{T}} y$ and with the *energy storage function* $U(E) \triangleq \mathbf{e}^{\mathrm{T}} E$, $E \in \overline{\mathbb{R}}_+^q$. This implies that (see [236] for details)

$$0 \leq U_{\mathrm{a}}(E_0) = U(E_0) = U_{\mathrm{r}}(E_0) < \infty, \quad E_0 \in \overline{\mathbb{R}}_+^q, \tag{3.6}$$

where

$$U_{\mathrm{a}}(E_0) \triangleq - \inf_{u(\cdot),\, K\geq k_0} \sum_{k=k_0}^{K-1} (\mathbf{e}^{\mathrm{T}}u(k) - \mathbf{e}^{\mathrm{T}}y(k)), \tag{3.7}$$

$$U_{\mathrm{r}}(E_0) \triangleq \inf_{u(\cdot),\, K\geq -k_0+1} \sum_{k=-K}^{k_0-1} (\mathbf{e}^{\mathrm{T}}u(k) - \mathbf{e}^{\mathrm{T}}y(k)), \tag{3.8}$$

and $E_0 = E(k_0) \in \overline{\mathbb{R}}_+^q$. Since $U_{\mathrm{a}}(E_0)$ is the maximum amount of stored energy which can be extracted from the discrete-time large-scale dynamical system $\mathcal{G}$ at any discrete-time instant K, and $U_{\mathrm{r}}(E_0)$ is the minimum amount of energy which can be delivered to the discrete-time large-scale dynamical system $\mathcal{G}$ to transfer it from a state of minimum potential $E(-K) = 0$ to a given state $E(k_0) = E_0$, it follows from (3.6) that the discrete-time large-scale dynamical system $\mathcal{G}$ can deliver to its surroundings all of its stored subsystem energies and can store all of the work done to all of its subsystems. In the case where $S(k) \equiv 0$, it follows from (3.5) and the fact that $\sigma_{ii}(E) \geq 0$, $E \in \overline{\mathbb{R}}_+^q$, $i = 1, \ldots, q$, that the zero solution $E(k) \equiv 0$ of the discrete-time large-scale dynamical system $\mathcal{G}$ with the energy balance equation (3.2) is Lyapunov stable with Lyapunov function $U(E)$ corresponding to the total energy in the system.

The next result shows that the large-scale dynamical system $\mathcal{G}$ is locally controllable.

Proposition 3.1. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2). Then for every equilibrium state $E_{\mathrm{e}} \in \overline{\mathbb{R}}_+^q$ and every $\varepsilon > 0$ and $T \in \mathbb{Z}_+$, there exist $S_{\mathrm{e}} \in \mathbb{R}^q$, $\alpha > 0$, and $\hat{T} \in \{0, \cdots, T\}$ such that for every $\hat{E} \in \overline{\mathbb{R}}_+^q$ with $\|\hat{E} - E_{\mathrm{e}}\| \leq \alpha T$, there exists $S : \{0, \cdots, \hat{T}\} \to \mathbb{R}^q$ such that $\|S(k) - S_{\mathrm{e}}\| \leq \varepsilon$, $k \in \{0, \cdots, \hat{T}\}$, and $E(k) = E_{\mathrm{e}} + \frac{(\hat{E}-E_{\mathrm{e}})}{\hat{T}}k$, $k \in \{0, \cdots, \hat{T}\}$.

Proof. Note that with $S_{\mathrm{e}} = d(E_{\mathrm{e}}) - w(E_{\mathrm{e}}) + E_{\mathrm{e}}$, the state $E_{\mathrm{e}} \in \overline{\mathbb{R}}_+^q$ is an equilibrium state of (3.2). Let $\theta > 0$ and $T \in \mathbb{Z}_+$, and define

$$M(\theta, T) \triangleq \sup_{E\in\overline{\mathcal{B}}_1(0),\, k\in\{0,\cdots,T\}} \|w(E_{\mathrm{e}} + k\theta E) - w(E_{\mathrm{e}}) - d(E_{\mathrm{e}} + k\theta E) + d(E_{\mathrm{e}}) - k\theta E\|. \tag{3.9}$$

Note that for every $T \in \mathbb{Z}_+$, $\lim_{\theta\to 0^+} M(\theta,T) = 0$. Next, let $\varepsilon > 0$ and $T \in \mathbb{Z}_+$ be given, and let $\alpha > 0$ be such that $M(\alpha,T) + \alpha \le \varepsilon$. (The existence of such an α is guaranteed since $M(\alpha,T) \to 0$ as $\alpha \to 0^+$). Now, let $\hat{E} \in \overline{\mathbb{R}}_+^q$ be such that $\|\hat{E} - E_\mathrm{e}\| \le \alpha T$. With $\hat{T} \triangleq \lceil \frac{\|\hat{E}-E_\mathrm{e}\|}{\alpha} \rceil \le T$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x, and

$$S(k) = -w(E(k)) + d(E(k)) + E(k) + \frac{\hat{E} - E_\mathrm{e}}{\lceil \frac{\|\hat{E}-E_\mathrm{e}\|}{\alpha} \rceil}, \quad k \in \{0, \cdots, \hat{T}\}, \tag{3.10}$$

it follows that

$$E(k) = E_\mathrm{e} + \frac{(\hat{E} - E_\mathrm{e})}{\lceil \frac{\|\hat{E}-E_\mathrm{e}\|}{\alpha} \rceil} k, \quad k \in \{0, \cdots, \hat{T}\}, \tag{3.11}$$

is a solution to (3.2). Now, noting that $E(\hat{T}) = \hat{E}$ and

$$\begin{aligned} \|S(k) - S_\mathrm{e}\| \le \; & \Bigg\| w\Big(E_\mathrm{e} + \frac{(\hat{E} - E_\mathrm{e})}{\lceil \frac{\|\hat{E}-E_\mathrm{e}\|}{\alpha} \rceil} k\Big) - w(E_\mathrm{e}) - d\Big(E_\mathrm{e} + \frac{(\hat{E} - E_\mathrm{e})}{\lceil \frac{\|\hat{E}-E_\mathrm{e}\|}{\alpha} \rceil} k\Big) \\ & \quad + d(E_\mathrm{e}) - \frac{(\hat{E} - E_\mathrm{e})}{\lceil \frac{\|\hat{E}-E_\mathrm{e}\|}{\alpha} \rceil} k \Bigg\| + \alpha \\ \le \; & M(\alpha, T) + \alpha \\ \le \; & \varepsilon, \quad k \in \{0, \cdots, \hat{T}\}, \end{aligned} \tag{3.12}$$

the result is immediate. □

It follows from Proposition 3.1 that the discrete-time large-scale dynamical system $\mathcal{G}$ with the energy balance equation (3.2) is *reachable* from and *controllable* to the origin in $\overline{\mathbb{R}}_+^q$. Recall that the discrete-time large-scale dynamical system $\mathcal{G}$ with the energy balance equation (3.2) is reachable from the origin in $\overline{\mathbb{R}}_+^q$ if, for all $E_0 = E(k_0) \in \overline{\mathbb{R}}_+^q$, there exists a finite time $k_\mathrm{i} \le k_0$ and an input $S(k)$ defined on $\{k_\mathrm{i}, \ldots, k_0\}$ such that the state $E(k)$, $k \ge k_\mathrm{i}$, can be driven from $E(k_\mathrm{i}) = 0$ to $E(k_0) = E_0$. Alternatively, $\mathcal{G}$ is controllable to the origin in $\overline{\mathbb{R}}_+^q$ if, for all $E_0 = E(k_0) \in \overline{\mathbb{R}}_+^q$, there exists a finite time $k_\mathrm{f} \ge k_0$ and an input $S(k)$ defined on $\{k_0, \ldots, k_\mathrm{f}\}$ such that the state $E(k)$, $k \ge k_0$, can be driven from $E(k_0) = E_0$ to $E(k_\mathrm{f}) = 0$. We let $\mathcal{U}_\mathrm{r}$ denote the set of all admissible bounded energy inputs to the discrete-time large-scale dynamical system $\mathcal{G}$ such that for any $K \ge -k_0$, the system energy state can be driven from $E(-K) = 0$ to $E(k_0) = E_0 \in \overline{\mathbb{R}}_+^q$ by $S(\cdot) \in \mathcal{U}_{\rm r}$, and we let $\mathcal{U}_{\rm c}$ denote the set of all admissible bounded energy inputs to the discrete-time large-scale dynamical system $\mathcal{G}$ such that for any $K \geq k_0$, the system energy state can be driven from $E(k_0) = E_0 \in \overline{\mathbb{R}}_+^q$ to $E(K) = 0$ by $S(\cdot) \in \mathcal{U}_{\rm c}$. Furthermore, let $\mathcal{U}$ be an input space that is a subset of bounded continuous $\mathbb{R}^q$-valued functions on $\mathbb{Z}$. The spaces $\mathcal{U}_{\rm r}$, $\mathcal{U}_{\rm c}$, and $\mathcal{U}$ are assumed to be closed under the shift operator, that is, if $S(\cdot) \in \mathcal{U}$ (respectively, $\mathcal{U}_{\rm c}$ or $\mathcal{U}_{\rm r}$), then the function S_K defined by $S_K(k) = S(k + K)$ is contained in $\mathcal{U}$ (respectively, $\mathcal{U}_{\rm c}$ or $\mathcal{U}_{\rm r}$) for all $K \geq 0$.

3.3. Nonconservation of Entropy and the Second Law of Thermodynamics

The nonlinear energy balance equation (3.2) can exhibit a full range of nonlinear behavior including bifurcations, limit cycles, and even chaos. However, a thermodynamically consistent energy flow model should ensure that the evolution of the system energy is diffusive (parabolic) in character with convergent subsystem energies. Hence, to ensure a thermodynamically consistent energy flow model we require the following axioms. For the statement of these axioms we first recall the following graph theoretic notions.

Definition 3.2 [19]. A *directed graph* $G(\mathcal{C})$ associated with the *connectivity matrix* $\mathcal{C} \in \mathbb{R}^{q \times q}$ has *vertices* $\{1, 2, \ldots, q\}$ and an *arc* from vertex i to vertex j, $i \neq j$, if and only if $\mathcal{C}_{(j,i)} \neq 0$. A *graph* $G(\mathcal{C})$ associated with the connectivity matrix $\mathcal{C} \in \mathbb{R}^{q \times q}$ is a directed graph for which the *arc set* is symmetric, that is, $\mathcal{C} = \mathcal{C}^{\rm T}$. We say that $G(\mathcal{C})$ is *strongly connected* if for any ordered pair of vertices (i, j), $i \neq j$, there exists a *path* (i.e., sequence of arcs) leading from i to j.

Recall that $\mathcal{C} \in \mathbb{R}^{q \times q}$ is *irreducible*, that is, there does not exist a permutation matrix such that $\mathcal{C}$ is cogredient to a lower-block triangular matrix, if and only if $G(\mathcal{C})$ is strongly connected (see Theorem 2.7 of [19]). Let $\phi_{ij}(E) \triangleq \sigma_{ij}(E) - \sigma_{ji}(E)$, $E \in \overline{\mathbb{R}}_+^q$, denote the net energy exchange between subsystems $\mathcal{G}_i$ and $\mathcal{G}_j$ of the discrete-time large-scale dynamical system $\mathcal{G}$.

Axiom *i*): For the connectivity matrix $\mathcal{C} \in \mathbb{R}^{q\times q}$ associated with the large-scale dynamical system $\mathcal{G}$ defined by

$$\mathcal{C}_{(i,j)} = \begin{cases} 0, & \text{if } \phi_{ij}(E) \equiv 0, \\ 1, & \text{otherwise}, \end{cases} \quad i \neq j, \quad i,j = 1,\ldots,q, \tag{3.13}$$

and

$$\mathcal{C}_{(i,i)} = -\sum_{k=1,\, k\neq i}^{q} \mathcal{C}_{(k,i)}, \quad i = j, \quad i = 1,\ldots,q, \tag{3.14}$$

rank $\mathcal{C} = q-1$, and for $\mathcal{C}_{(i,j)} = 1$, $i \neq j$, $\phi_{ij}(E) = 0$ if and only if $E_i = E_j$.

Axiom *ii*): For $i,j = 1,\ldots,q$, $(E_i - E_j)\phi_{ij}(E) \leq 0$, $E \in \overline{\mathbb{R}}_+^q$.

Axiom *iii*): For $i,j = 1,\ldots,q$, $\frac{\Delta E_i - \Delta E_j}{E_i - E_j} \geq -1$, $E_i \neq E_j$.

The fact that $\phi_{ij}(E) = 0$ if and only if $E_i = E_j$, $i \neq j$, implies that subsystems $\mathcal{G}_i$ and $\mathcal{G}_j$ of $\mathcal{G}$ are *connected*; alternatively, $\phi_{ij}(E) \equiv 0$ implies that $\mathcal{G}_i$ and $\mathcal{G}_j$ are *disconnected*. Axiom i) implies that if the energies in the connected subsystems $\mathcal{G}_i$ and $\mathcal{G}_j$ are equal, then energy exchange between these subsystems is not possible. This is a statement consistent with the *zeroth law of thermodynamics* which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium. Furthermore, it follows from the fact that $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$ and rank $\mathcal{C} = q-1$ that the connectivity matrix $\mathcal{C}$ is irreducible which implies that for any pair of subsystems $\mathcal{G}_i$ and $\mathcal{G}_j$, $i \neq j$, of $\mathcal{G}$ there exists a sequence of connected subsystems of $\mathcal{G}$ that connect $\mathcal{G}_i$ and $\mathcal{G}_j$. Axiom ii) implies that energy is exchanged from more energetic subsystems to less energetic subsystems and is consistent with the *second law of thermodynamics* which states that heat (energy) must flow in the direction of lower temperatures. Furthermore, note that $\phi_{ij}(E) = -\phi_{ji}(E)$, $E \in \overline{\mathbb{R}}_+^q$, $i \neq j$, $i,j = 1,\ldots,q$, which implies conservation of energy between lossless subsystems.

With $S(k) \equiv 0$, Axioms i) and ii) along with the fact that $\phi_{ij}(E) = -\phi_{ji}(E)$, $E \in \overline{\mathbb{R}}_+^q$, $i \neq j$, $i,j = 1,\ldots,q$, imply that at a given instant of time energy can only be transported, stored, or dissipated but not created and the maximum amount of energy that can be transported and/or dissipated from a subsystem cannot exceed the energy in the subsystem. Finally, Axiom iii) implies that for any pair of connected subsystems $\mathcal{G}_i$ and $\mathcal{G}_j$, $i \neq j$, the energy difference between consecutive time instants is monotonic; that is, $[E_i(k+1) - E_j(k+1)][E_i(k) - E_j(k)] \geq 0$ for all $E_i \neq E_j$, $k \geq k_0$, $i, j = 1, \ldots, q$.

Next, we establish a Clausius-type inequality for our thermodynamically consistent energy flow model.

Proposition 3.2. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2) and assume that Axioms i), ii), and iii) hold. Then for all $E_0 \in \overline{\mathbb{R}}_+^q$, $k_\mathrm{f} \geq k_0$, and $S(\cdot) \in \mathcal{U}$ such that $E(k_\mathrm{f}) = E(k_0) = E_0$,

$$\sum_{k=k_0}^{k_\mathrm{f}-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)} = \sum_{k=k_0}^{k_\mathrm{f}-1} \sum_{i=1}^{q} \frac{Q_i(k)}{c + E_i(k+1)} \leq 0, \tag{3.15}$$

where $c > 0$, $Q_i(k) \triangleq S_i(k) - \sigma_{ii}(E(k))$, $i = 1, \ldots, q$, is the amount of net energy (heat) received by the ith subsystem at the kth instant, and $E(k)$, $k \geq k_0$, is the solution to (3.2) with initial condition $E(k_0) = E_0$. Furthermore, equality holds in (3.15) if and only if $\Delta E_i(k) = 0$, $i = 1, \ldots, q$, and $E_i(k) = E_j(k)$, $i, j = 1, \ldots, q$, $i \neq j$, $k \in \{k_0, \ldots, k_\mathrm{f} - 1\}$.

Proof. Since $E(k) \geq\geq 0$, $k \geq k_0$, and $\phi_{ij}(E) = -\phi_{ji}(E)$, $E \in \overline{\mathbb{R}}_+^q$, $i \neq j$, $i, j = 1, \ldots, q$, it follows from (3.2), Axioms ii) and iii), and the fact that $\frac{x}{x+1} \leq \log_e(1+x)$, $x > -1$, that

$$\begin{aligned}
\sum_{k=k_0}^{k_\mathrm{f}-1} \sum_{i=1}^{q} \frac{Q_i(k)}{c + E_i(k+1)} &= \sum_{k=k_0}^{k_\mathrm{f}-1} \sum_{i=1}^{q} \frac{\Delta E_i(k) - \sum_{j=1,\, j\neq i}^{q} \phi_{ij}(E(k))}{c + E_i(k+1)} \\
&= \sum_{k=k_0}^{k_\mathrm{f}-1} \sum_{i=1}^{q} \left[\frac{\Delta E_i(k)}{c + E_i(k)}\right] \left[1 + \frac{\Delta E_i(k)}{c + E_i(k)}\right]^{-1} \\
&\quad - \sum_{k=k_0}^{k_\mathrm{f}-1} \sum_{i=1}^{q} \sum_{j=1,\, j\neq i}^{q} \frac{\phi_{ij}(E(k))}{c + E_i(k+1)} \\
&\leq \sum_{i=1}^{q} \log_e \left(\frac{c + E_i(k_\mathrm{f})}{c + E_i(k_0)}\right) - \sum_{k=k_0}^{k_\mathrm{f}-1} \sum_{i=1}^{q} \sum_{j=1,\, j\neq i}^{q} \frac{\phi_{ij}(E(k))}{c + E_i(k+1)}
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{k=k_0}^{k_\mathrm{f}-1}\sum_{i=1}^{q-1}\sum_{j=i+1}^{q}\left(\frac{\phi_{ij}(E(k))}{c+E_i(k+1)}-\frac{\phi_{ij}(E(k))}{c+E_j(k+1)}\right)\\
&= -\sum_{k=k_0}^{k_\mathrm{f}-1}\sum_{i=1}^{q-1}\sum_{j=i+1}^{q}\frac{\phi_{ij}(E(k))(E_j(k+1)-E_i(k+1))}{(c+E_i(k+1))(c+E_j(k+1))}\\
&\leq 0, \qquad (3.16)
\end{aligned}
$$

which proves (3.15).

Alternatively, equality holds in (3.15) if and only if $\sum_{k=k_0}^{k_\mathrm{f}-1}\frac{\Delta E_i(k)}{c+E_i(k+1)} = 0$, $i = 1,\ldots,q$, and $\phi_{ij}(E(k))(E_j(k+1) - E_i(k+1)) = 0$, $i,j = 1,\ldots,q$, $i \neq j$, $k \geq k_0$. Moreover, $\sum_{k=k_0}^{k_\mathrm{f}-1}\frac{\Delta E_i(k)}{c+E_i(k+1)} = 0$ is equivalent to $\Delta E_i(k) = 0$, $i = 1,\ldots,q$, $k \in \{k_0,\ldots,k_\mathrm{f}-1\}$. Hence, $\phi_{ij}(E(k))(E_j(k+1) - E_i(k+1)) = \phi_{ij}(E(k))(E_j(k) - E_i(k)) = 0$, $i,j = 1,\ldots,q$, $i \neq j$, $k \geq k_0$. Thus, it follows from Axioms $i) - iii)$ that equality holds in (3.15) if and only if $\Delta E_i = 0$, $i = 1,\ldots,q$, and $E_j = E_i$, $i,j = 1,\ldots,q$, $i \neq j$. □

Inequality (3.15) is analogous to Clausius' inequality for reversible and irreversible thermodynamics as applied to discrete-time large-scale dynamical systems. It follows from Axiom $i)$ and (3.2) that for the *isolated* discrete-time large-scale dynamical system $\mathcal{G}$, that is, $S(k) \equiv 0$ and $d(E(k)) \equiv 0$, the energy states given by $E_\mathrm{e} = \alpha\mathbf{e}$, $\alpha \geq 0$, correspond to the equilibrium energy states of $\mathcal{G}$. Thus, we can define an *equilibrium process* as a process where the trajectory of the discrete-time large-scale dynamical system $\mathcal{G}$ stays at the equilibrium point of the isolated system $\mathcal{G}$. The input that can generate such a trajectory can be given by $S(k) = d(E(k))$, $k \geq k_0$. Alternatively, a *nonequilibrium process* is a process that is not an equilibrium one. Hence, it follows from Axiom $i)$ that for an equilibrium process $\phi_{ij}(E(k)) \equiv 0$, $k \geq k_0$, $i \neq j$, $i,j = 1,\ldots,q$, and thus, by Proposition 3.2 and $\Delta E_i = 0$, $i = 1,\ldots,q$, inequality (3.15) is satisfied as an equality. Alternatively, for a nonequilibrium process it follows from Axioms $i) - iii)$ that (3.15) is satisfied as a strict inequality.

Next, we give a deterministic definition of entropy for the discrete-time large-scale dynamical system $\mathcal{G}$ that is consistent with the classical thermodynamic definition of entropy.

Definition 3.3. For the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2), a function $\mathcal{S} : \overline{\mathbb{R}}^q_+ \to \mathbb{R}$ satisfying

$$\mathcal{S}(E(k_2)) \geq \mathcal{S}(E(k_1)) + \sum_{k=k_1}^{k_2-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)}, \tag{3.17}$$

for any $k_2 \geq k_1 \geq k_0$ and $S(\cdot) \in \mathcal{U}$, is called the *entropy* of $\mathcal{G}$.

Next, we show that (3.15) guarantees the existence of an entropy function for $\mathcal{G}$. For this result define the *available entropy* of the large-scale dynamical system $\mathcal{G}$ by

$$\mathcal{S}_{\mathrm{a}}(E_0) \triangleq - \sup_{S(\cdot)\in\mathcal{U}_{\mathrm{c}},\, K\geq k_0} \sum_{k=k_0}^{K-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)}, \tag{3.18}$$

where $E(k_0) = E_0 \in \overline{\mathbb{R}}^q_+$ and $E(K) = 0$, and define the *required entropy supply* of the large-scale dynamical system $\mathcal{G}$ by

$$\mathcal{S}_{\mathrm{r}}(E_0) \triangleq \sup_{S(\cdot)\in\mathcal{U}_{\mathrm{r}},\, K\geq -k_0+1} \sum_{k=-K}^{k_0-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)}, \tag{3.19}$$

where $E(-K) = 0$ and $E(k_0) = E_0 \in \overline{\mathbb{R}}^q_+$. Note that the available entropy $\mathcal{S}_{\mathrm{a}}(E_0)$ is the minimum amount of scaled heat (entropy) that can be extracted from the large-scale dynamical system $\mathcal{G}$ in order to transfer it from an initial state $E(k_0) = E_0$ to $E(K) = 0$. Alternatively, the required entropy supply $\mathcal{S}_{\mathrm{r}}(E_0)$ is the maximum amount of scaled heat (entropy) that can be delivered to $\mathcal{G}$ to transfer it from the origin to a given initial state $E(k_0) = E_0$.

Theorem 3.1. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2) and assume that Axioms *ii*) and *iii*) hold. Then there exists an entropy function for $\mathcal{G}$. Moreover, $\mathcal{S}_{\mathrm{a}}(E)$, $E \in \overline{\mathbb{R}}^q_+$, and $\mathcal{S}_{\mathrm{r}}(E)$, $E \in \overline{\mathbb{R}}^q_+$, are possible entropy functions for $\mathcal{G}$ with $\mathcal{S}_{\mathrm{a}}(0) = \mathcal{S}_{\mathrm{r}}(0) = 0$. Finally, all entropy functions $\mathcal{S}(E)$, $E \in \overline{\mathbb{R}}^q_+$, for $\mathcal{G}$ satisfy

$$\mathcal{S}_{\mathrm{r}}(E) \leq \mathcal{S}(E) - \mathcal{S}(0) \leq \mathcal{S}_{\mathrm{a}}(E), \quad E \in \overline{\mathbb{R}}^q_+. \tag{3.20}$$

Proof. Since, by Proposition 3.1, $\mathcal{G}$ is controllable to and reachable from the origin in $\overline{\mathbb{R}}_+^q$, it follows from (3.18) and (3.19) that $\mathcal{S}_{\rm a}(E_0) < \infty$, $E_0 \in \overline{\mathbb{R}}_+^q$, and $\mathcal{S}_{\rm r}(E_0) > -\infty$, $E_0 \in \overline{\mathbb{R}}_+^q$, respectively. Next, let $E_0 \in \overline{\mathbb{R}}_+^q$ and let $S(\cdot) \in \mathcal{U}$ be such that $E(k_{\rm i}) = E(k_{\rm f}) = 0$ and $E(k_0) = E_0$, where $k_{\rm i} \le k_0 \le k_{\rm f}$. In this case, it follows from (3.15) that

$$\sum_{k=k_{\rm i}}^{k_{\rm f}-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)} \le 0, \tag{3.21}$$

or, equivalently,

$$\sum_{k=k_{\rm i}}^{k_0-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)} \le -\sum_{k=k_0}^{k_{\rm f}-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)}. \tag{3.22}$$

Now, taking the supremum on both sides of (3.22) over all $S(\cdot) \in \mathcal{U}_{\rm r}$ and $k_{\rm i} + 1 \le k_0$, we obtain

$$\begin{aligned} \mathcal{S}_{\rm r}(E_0) &= \sup_{S(\cdot)\in\mathcal{U}_{\rm r},\, k_{\rm i}+1\le k_0} \sum_{k=k_{\rm i}}^{k_0-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)} \\ &\le -\sum_{k=k_0}^{k_{\rm f}-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)}. \end{aligned} \tag{3.23}$$

Next, taking the infimum on both sides of (3.23) over all $S(\cdot) \in \mathcal{U}_{\rm c}$ and $k_{\rm f} \ge k_0$ we obtain $\mathcal{S}_{\rm r}(E_0) \le \mathcal{S}_{\rm a}(E_0)$, $E_0 \in \overline{\mathbb{R}}_+^q$, which implies that $-\infty < \mathcal{S}_{\rm r}(E_0) \le \mathcal{S}_{\rm a}(E_0) < +\infty$, $E_0 \in \overline{\mathbb{R}}_+^q$. Hence, the function $\mathcal{S}_{\rm a}(\cdot)$ and $\mathcal{S}_{\rm r}(\cdot)$ are well defined.

Next, it follows from the definition of $\mathcal{S}_{\rm a}(\cdot)$ that, for any $K \ge k_1$ and $S(\cdot) \in \mathcal{U}_{\rm c}$ such that $E(k_1) \in \overline{\mathbb{R}}_+^q$ and $E(K) = 0$,

$$-\mathcal{S}_{\rm a}(E(k_1)) \ge \sum_{k=k_1}^{k_2-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)} + \sum_{k=k_2}^{K-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)}, \quad k_1 \le k_2 \le K, \tag{3.24}$$

and hence,

$$\begin{aligned} -\mathcal{S}_{\rm a}(E(k_1)) &\ge \sum_{k=k_1}^{k_2-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)} + \sup_{S(\cdot)\in\mathcal{U}_{\rm c},\, K\ge k_2} \sum_{k=k_2}^{K-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)} \\ &= \sum_{k=k_1}^{k_2-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)} - \mathcal{S}_{\rm a}(E(k_2)), \end{aligned} \tag{3.25}$$

which implies that $\mathcal{S}_{\rm a}(E)$, $E \in \overline{\mathbb{R}}_+^q$, satisfies (3.17). Thus, $\mathcal{S}_{\rm a}(E)$, $E \in \overline{\mathbb{R}}_+^q$, is a possible entropy function for $\mathcal{G}$. Note that with $E(k_0) = E(K) = 0$ it follows from (3.15) that the supremum in (3.18) is taken over the set of nonpositive values with one of the values being zero for $S(k) \equiv 0$. Thus, $\mathcal{S}_{\rm a}(0) = 0$. Similarly, it can be shown that $\mathcal{S}_{\rm r}(E)$, $E \in \overline{\mathbb{R}}_+^q$, given by (3.19) satisfies (3.17) and hence is a possible entropy function for the system $\mathcal{G}$ with $\mathcal{S}_{\rm r}(0) = 0$.

Next, suppose there exists an entropy function $\mathcal{S} : \overline{\mathbb{R}}_+^q \to \mathbb{R}$ for $\mathcal{G}$ and let $E(k_2) = 0$ in (3.17). Then it follows from (3.17) that

$$\mathcal{S}(E(k_1)) - \mathcal{S}(0) \le -\sum_{k=k_1}^{k_2-1}\sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)}, \tag{3.26}$$

for all $k_2 \ge k_1$ and $S(\cdot) \in \mathcal{U}_{\rm c}$, which implies that

$$\begin{aligned} \mathcal{S}(E(k_1)) - \mathcal{S}(0) &\le \inf_{S(\cdot)\in\mathcal{U}_{\rm c},\, k_2\ge k_1} \left[-\sum_{k=k_1}^{k_2-1}\sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)}\right] \\ &= -\sup_{S(\cdot)\in\mathcal{U}_{\rm c},\, k_2\ge k_1} \sum_{k=k_1}^{k_2-1}\sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)} \\ &= \mathcal{S}_{\rm a}(E(k_1)). \end{aligned} \tag{3.27}$$

Since $E(k_1)$ is arbitrary, it follows that $\mathcal{S}(E) - \mathcal{S}(0) \le \mathcal{S}_{\rm a}(E)$, $E \in \overline{\mathbb{R}}_+^q$. Alternatively, let $E(k_1) = 0$ in (3.17). Then it follows from (3.17) that

$$\mathcal{S}(E(k_2)) - \mathcal{S}(0) \ge \sum_{k=k_1}^{k_2-1}\sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)}, \tag{3.28}$$

for all $k_1 + 1 \le k_2$ and $S(\cdot) \in \mathcal{U}_{\rm r}$. Hence,

$$\mathcal{S}(E(k_2)) - \mathcal{S}(0) \ge \sup_{S(\cdot)\in\mathcal{U}_{\rm r},\, k_1+1\le k_2} \sum_{k=k_1}^{k_2-1}\sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)} = \mathcal{S}_{\rm r}(E(k_2)), \tag{3.29}$$

which, since $E(k_2)$ is arbitrary, implies that $\mathcal{S}_{\rm r}(E) \le \mathcal{S}(E) - \mathcal{S}(0)$, $E \in \overline{\mathbb{R}}_+^q$. Thus, all entropy functions for $\mathcal{G}$ satisfy (3.20). $\square$

Remark 3.1. It is important to note that inequality (3.15) is equivalent to the existence of an entropy function for $\mathcal{G}$. Sufficiency is simply a statement of Theorem 3.1 while necessity follows from (3.17) with $E(k_2) = E(k_1)$. For nonequilibrium process with energy balance equation (3.2), Definition 3.3 does not provide enough information to define the entropy uniquely. This difficulty has long been pointed out in [172] for thermodynamic systems. A similar remark holds for the definition of ectropy introduced below.

The next proposition gives a closed-form expression for the entropy of $\mathcal{G}$.

Proposition 3.3. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2) and assume that Axioms *ii*) and *iii*) hold. Then the function $\mathcal{S} : \overline{\mathbb{R}}_+^q \to \mathbb{R}$ given by

$$\mathcal{S}(E) = \mathbf{e}^{\mathrm{T}}\mathbf{log}_e(c\mathbf{e} + E) - q\log_e c, \quad E \in \overline{\mathbb{R}}_+^q, \tag{3.30}$$

where $c > 0$ and $\mathbf{log}_e(c\mathbf{e} + E)$ denotes the vector natural logarithm given by $[\log_e(c + E_1), \ldots, \log_e(c + E_q)]^{\mathrm{T}}$, is an entropy function of $\mathcal{G}$.

Proof. Since $E(k) \geq\geq 0$, $k \geq k_0$, and $\phi_{ij}(E) = -\phi_{ji}(E)$, $E \in \overline{\mathbb{R}}_+^q$, $i \neq j$, $i, j = 1, \ldots, q$, it follows that

$$\begin{aligned}
\Delta\mathcal{S}(E(k)) &= \sum_{i=1}^{q} \log_e\left[1 + \frac{\Delta E_i(k)}{c + E_i(k)}\right] \\
&\geq \sum_{i=1}^{q} \left[\frac{\Delta E_i(k)}{c + E_i(k)}\right]\left[1 + \frac{\Delta E_i(k)}{c + E_i(k)}\right]^{-1} \\
&= \sum_{i=1}^{q} \frac{\Delta E_i(k)}{c + E_i(k) + \Delta E_i(k)} \\
&= \sum_{i=1}^{q} \frac{\Delta E_i(k)}{c + E_i(k+1)} \\
&= \sum_{i=1}^{q} \left[\frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)} + \sum_{j=1,\, j\neq i}^{q} \frac{\phi_{ij}(E(k))}{c + E_i(k+1)}\right] \\
&= \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)} + \sum_{i=1}^{q-1}\sum_{j=i+1}^{q} \left(\frac{\phi_{ij}(E(k))}{c + E_i(k+1)} - \frac{\phi_{ij}(E(k))}{c + E_j(k+1)}\right) \\
&= \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)} + \sum_{i=1}^{q-1}\sum_{j=i+1}^{q} \frac{\phi_{ij}(E(k))(E_j(k+1) - E_i(k+1))}{(c + E_i(k+1))(c + E_j(k+1))}
\end{aligned}$$

$$\geq \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + E_i(k+1)}, \quad k \geq k_0, \tag{3.31}$$

where in (3.31) we used the fact that $\log_e(1+x) \geq \frac{x}{x+1}, x > -1$. Now, summing (3.31) over $\{k_1, \ldots, k_2 - 1\}$ yields (3.17). $\square$

Remark 3.2. Note that it follows from the first equality in (3.31) that the entropy function given by (3.30) satisfies (3.17) as an equality for an equilibrium process and as a strict inequality for a nonequilibrium process.

The entropy expression given by (3.30) is identical in form to the Boltzmann entropy for statistical thermodynamics. Due to the fact that the entropy is indeterminate to the extent of an additive constant, we can place the constant $q \log_e c$ to zero by taking $c = 1$. Since $\mathcal{S}(E)$ given by (3.30) achieves a maximum when all the subsystem energies E_i, $i = 1, \ldots, q$, are equal, entropy can be thought of as a measure of the tendency of a system to lose the ability to do useful work, and lose order and to settle to a more homogenous state.

3.4. Nonconservation of Ectropy

In this section, we introduce a *new* and dual notion to entropy, namely ectropy, describing the status quo of the discrete-time large-scale dynamical system $\mathcal{G}$. First, however, we present a dual inequality to inequality (3.15) that holds for our thermodynamically consistent energy flow model.

Proposition 3.4. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2) and assume that Axioms *i*), *ii*), and *iii*) hold. Then for all $E_0 \in \overline{\mathbb{R}}_+^q$, $k_\mathrm{f} \geq k_0$, and $S(\cdot) \in \mathcal{U}$ such that $E(k_\mathrm{f}) = E(k_0) = E_0$,

$$\sum_{k=k_0}^{k_\mathrm{f}-1} \sum_{i=1}^{q} E_i(k+1)[S_i(k) - \sigma_{ii}(E(k))] = \sum_{k=k_0}^{k_\mathrm{f}-1} \sum_{i=1}^{q} E_i(k+1) Q_i(k) \geq 0, \tag{3.32}$$

where $E(k)$, $k \geq k_0$, is the solution to (3.2) with initial condition $E(k_0) = E_0$. Furthermore, equality holds in (3.32) if and only if $\Delta E_i = 0$ and $E_i = E_j$, $i, j = 1, \ldots, q$, $i \neq j$.

Proof. Since $E(k) \geq\geq 0$, $k \geq k_0$, and $\phi_{ij}(E) = -\phi_{ji}(E)$, $E \in \overline{\mathbb{R}}_+^q$, $i \neq j$, $i, j = 1, \ldots, q$, it follows from (3.2) and Axioms *ii*) and *iii*) that

$$\begin{aligned}
2\sum_{k=k_0}^{k_\mathrm{f}-1}\sum_{i=1}^{q} E_i(k+1)Q_i(k) &= \sum_{k=k_0}^{k_\mathrm{f}-1}\sum_{i=1}^{q} E_i^2(k+1) - E_i^2(k) \\
&\quad -2\sum_{k=k_0}^{k_\mathrm{f}-1}\sum_{i=1}^{q}\sum_{j=1,\,j\neq i}^{q} E_i(k+1)\phi_{ij}(E(k)) \\
&\quad +\sum_{k=k_0}^{k_\mathrm{f}-1}\sum_{i=1}^{q}\Big[\sum_{j=1,\,j\neq i}^{q} \phi_{ij}(E(k)) + S_i(k) - \sigma_{ii}(E(k))\Big]^2 \\
&= E^\mathrm{T}(k_\mathrm{f})E(k_\mathrm{f}) - E^\mathrm{T}(k_0)E(k_0) \\
&\quad -2\sum_{k=k_0}^{k_\mathrm{f}-1}\sum_{i=1}^{q}\sum_{j=1,\,j\neq i}^{q} E_i(k+1)\phi_{ij}(E(k)) \\
&\quad +\sum_{k=k_0}^{k_\mathrm{f}-1}\sum_{i=1}^{q}\Big[\sum_{j=1,\,j\neq i}^{q} \phi_{ij}(E(k)) + S_i(k) - \sigma_{ii}(E(k))\Big]^2 \\
&= -2\sum_{k=k_0}^{k_\mathrm{f}-1}\sum_{i=1}^{q-1}\sum_{j=i+1}^{q} \phi_{ij}(E(k))(E_i(k+1) - E_j(k+1)) \\
&\quad +\sum_{k=k_0}^{k_\mathrm{f}-1}\sum_{i=1}^{q}\Big[\sum_{j=1,\,j\neq i}^{q} \phi_{ij}(E(k)) + S_i(k) - \sigma_{ii}(E(k))\Big]^2 \\
&\geq 0,
\end{aligned} \tag{3.33}$$

which proves (3.32).

Alternatively, equality holds in (3.32) if and only if $\phi_{ij}(E(k))(E_i(k+1) - E_j(k+1)) = 0$ and $\sum_{j=1,\,j\neq i}^{q} \phi_{ij}(E(k)) + S_i(k) - \sigma_{ii}(E(k)) = 0$, $i, j = 1, \ldots, q$, $i \neq j$, $k \geq k_0$. Next, $\sum_{j=1,\,j\neq i}^{q} \phi_{ij}(E(k)) + S_i(k) - \sigma_{ii}(E(k)) = 0$ if and only if $\Delta E_i = 0$, $i = 1, \ldots, q$, $k \geq k_0$. Hence, $\phi_{ij}(E(k))(E_j(k+1) - E_i(k+1)) = \phi_{ij}(E(k))(E_j(k) - E_i(k)) = 0$, $i, j = 1, \ldots, q$, $i \neq j$, $k \geq k_0$. Thus, it follows from Axioms $i) - iii)$ that equality holds in (3.32) if and only if $\Delta E_i = 0$, $i = 1, \ldots, q$, and $E_j = E_i$, $i, j = 1, \ldots, q$, $i \neq j$. $\square$

Note that inequality (3.32) is satisfied as an equality for an equilibrium process and as a strict inequality for a nonequilibrium process. Next, we present the definition of ectropy for the discrete-time large-scale dynamical system $\mathcal{G}$.

Definition 3.4. For the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2), a function $\mathcal{E} : \overline{\mathbb{R}}_+^q \to \mathbb{R}$ satisfying

$$\mathcal{E}(E(k_2)) \leq \mathcal{E}(E(k_1)) + \sum_{k=k_1}^{k_2-1} \sum_{i=1}^{q} E_i(k+1)[S_i(k) - \sigma_{ii}(E(k))], \tag{3.34}$$

for any $k_2 \geq k_1 \geq k_0$ and $S(\cdot) \in \mathcal{U}$, is called the *ectropy* of $\mathcal{G}$.

For the next result define the *available ectropy* of the large-scale dynamical system $\mathcal{G}$ by

$$\mathcal{E}_{\mathrm{a}}(E_0) \triangleq - \inf_{S(\cdot)\in\mathcal{U}_{\mathrm{c}},\, K\geq k_0} \sum_{k=k_0}^{K-1} \sum_{i=1}^{q} E_i(k+1)[S_i(k) - \sigma_{ii}(E(k))], \tag{3.35}$$

where $E(k_0) = E_0 \in \overline{\mathbb{R}}_+^q$ and $E(K) = 0$, and the *required ectropy supply* of the large-scale dynamical system $\mathcal{G}$ by

$$\mathcal{E}_{\mathrm{r}}(E_0) \triangleq \inf_{S(\cdot)\in\mathcal{U}_{\mathrm{r}},\, K\geq -k_0+1} \sum_{k=-K}^{k_0-1} \sum_{i=1}^{q} E_i(k+1)[S_i(k) - \sigma_{ii}(E(k))], \tag{3.36}$$

where $E(-K) = 0$ and $E(k_0) = E_0 \in \overline{\mathbb{R}}_+^q$. Note that the available ectropy $\mathcal{E}_{\mathrm{a}}(E_0)$ is the maximum amount of scaled heat (ectropy) that can be extracted from the large-scale dynamical system $\mathcal{G}$ in order to transfer it from an initial state $E(k_0) = E_0$ to $E(K) = 0$. Alternatively, the required ectropy supply $\mathcal{E}_{\mathrm{r}}(E_0)$ is the minimum amount of scaled heat (ectropy) that can be delivered to $\mathcal{G}$ to transfer it from an initial state $E(-K) = 0$ to a given state $E(k_0) = E_0$.

Theorem 3.2. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2) and assume that Axioms *ii*) and *iii*) hold. Then there exists an ectropy function for $\mathcal{G}$. Moreover, $\mathcal{E}_{\mathrm{a}}(E)$, $E \in \overline{\mathbb{R}}_+^q$, and $\mathcal{E}_{\mathrm{r}}(E)$, $E \in \overline{\mathbb{R}}_+^q$, are possible ectropy functions for $\mathcal{G}$ with $\mathcal{E}_{\rm a}(0) = \mathcal{E}_{\rm r}(0) = 0$. Finally, all ectropy functions $\mathcal{E}(E)$, $E \in \overline{\mathbb{R}}_+^q$, for $\mathcal{G}$ satisfy

$$\mathcal{E}_{\rm a}(E) \leq \mathcal{E}(E) - \mathcal{E}(0) \leq \mathcal{E}_{\rm r}(E), \quad E \in \overline{\mathbb{R}}_+^q. \tag{3.37}$$

Proof. Since, by Proposition 3.1, $\mathcal{G}$ is controllable to and reachable from the origin in $\overline{\mathbb{R}}_+^q$ it follows from (3.35) and (3.36) that $\mathcal{E}_{\rm a}(E_0) > -\infty$, $E_0 \in \overline{\mathbb{R}}_+^q$, and $\mathcal{E}_{\rm r}(E_0) < \infty$, $E_0 \in \overline{\mathbb{R}}_+^q$, respectively. Next, let $E_0 \in \overline{\mathbb{R}}_+^q$ and let $S(\cdot) \in \mathcal{U}$ be such that $E(k_{\rm i}) = E(k_{\rm f}) = 0$ and $E(k_0) = E_0$, where $k_{\rm i} \leq k_0 \leq k_{\rm f}$. In this case, it follows from (3.32) that

$$\sum_{k=k_{\rm i}}^{k_{\rm f}-1} \sum_{i=1}^{q} E_i(k+1)[S_i(k) - \sigma_{ii}(E(k))] \geq 0, \tag{3.38}$$

or, equivalently,

$$\sum_{k=k_{\rm i}}^{k_0-1} \sum_{i=1}^{q} E_i(k+1)[S_i(k) - \sigma_{ii}(E(k))] \geq - \sum_{k=k_0}^{k_{\rm f}-1} \sum_{i=1}^{q} E_i(k+1)[S_i(k) - \sigma_{ii}(E(k))]. \tag{3.39}$$

Now, taking the infimum on both sides of (3.39) over all $S(\cdot) \in \mathcal{U}_{\rm r}$ and $k_{\rm i} + 1 \leq k_0$ yields

$$\begin{aligned} \mathcal{E}_{\rm r}(E_0) &= \inf_{S(\cdot)\in\mathcal{U}_{\rm r},\, k_{\rm i}+1\leq k_0} \sum_{k=k_{\rm i}}^{k_0-1} \sum_{i=1}^{q} E_i(k+1)[S_i(k) - \sigma_{ii}(E(k))] \\ &\geq - \sum_{k=k_0}^{k_{\rm f}-1} \sum_{i=1}^{q} E_i(k+1)[S_i(k) - \sigma_{ii}(E(k))]. \end{aligned} \tag{3.40}$$

Next, taking the supremum on both sides of (3.40) over all $S(\cdot) \in \mathcal{U}_{\rm c}$ and $k_{\rm f} \geq k_0$ we obtain $\mathcal{E}_{\rm r}(E_0) \geq \mathcal{E}_{\rm a}(E_0)$, $E_0 \in \overline{\mathbb{R}}_+^q$, which implies that $-\infty < \mathcal{E}_{\rm a}(E_0) \leq \mathcal{E}_{\rm r}(E_0) < \infty$, $E_0 \in \overline{\mathbb{R}}_+^q$. Hence, the functions $\mathcal{E}_{\rm a}(\cdot)$ and $\mathcal{E}_{\rm r}(\cdot)$ are well defined.

Next, it follows from the definition of $\mathcal{E}_{\rm a}(\cdot)$ that, for any $K \geq k_1$ and $S(\cdot) \in \mathcal{U}_{\rm c}$ such that $E(k_1) \in \overline{\mathbb{R}}_+^q$ and $E(K) = 0$,

$$\begin{aligned} -\mathcal{E}_{\rm a}(E(k_1)) \leq & \sum_{k=k_1}^{k_2-1} \sum_{i=1}^{q} E_i(k+1)[S_i(k) - \sigma_{ii}(E(k))] \\ & + \sum_{k=k_2}^{K-1} \sum_{i=1}^{q} E_i(k+1)[S_i(k) - \sigma_{ii}(E(k))], \quad k_1 \leq k_2 \leq K, \end{aligned} \tag{3.41}$$

and hence,

$$
\begin{aligned}
-\mathcal{E}_{\mathrm{a}}(E(k_1)) \leq & \sum_{k=k_1}^{k_2-1}\sum_{i=1}^{q} E_i(k+1)[S_i(k)-\sigma_{ii}(E(k))] \\
& + \inf_{S(\cdot)\in\mathcal{U}_{\mathrm{c}},\,K\geq k_2} \sum_{k=k_2}^{K-1}\sum_{i=1}^{q} E_i(k+1)[S_i(k)-\sigma_{ii}(E(k))] \\
= & \sum_{k=k_1}^{k_2-1}\sum_{i=1}^{q} E_i(k+1)[S_i(k)-\sigma_{ii}(E(k))] - \mathcal{E}_{\mathrm{a}}(E(k_2)),
\end{aligned} \tag{3.42}
$$

which implies that $\mathcal{E}_{\mathrm{a}}(E)$, $E \in \overline{\mathbb{R}}_+^q$, satisfies (3.34). Thus, $\mathcal{E}_{\mathrm{a}}(E)$, $E \in \overline{\mathbb{R}}_+^q$, is a possible ectropy function for the system $\mathcal{G}$. Note that with $E(k_0) = E(K) = 0$ it follows from (3.32) that the infimum in (3.35) is taken over the set of nonnegative values with one of the values being zero for $S(k) \equiv 0$. Thus, $\mathcal{E}_{\mathrm{a}}(0) = 0$. Similarly, it can be shown that $\mathcal{E}_{\mathrm{r}}(E)$, $E \in \overline{\mathbb{R}}_+^q$, given by (3.36) satisfies (3.34), and hence, is a possible ectropy function for the system $\mathcal{G}$ with $\mathcal{E}_{\mathrm{r}}(0) = 0$.

Next, suppose there exists an ectropy function $\mathcal{E} : \overline{\mathbb{R}}_+^q \to \mathbb{R}$ for $\mathcal{G}$ and let $E(k_2) = 0$ in (3.34). Then it follows from (3.34) that

$$
\mathcal{E}(E(k_1)) - \mathcal{E}(0) \geq -\sum_{k=k_1}^{k_2-1}\sum_{i=1}^{q} E_i(k+1)[S_i(k)-\sigma_{ii}(E(k))], \tag{3.43}
$$

for all $k_2 \geq k_1$ and $S(\cdot) \in \mathcal{U}_{\mathrm{c}}$, which implies that

$$
\begin{aligned}
\mathcal{E}(E(k_1)) - \mathcal{E}(0) \geq & \sup_{S(\cdot)\in\mathcal{U}_{\mathrm{c}},\,k_2\geq k_1} \Big[-\sum_{k=k_1}^{k_2-1}\sum_{i=1}^{q} E_i(k+1)[S_i(k)-\sigma_{ii}(E(k))]\Big] \\
= & -\inf_{S(\cdot)\in\mathcal{U}_{\mathrm{c}},\,k_2\geq k_1} \sum_{k=k_1}^{k_2-1}\sum_{i=1}^{q} E_i(k+1)[S_i(k)-\sigma_{ii}(E(k))] \\
= & \ \mathcal{E}_{\mathrm{a}}(E(k_1)).
\end{aligned} \tag{3.44}
$$

Since $E(k_1)$ is arbitrary, it follows that $\mathcal{E}(E) - \mathcal{E}(0) \geq \mathcal{E}_{\mathrm{a}}(E)$, $E \in \overline{\mathbb{R}}_+^q$. Alternatively, let $E(k_1) = 0$ in (3.34). Then it follows from (3.34) that

$$
\mathcal{E}(E(k_2)) - \mathcal{E}(0) \leq \sum_{k=k_1}^{k_2-1}\sum_{i=1}^{q} E_i(k+1)[S_i(k)-\sigma_{ii}(E(k))], \tag{3.45}
$$

for all $k_1 + 1 \leq k_2$ and $S(\cdot) \in \mathcal{U}_{\mathrm{r}}$. Hence,

$$\begin{aligned}\mathcal{E}(E(k_2)) - \mathcal{E}(0) &\leq \inf_{S(\cdot)\in\mathcal{U}_{\mathrm{r}},\, k_1+1\leq k_2} \sum_{k=k_1}^{k_2-1}\sum_{i=1}^{q} E_i(k+1)[S_i(k) - \sigma_{ii}(E(k))] \\ &= \mathcal{E}_{\mathrm{r}}(E(k_2)),\end{aligned} \tag{3.46}$$

which, since $E(k_2)$ is arbitrary, implies that $\mathcal{E}_{\mathrm{r}}(E) \geq \mathcal{E}(E) - \mathcal{E}(0)$, $E \in \overline{\mathbb{R}}_+^q$. Thus, all ectropy functions for $\mathcal{G}$ satisfy (3.37). $\square$

The next proposition gives a closed-form expression for the ectropy of $\mathcal{G}$.

Proposition 3.5. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2) and assume that Axioms *ii*) and *iii*) hold. Then the function $\mathcal{E} : \overline{\mathbb{R}}_+^q \to \mathbb{R}$ given by

$$\mathcal{E}(E) = \tfrac{1}{2}E^{\mathrm{T}}E, \quad E \in \overline{\mathbb{R}}_+^q, \tag{3.47}$$

is an ectropy function of $\mathcal{G}$.

Proof. Since $E(k) \geq\geq 0$, $k \geq k_0$, and $\phi_{ij}(E) = -\phi_{ji}(E)$, $E \in \overline{\mathbb{R}}_+^q$, $i \neq j$, $i,j = 1,\ldots,q$, it follows that

$$\begin{aligned}\Delta\mathcal{E}(E(k)) &= \tfrac{1}{2}E^{\mathrm{T}}(k+1)E(k+1) - \tfrac{1}{2}E^{\mathrm{T}}(k)E(k) \\ &= \sum_{i=1}^{q} E_i(k+1)[S_i(k) - \sigma_{ii}(E(k))] \\ &\quad -\tfrac{1}{2}\sum_{i=1}^{q}\Big[\sum_{j=1,\,j\neq i}^{q}\phi_{ij}(E(k)) + S_i(k) - \sigma_{ii}(E(k))\Big]^2 \\ &\quad +\sum_{i=1}^{q}\sum_{j=1,\,j\neq i}^{q} E_i(k+1)\phi_{ij}(E(k)) \\ &= \sum_{i=1}^{q} E_i(k+1)[S_i(k) - \sigma_{ii}(E(k))] \\ &\quad -\tfrac{1}{2}\sum_{i=1}^{q}\Big[\sum_{j=1,\,j\neq i}^{q}\phi_{ij}(E(k)) + S_i(k) - \sigma_{ii}(E(k))\Big]^2\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{q-1}\sum_{j=i+1}^{q}(E_i(k+1)-E_j(k+1))\phi_{ij}(E(k)) \\
&\leq \sum_{i=1}^{q}E_i(k+1)[S_i(k)-\sigma_{ii}(E(k))], \quad k\geq k_0.
\end{aligned}
\tag{3.48}
$$

Now, summing (3.48) over $\{k_1,\ldots,k_2-1\}$ yields (3.34). $\square$

Remark 3.3. Note that it follows from the last equality in (3.48) that the ectropy function given by (3.47) satisfies (3.34) as an equality for an equilibrium process and as a strict inequality for a nonequilibrium process.

It follows from (3.47) that ectropy is a measure of the extent to which the system energy deviates from a homogeneous state. Thus, ectropy is the dual of entropy and is a measure of the tendency of the discrete-time large-scale dynamical system $\mathcal{G}$ to do useful work and grow more organized.

3.5. Semistability of Thermodynamic Models

Inequality (3.17) is analogous to Clausius' inequality for equilibrium and nonequilibrium thermodynamics as applied to discrete-time large-scale dynamical systems; while inequality (3.34) is an anti Clausius' inequality. Moreover, for the ectropy function defined by (3.47), inequality (3.48) shows that a thermodynamically consistent discrete-time large-scale dynamical system is *dissipative* [236] with with respect to the supply rate $E^{\mathrm{T}}S$ and with storage function corresponding to the system ectropy $\mathcal{E}(E)$. For the entropy function given by (3.30) note that $\mathcal{S}(0)=0$, or, equivalently, $\lim_{E\to 0}\mathcal{S}(E)=0$, which is consistent with the *third law of thermodynamics* (Nernst's theorem) which states that the entropy of every system at absolute zero can always be taken to be equal to zero.

For the isolated discrete-time large-scale dynamical system $\mathcal{G}$, (3.17) yields the fundamental inequality

$$\mathcal{S}(E(k_2)) \geq \mathcal{S}(E(k_1)), \quad k_2 \geq k_1. \tag{3.49}$$

Inequality (3.49) implies that, for any dynamical change in an isolated (i.e., $S(k) \equiv 0$ and $d(E(k)) \equiv 0$) discrete-time large-scale system, the entropy of the final state can never be less than the entropy of the initial state. It is important to stress that this result holds for an isolated dynamical system. It is, however, possible with energy supplied from an external dynamical system (e.g., a controller) to reduce the entropy of the discrete-time large-scale dynamical system. The entropy of both systems taken together, however, cannot decrease. The above observations imply that when an isolated discrete-time large-scale dynamical system with thermodynamically consistent energy flow characteristics (i.e., Axioms $i) - iii)$ hold) is at a state of maximum entropy consistent with its energy, it cannot be subject to any further dynamical change since any such change would result in a decrease of entropy. This of course implies that the state of *maximum entropy* is the stable state of an isolated system and this state has to be semistable.

Analogously, it follows from (3.34) that for an isolated discrete-time large-scale dynamical system $\mathcal{G}$ the fundamental inequality

$$\mathcal{E}(E(k_2)) \leq \mathcal{E}(E(k_1)), \quad k_2 \geq k_1, \tag{3.50}$$

is satisfied, which implies that the ectropy of the final state of $\mathcal{G}$ is always less than or equal to the ectropy of the initial state of $\mathcal{G}$. Hence, for the isolated large-scale dynamical system $\mathcal{G}$ the entropy increases if and only if the ectropy decreases. Thus, the state of *minimum ectropy* is the stable state of an isolated system and this equilibrium state has to be semistable. The next theorem concretizes the above observations.

Theorem 3.3. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2) with $S(k) \equiv 0$ and $d(E) \equiv 0$, and assume that Axioms $i) - iii)$ hold. Then for every $\alpha \geq 0$, $\alpha\mathbf{e}$ is a Lyapunov equilibrium state of (3.2). Furthermore,

$E(k) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}E(k_0)$ as $k \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}E(k_0)$ is a semistable equilibrium state. Finally, if for some $m \in \{1, \ldots, q\}$, $\sigma_{mm}(E) \geq 0$, $E \in \overline{\mathbb{R}}_+^q$, and $\sigma_{mm}(E) = 0$ if and only if $E_m = 0$,[2] then the zero solution $E(k) \equiv 0$ to (3.2) is a globally asymptotically stable equilibrium state of (3.2).

Proof. It follows from Axiom $i)$ that $\alpha\mathbf{e} \in \overline{\mathbb{R}}_+^q$, $\alpha \geq 0$, is an equilibrium state for (3.2). To show Lyapunov stability of the equilibrium state $\alpha\mathbf{e}$ consider the system shifted ectropy $\mathcal{E}_{\mathrm{s}}(E) = \frac{1}{2}(E - \alpha\mathbf{e})^{\mathrm{T}}(E - \alpha\mathbf{e})$ as a Lyapunov function candidate. Now, since $\phi_{ij}(E) = -\phi_{ji}(E)$, $E \in \overline{\mathbb{R}}_+^q$, $i \neq j$, $i, j = 1, \ldots, q$, and $\mathbf{e}^{\mathrm{T}}E(k+1) = \mathbf{e}^{\mathrm{T}}E(k)$, $k \geq k_0$, it follows from Axioms $ii)$ and $iii)$ that

$$\begin{aligned}
\Delta\mathcal{E}_{\mathrm{s}}(E(k)) &= \tfrac{1}{2}(E(k+1) - \alpha\mathbf{e})^{\mathrm{T}}(E(k+1) - \alpha\mathbf{e}) - \tfrac{1}{2}(E(k) - \alpha\mathbf{e})^{\mathrm{T}}(E(k) - \alpha\mathbf{e}) \\
&= \sum_{i=1}^{q} \sum_{j=1,\, j\neq i}^{q} E_i(k+1)\phi_{ij}(E(k)) - \tfrac{1}{2}\sum_{i=1}^{q}\Big[\sum_{j=1,\, j\neq i}^{q} \phi_{ij}(E(k))\Big]^2 \\
&= \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} (E_i(k+1) - E_j(k+1))\phi_{ij}(E(k)) - \tfrac{1}{2}\sum_{i=1}^{q}\Big[\sum_{j=1,\, j\neq i}^{q} \phi_{ij}(E(k))\Big]^2 \\
&\leq 0, \quad E(k) \in \overline{\mathbb{R}}_+^q, \quad k \geq k_0,
\end{aligned} \tag{3.51}$$

which establishes Lyapunov stability of the equilibrium state $\alpha\mathbf{e}$.

To show that $\alpha\mathbf{e}$ is semistable, note that

$$\begin{aligned}
\Delta\mathcal{E}_{\mathrm{s}}(E(k)) &= \sum_{i=1}^{q} \sum_{j=1,\, j\neq i}^{q} E_i(k)\phi_{ij}(E(k)) + \tfrac{1}{2}\sum_{i=1}^{q}\Big[\sum_{j=1,\, j\neq i}^{q} \phi_{ij}(E(k))\Big]^2 \\
&\geq \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} (E_i(k) - E_j(k))\phi_{ij}(E(k)) \\
&= \sum_{i=1}^{q-1} \sum_{j\in\mathcal{K}_i} (E_i(k) - E_j(k))\phi_{ij}(E(k)), \quad E(k) \in \overline{\mathbb{R}}_+^q, \quad k \geq k_0,
\end{aligned} \tag{3.52}$$

where $\mathcal{K}_i \triangleq \mathcal{N}_i \setminus \cup_{l=1}^{i-1}\{l\}$ and $\mathcal{N}_i \triangleq \{j \in \{1, \ldots, q\} : \phi_{ij}(E) = 0$ if and only if $E_i = E_j\}$, $i = 1, \ldots, q$.

[2]The assumption $\sigma_{mm}(E) \geq 0$, $E \in \overline{\mathbb{R}}_+^q$, and $\sigma_{mm}(E) = 0$ if and only if $E_m = 0$ for some $m \in \{1, \ldots, q\}$ implies that if the mth subsystem possesses no energy, then this subsystem cannot dissipate energy to the environment. Conversely, if the mth subsystem does not dissipate energy to the environment, then this subsystem has no energy.

Next, we show that $\Delta\mathcal{E}_{\rm s}(E) = 0$ if and only if $(E_i - E_j)\phi_{ij}(E) = 0$, $i = 1,\ldots,q$, $j \in \mathcal{K}_i$. First, assume that $(E_i - E_j)\phi_{ij}(E) = 0$, $i = 1,\ldots,q$, $j \in \mathcal{K}_i$. Then it follows from (3.52) that $\Delta\mathcal{E}_{\rm s}(E) \geq 0$. However, it follows from (3.51) that $\Delta\mathcal{E}_{\rm s}(E) \leq 0$. Hence, $\Delta\mathcal{E}_{\rm s}(E) = 0$. Conversely, assume $\Delta\mathcal{E}_{\rm s}(E) = 0$. In this case, it follows from (3.51) that $(E_i(k+1) - E_j(k+1))\phi_{ij}(E(k)) = 0$ and $\sum_{j=1,\,j\neq i}^{q}\phi_{ij}(E(k)) = 0$, $k \geq k_0$, $i,j = 1,\ldots,q$, $i \neq j$. Since

$$\begin{aligned}
[E_i(k+1) - E_j(k+1)]\phi_{ij}(E(k)) &= [E_i(k) - E_j(k)]\phi_{ij}(E(k)) \\
&\quad + \Big[\sum_{h=1,\,h\neq i}^{q}\phi_{ih}(E(k)) - \sum_{l=1,\,l\neq j}^{q}\phi_{jl}(E(k))\Big]\phi_{ij}(E(k)) \\
&= [E_i(k) - E_j(k)]\phi_{ij}(E(k)), \\
&\qquad k \geq k_0, \quad i,j = 1,\ldots,q, \quad i \neq j,
\end{aligned} \tag{3.53}$$

it follows that $(E_i - E_j)\phi_{ij}(E) = 0$, $i = 1,\ldots,q$, $j \in \mathcal{K}_i$.

Let $\mathcal{R} \triangleq \{E \in \overline{\mathbb{R}}_+^q : \Delta\mathcal{E}_{\rm s}(E) = 0\} = \{E \in \overline{\mathbb{R}}_+^q : (E_i - E_j)\phi_{ij}(E) = 0, i = 1,\ldots,q, j \in \mathcal{K}_i\}$. Now, by Axiom i) the directed graph associated with the connectivity matrix $\mathcal{C}$ for the discrete-time large-scale dynamical system $\mathcal{G}$ is strongly connected which implies that $\mathcal{R} = \{E \in \overline{\mathbb{R}}_+^q : E_1 = \cdots = E_q\}$. Since the set $\mathcal{R}$ consists of the equilibrium states of (3.2), it follows that the largest invariant set $\mathcal{M}$ contained in $\mathcal{R}$ is given by $\mathcal{M} = \mathcal{R}$. Hence, it follows from LaSalle's invariant set theorem that for any initial condition $E(k_0) \in \overline{\mathbb{R}}_+^q$, $E(k) \to \mathcal{M}$ as $k \to \infty$, and hence, $\alpha\mathbf{e}$ is a semistable equilibrium state of (3.2). Next, note that since $\mathbf{e}^{\rm T}E(k) = \mathbf{e}^{\rm T}E(k_0)$ and $E(k) \to \mathcal{M}$ as $k \to \infty$, it follows that $E(k) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\rm T}E(k_0)$ as $k \to \infty$. Hence, with $\alpha = \frac{1}{q}\mathbf{e}^{\rm T}E(k_0)$, $\alpha\mathbf{e} = \frac{1}{q}\mathbf{e}\mathbf{e}^{\rm T}E(k_0)$ is a semistable equilibrium state of (3.2).

Finally, to show that in the case where for some $m \in \{1,\ldots,q\}$, $\sigma_{mm}(E) \geq 0$, $E \in \overline{\mathbb{R}}_+^q$, and $\sigma_{mm}(E) = 0$ if and only if $E_m = 0$, the zero solution $E(k) \equiv 0$ to (3.2) is globally asymptotically stable consider the system ectropy $\mathcal{E}(E) = \frac{1}{2}E^{\rm T}E$ as a candidate Lyapunov function. Note that $\mathcal{E}(0) = 0$, $\mathcal{E}(E) > 0$, $E \in \overline{\mathbb{R}}_+^q$, $E \neq 0$, and $\mathcal{E}(E)$ is radially unbounded.

Now, the Lyapunov difference is given by

$$
\begin{aligned}
\Delta\mathcal{E}(E(k)) &= \tfrac{1}{2}E^{\mathrm{T}}(k+1)E(k+1) - \tfrac{1}{2}E^{\mathrm{T}}(k)E(k) \\
&= -E_m(k+1)\sigma_{mm}(E(k)) - \tfrac{1}{2}\Big[\sum_{j=1,\, j\neq m}^{q}\phi_{mj}(E(k)) - \sigma_{mm}(E(k))\Big]^2 \\
&\quad -\tfrac{1}{2}\sum_{i=1, i\neq m}^{q}\Big[\sum_{j=1,\, j\neq i}^{q}\phi_{ij}(E(k))\Big]^2 + \sum_{i=1}^{q}\sum_{j=1,\, j\neq i}^{q}E_i(k+1)\phi_{ij}(E(k)) \\
&= -E_m(k+1)\sigma_{mm}(E(k)) - \tfrac{1}{2}\Big[\sum_{j=1,\, j\neq m}^{q}\phi_{mj}(E(k)) - \sigma_{mm}(E(k))\Big]^2 \\
&\quad -\tfrac{1}{2}\sum_{i=1, i\neq m}^{q}\Big[\sum_{j=1,\, j\neq i}^{q}\phi_{ij}(E(k))\Big]^2 + \sum_{i=1}^{q-1}\sum_{j=i+1}^{q}(E_i(k+1) - E_j(k+1))\phi_{ij}(E(k)) \\
&\leq 0, \quad E(k) \in \overline{\mathbb{R}}_+^q, \quad k \geq k_0,
\end{aligned} \tag{3.54}
$$

which shows that the zero solution $E(k) \equiv 0$ to (3.2) is Lyapunov stable.

To show global asymptotic stability of the zero equilibrium state, note that

$$
\begin{aligned}
\Delta\mathcal{E}(E(k)) &= \sum_{i=1}^{q-1}\sum_{j=i+1}^{q}(E_i(k) - E_j(k))\phi_{ij}(E(k)) + \tfrac{1}{2}\sum_{i=1, i\neq m}^{q}\Big[\sum_{j=1,\, j\neq i}^{q}\phi_{ij}(E(k))\Big]^2 \\
&\quad -E_m(k)\sigma_{mm}(E(k)) + \tfrac{1}{2}\Big[\sum_{j=1,\, j\neq m}^{q}\phi_{mj}(E(k)) - \sigma_{mm}(E(k))\Big]^2 \\
&\geq \sum_{i=1}^{q-1}\sum_{j\in\mathcal{K}_i}(E_i(k) - E_j(k))\phi_{ij}(E(k)) - E_m(k)\sigma_{mm}(E(k)), \\
&\qquad E(k) \in \overline{\mathbb{R}}_+^q, \quad k \geq k_0.
\end{aligned} \tag{3.55}
$$

Next, we show that $\Delta\mathcal{E}(E) = 0$ if and only if $(E_i - E_j)\phi_{ij}(E) = 0$ and $\sigma_{mm}(E) = 0$, $i = 1, \ldots, q$, $j \in \mathcal{K}_i$, $m \in \{1, \ldots, q\}$. First, assume that $(E_i - E_j)\phi_{ij}(E) = 0$ and $\sigma_{mm}(E) = 0$, $i = 1, \ldots, q$, $j \in \mathcal{K}_i$, $m \in \{1, \ldots, q\}$. Then it follows from (3.55) that $\Delta\mathcal{E}(E) \geq 0$. However, it follows from (3.54) that $\Delta\mathcal{E}(E) \leq 0$. Thus, $\Delta\mathcal{E}(E) = 0$. Conversely, assume $\Delta\mathcal{E}(E) = 0$. Then it follows from (3.54) that $(E_i(k+1) - E_j(k+1))\phi_{ij}(E(k)) = 0$, $i, j = 1, \ldots, q$, $i \neq j$, $\sum_{j=1,\, j\neq i}^{q}\phi_{ij}(E(k)) = 0$, $i = 1, \ldots, q$, $i \neq m$, $k \geq k_0$, and $\sigma_{mm}(E) = 0$, $m \in \{1, \ldots, q\}$. Note that in this case it follows that $\sigma_{mm}(E) = \sum_{j=1, j\neq m}^{q}\phi_{mj}(E) = 0$, and hence,

$$
[E_i(k+1) - E_j(k+1)]\phi_{ij}(E(k)) = [E_i(k) - E_j(k)]\phi_{ij}(E(k)),
$$

$$k \geq k_0, \quad i,j = 1,\ldots,q, \quad i \neq j, \tag{3.56}$$

which implies that $(E_i - E_j)\phi_{ij}(E) = 0$, $i = 1,\ldots,q$, $j \in \mathcal{K}_i$. Hence, $(E_i - E_j)\phi_{ij}(E) = 0$ and $\sigma_{mm}(E) = 0$, $i = 1,\ldots,q$, $j \in \mathcal{K}_i$, $m \in \{1,\ldots,q\}$ if and only if $\Delta\mathcal{E}(E) = 0$.

Let $\mathcal{R} \triangleq \{E \in \overline{\mathbb{R}}_+^q : \Delta\mathcal{E}(E) = 0\} = \{E \in \overline{\mathbb{R}}_+^q : \sigma_{mm}(E) = 0,\ m \in \{1,\ldots,q\}\} \cap \{E \in \overline{\mathbb{R}}_+^q : (E_i - E_j)\phi_{ij}(E) = 0,\ i = 1,\ldots,q,\ j \in \mathcal{K}_i\}$. Now, since Axiom i) holds and $\sigma_{mm}(E) = 0$ if and only if $E_m = 0$ it follows that $\mathcal{R} = \{E \in \overline{\mathbb{R}}_+^q : E_m = 0,\ m \in \{1,\ldots,q\}\} \cap \{E \in \overline{\mathbb{R}}_+^q : E_1 = E_2 = \cdots = E_q\} = \{0\}$ and the largest invariant set $\mathcal{M}$ contained in $\mathcal{R}$ is given by $\mathcal{M} = \{0\}$. Hence, it follows from LaSalle's invariant set theorem that for any initial condition $E(k_0) \in \overline{\mathbb{R}}_+^q$, $E(k) \to \mathcal{M} = \{0\}$ as $k \to \infty$, which proves global asymptotic stability of the zero equilibrium state of (3.2). $\square$

Remark 3.4. It is important to note that Axiom iii) involving monotonicity of solutions is explicitly used to prove semistability for discrete-time compartmental dynamical systems. However, Axiom iii) is a sufficient condition and not necessary for guaranteeing semistability. Replacing the monotonicity condition with $\sum_{i=1,j=1,i\neq j}^{q} \alpha_{ij}(E) f_{ij}(E) \geq 0$, where

$$\alpha_{ij}(E) \triangleq \begin{cases} \frac{\phi_{ij}(E)}{E_j - E_i}, & E_i \neq E_j \\ 0, & E_i = E_j \end{cases} \tag{3.57}$$

$$f_{ij}(E) \triangleq [E_i(k) - E_j(k)][E_i(k+1) - E_j(k+1)], \tag{3.58}$$

provides a weaker sufficient condition for guaranteeing semistability. However, in this case, to ensure that the entropy of $\mathcal{G}$ is monotonically increasing, we additionally require that $\sum_{i=1,j=1,i\neq j}^{q} \beta_{ij}(E) f_{ij}(E) \geq 0$, where

$$\beta_{ij}(E) \triangleq \begin{cases} \frac{1}{(c+E_i(k+1))(c+E_j(k+1))} \cdot \frac{\phi_{ij}(E(k))}{E_j(k) - E_i(k)}, & E_i \neq E_j \\ 0, & E_i = E_j \end{cases}. \tag{3.59}$$

Thus, a weaker condition for Axiom iii) which combines $\sum_{i,j=1,i\neq j}^{q} \alpha_{ij}(E) f_{ij}(E) \geq 0$ and $\sum_{i,j=1,i\neq j}^{q} \beta_{ij}(E) f_{ij}(E) \geq 0$, is $\sum_{i=1,j=1,i\neq j}^{q} \gamma_{ij}(E) f_{ij}(E) \geq 0$, where $\gamma_{ij}(E) \triangleq \alpha_{ij}(E) + \beta_{ij}(E) - \mathrm{sgn}(f_{ij}(E))|\alpha_{ij}(E) - \beta_{ij}(E)|$ and $\mathrm{sgn}(f_{ij}(E)) \triangleq |f_{ij}(E)|/f_{ij}(E)$.

In Theorem 3.3 we used the shifted ectropy function to show that for the isolated (i.e., $S(k) \equiv 0$ and $d(E) \equiv 0$) discrete-time large-scale dynamical system $\mathcal{G}$ with Axioms $i) - iii)$, $E(k) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}E(k_0)$ as $k \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}E(k_0)$ is a semistable equilibrium state. This result can also be arrived at using the system entropy for the isolated discrete-time large-scale dynamical system $\mathcal{G}$ with Axioms $i) - iii)$. To see this, note that since $\mathbf{e}^{\mathrm{T}}w(E) = \mathbf{e}^{\mathrm{T}}E$, $E \in \overline{\mathbb{R}}_+^q$, it follows that $\mathbf{e}^{\mathrm{T}}\Delta E(k) = 0$, $k \geq k_0$. Hence, $\mathbf{e}^{\mathrm{T}}E(k) = \mathbf{e}^{\mathrm{T}}E(k_0)$, $k \geq k_0$. Furthermore, since $E(k) \geq\geq 0$, $k \geq k_0$, it follows that $0 \leq\leq E(k) \leq\leq \mathbf{e}\mathbf{e}^{\mathrm{T}}E(k_0)$, $k \geq k_0$, which implies that all solutions to (3.2) are bounded. Next, since by (3.49) the entropy $\mathcal{S}(E(k))$, $k \geq k_0$, of $\mathcal{G}$ is monotonically increasing and $E(k)$, $k \geq k_0$, is bounded, the result follows by using similar arguments as in Theorem 3.3 and using the fact that $\frac{x}{1+x} \leq \log_e(1+x) \leq x$ for all $x > -1$.

3.6. Energy Equipartition

Theorem 3.3 implies that the steady-state value of the energy in each subsystem $\mathcal{G}_i$ of the isolated large-scale dynamical system $\mathcal{G}$ is equal; that is, the steady-state energy of the isolated discrete-time large-scale dynamical system $\mathcal{G}$ given by $E_\infty = \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}E(k_0) = \left[\frac{1}{q}\sum_{i=1}^{q} E_i(k_0)\right]\mathbf{e}$ is uniformly distributed over all subsystems of $\mathcal{G}$. This phenomenon is known as *equipartition of energy*[3] [25,26,116,165,200] and is an emergent behavior in thermodynamic systems. The next proposition shows that among all possible energy distributions in the discrete-time large-scale dynamical system $\mathcal{G}$, energy equipartition corresponds to the minimum value of the system's ectropy and the maximum value of the system's entropy (see Figure 3.2).

Proposition 3.6. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2), let $\mathcal{E} : \overline{\mathbb{R}}_+^q \to \mathbb{R}$ and $\mathcal{S} : \overline{\mathbb{R}}_+^q \to \mathbb{R}$ denote the ectropy and entropy of $\mathcal{G}$ given by (3.47) and (3.30), respectively, and define $\mathcal{D}_{\mathrm{c}} \triangleq \{E \in \overline{\mathbb{R}}_+^q : \mathbf{e}^{\mathrm{T}}E = \beta\}$, where

[3] The phenomenon of equipartition of energy is closely related to the notion of a *monotemperaturic* system discussed in [39].

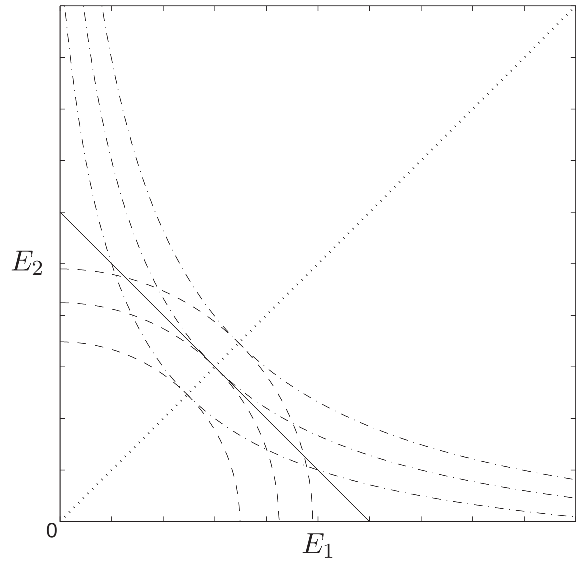

Figure 3.2: Thermodynamic equilibria ($\cdots$), constant energy surfaces (———), constant ectropy surfaces ($-\ -\ -$), and constant entropy surfaces ($-\cdot-\cdot-$)

$\beta \geq 0$. Then,

$$\arg\min_{E\in\mathcal{D}_{\rm c}}(\mathcal{E}(E)) = \arg\max_{E\in\mathcal{D}_{\rm c}}(\mathcal{S}(E)) = E^* = \frac{\beta}{q}\mathbf{e}. \tag{3.60}$$

Furthermore, $\mathcal{E}_{\min} \triangleq \mathcal{E}(E^*) = \frac{1}{2}\frac{\beta^2}{q}$ and $\mathcal{S}_{\max} \triangleq \mathcal{S}(E^*) = q\log_e(c+\frac{\beta}{q}) - q\log_e c$.

Proof. The existence and uniqueness of E^* follows from the fact that $\mathcal{E}(E)$ and $-\mathcal{S}(E)$ are strictly convex continuous functions on the compact set $\mathcal{D}_{\rm c}$. To minimize $\mathcal{E}(E) = \frac{1}{2}E^{\rm T}E$, $E \in \overline{\mathbb{R}}_+^q$, subject to $E \in \mathcal{D}_{\rm c}$ form the Lagrangian $\mathcal{L}(E,\lambda) = \frac{1}{2}E^{\rm T}E + \lambda(\mathbf{e}^{\rm T}E - \beta)$, where $\lambda \in \mathbb{R}$ is the Lagrange multiplier. If E^* solves this minimization problem, then

$$0 = \left.\frac{\partial\mathcal{L}}{\partial E}\right|_{E=E^*} = E^{*\rm T} + \lambda\mathbf{e}^{\rm T}, \tag{3.61}$$

and hence, $E^* = -\lambda\mathbf{e}$. Now, it follows from $\mathbf{e}^{\rm T}E = \beta$ that $\lambda = -\frac{\beta}{q}$, which implies that $E^* = \frac{\beta}{q}\mathbf{e} \in \overline{\mathbb{R}}_+^q$. The fact that E^* minimizes the ectropy on the compact set $\mathcal{D}_{\rm c}$ can be shown by computing the Hessian of the ectropy for the constrained parameter optimization problem and showing that the Hessian is positive definite at E^*. $\mathcal{E}_{\min} = \frac{1}{2}\frac{\beta^2}{q}$ is now immediate.

Analogously, to maximize $\mathcal{S}(E) = \mathbf{e}^{\rm T}\mathbf{log}_e(c\mathbf{e}+E) - q\log_e c$ on the compact set $\mathcal{D}_{\rm c}$, form the Lagrangian $\mathcal{L}(E,\lambda) \triangleq \sum_{i=1}^q \log_e(c+E_i) + \lambda(\mathbf{e}^{\rm T}E - \beta)$, where $\lambda \in \mathbb{R}$ is a Largange multiplier. If E^* solves this maximization problem, then

$$0 = \frac{\partial \mathcal{L}}{\partial E}\bigg|_{E=E^*} = \left[\frac{1}{c+E_1^*} + \lambda, \ldots, \frac{1}{c+E_q^*} + \lambda\right]. \tag{3.62}$$

Thus, $\lambda = -\frac{1}{c+E_i^*}$, $i = 1, \ldots, q$. If $\lambda = 0$, then the only value of E^* that satisfies (3.62) is $E^* = \infty$, which does not satisfy the constraint equation $\mathbf{e}^{\mathrm{T}}E = \beta$ for finite $\beta \geq 0$. Hence, $\lambda \neq 0$ and $E_i^* = -(\frac{1}{\lambda} + c)$, $i = 1, \ldots, q$, which implies $E^* = -(\frac{1}{\lambda} + c)\mathbf{e}$. Now, it follows from $\mathbf{e}^{\mathrm{T}}E = \beta$ that $-(\frac{1}{\lambda} + c) = \frac{\beta}{q}$, and hence, $E^* = \frac{\beta}{q}\mathbf{e} \in \overline{\mathbb{R}}_+^q$. The fact that E^* maximizes the entropy on the compact set $\mathcal{D}_{\mathrm{c}}$ can be shown by computing the Hessian and showing that it is negative definite at E^*. $\mathcal{S}_{\max} = q\log_e(c + \frac{\beta}{q}) - q\log_e c$ is now immediate. $\square$

It follows from (3.49), (3.50), and Proposition 3.6 that conservation of energy necessarily implies nonconservation of ectropy and entropy. Hence, in an isolated discrete-time large-scale dynamical system $\mathcal{G}$ all the energy, though always conserved, will eventually be degraded (diluted) to the point where it cannot produce any useful work. Hence, all motion would cease and the dynamical system would be fated to a state of eternal rest (semistability) wherein all subsystems will posses identical energies (energy equipartition). Ectropy would be a minimum and entropy would be a maximum giving rise to a state of absolute disorder. This is precisely what is known in theoretical physics as the *heat death of the universe* [104].

3.7. Entropy Increase and the Second Law of Thermodynamics

In the preceding discussion it was assumed that our discrete-time large-scale nonlinear dynamical system model is such that energy is exchanged from more energetic subsystems to less energetic subsystems, that is, heat (energy) flows in the direction of lower temperatures. Although this universal phenomenon can be predicted with virtual certainty, it follows as a manifestation of entropy and ectropy nonconservation for the case of two subsystems.

To see this, consider the isolated (i.e., $S(k) \equiv 0$ and $d(E) \equiv 0$) discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2) and assume that the system entropy is monotonically increasing and hence $\Delta\mathcal{S}(E(k)) \geq 0$, $k \geq k_0$. Now, since

$$\begin{aligned}
0 \leq\ & \Delta\mathcal{S}(E(k)) \\
=\ & \sum_{i=1}^{q} \log_e \left[1 + \frac{\Delta E_i(k)}{c + E_i(k)}\right] \\
\leq\ & \sum_{i=1}^{q} \frac{\Delta E_i(k)}{c + E_i(k)} \\
=\ & \sum_{i=1}^{q} \sum_{j=1,\, j\neq i}^{q} \frac{\phi_{ij}(E(k))}{c + E_i(k)} \\
=\ & \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} \left[\frac{\phi_{ij}(E(k))}{c + E_i(k)} - \frac{\phi_{ij}(E(k))}{c + E_j(k)}\right] \\
=\ & \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} \frac{\phi_{ij}(E(k))(E_j(k) - E_i(k))}{(c + E_i(k))(c + E_j(k))}, \quad k \geq k_0,
\end{aligned} \tag{3.63}$$

it follows that for $q = 2$, $(E_1 - E_2)\phi_{12}(E) \leq 0$, $E \in \overline{\mathbb{R}}_+^2$, which implies that energy (heat) flows naturally from a more energetic subsystem (hot object) to a less energetic subsystem (cooler object). The universality of this emergent behavior thus follows from the fact that entropy (respectively, ectropy) transfer, accompanying energy transfer, always increases (respectively, decreases).

In the case where we have multiple subsystems, it is clear from (3.63) that entropy and ectropy nonconservation does not necessarily imply Axiom *ii*). However, if we invoke the additional condition (Axiom *iv*)) that if for any pair of connected subsystems $\mathcal{G}_k$ and $\mathcal{G}_l$, $k \neq l$, with energies $E_k \geq E_l$ (respectively, $E_k \leq E_l$), and for any other pair of connected subsystems $\mathcal{G}_m$ and $\mathcal{G}_n$, $m \neq n$, with energies $E_m \geq E_n$ (respectively, $E_m \leq E_n$) the inequality $\phi_{kl}(E)\phi_{mn}(E) \geq 0$, $E \in \overline{\mathbb{R}}_+^q$, holds, then nonconservation of entropy and ectropy in the isolated discrete-time large-scale dynamical system $\mathcal{G}$ implies Axiom *ii*). The above inequality postulates that the direction of energy exchange for any pair of *energy similar* subsystems is consistent; that is, if for a given pair of connected subsystems at given different energy levels the energy flows in a certain direction, then for any other pair of connected subsystems with the same energy level, the energy flow direction is consistent with the original pair of subsystems. Note that this assumption does *not* specify the direction of energy flow between subsystems.

To see that $\Delta\mathcal{S}(E(k)) \geq 0$, $k \geq k_0$, along with Axiom *iv*) implies Axiom *ii*) note that since (3.63) holds for all $k \geq k_0$ and $E(k_0) \in \overline{\mathbb{R}}_+^q$ is arbitrary, (3.63) implies

$$\sum_{i=1}^{q} \sum_{j\in\mathcal{K}_i} \frac{\phi_{ij}(E)(E_j - E_i)}{(c+E_i)(c+E_j)} \geq 0, \quad E \in \overline{\mathbb{R}}_+^q. \tag{3.64}$$

Now, it follows from (3.64) that for any fixed system energy level $E \in \overline{\mathbb{R}}_+^q$ there exists at least one pair of connected subsystems $\mathcal{G}_k$ and $\mathcal{G}_l$, $k \neq l$, such that $\phi_{kl}(E)(E_l - E_k) \geq 0$. Thus, if $E_k \geq E_l$ (respectively, $E_k \leq E_l$), then $\phi_{kl}(E) \leq 0$ (respectively, $\phi_{kl}(E) \geq 0$). Furthermore, it follows from Axiom *iv*) that for any other pair of connected subsystems $\mathcal{G}_m$ and $\mathcal{G}_n$, $m \neq n$, with $E_m \geq E_n$ (respectively, $E_m \leq E_n$) the inequality $\phi_{mn}(E) \leq 0$ (respectively, $\phi_{mn}(E) \geq 0$) holds which implies that

$$\phi_{mn}(E)(E_n - E_m) \geq 0, \quad m \neq n. \tag{3.65}$$

Thus, it follows from (3.65) that energy (heat) flows naturally from more energetic subsystems (hot objects) to less energetic subsystems (cooler objects). Of course, since in the isolated discrete-time large-scale dynamical system $\mathcal{G}$ ectropy decreases if and only if entropy increases, the same result can be arrived at by considering the ectropy of $\mathcal{G}$. Since Axiom *ii*) holds, it follows from the conservation of energy and the fact that the discrete-time large-scale dynamical system $\mathcal{G}$ is strongly connected that nonconservation of entropy and ectropy necessarily implies energy equipartition.

Finally, we close this section by showing that our definition of entropy given by (3.30) satisfies the eight criteria established in [90] for the acceptance of an analytic expression for representing a system entropy function. In particular, note that for a dynamical system $\mathcal{G}$: *i*) $\mathcal{S}(E)$ is well defined for every state $E \in \overline{\mathbb{R}}_+^q$ as long as $c > 0$. *ii*) If $\mathcal{G}$ is isolated, then $\mathcal{S}(E(k))$ is a nondecreasing function of time. *iii*) If $\mathcal{S}_i(E_i) = \log_e(c + E_i) - \log_e c$ is the entropy of the ith subsystem of the system $\mathcal{G}$, then $\mathcal{S}(E) = \sum_{i=1}^{q} \mathcal{S}_i(E_i) = \mathbf{e}^{\mathrm{T}}\mathbf{log}_e(c\mathbf{e} + E) - q\log_e c$

and hence the system entropy $\mathcal{S}(E)$ is an additive quantity over all subsystems. *iv*) For the system $\mathcal{G}$, $\mathcal{S}(E) \geq 0$ for all $E \in \overline{\mathbb{R}}_+^q$. *v*) It follows from Proposition 3.6 that for a given value $\beta \geq 0$ of the total energy of the system $\mathcal{G}$, one and only one state, namely, $E^* = \frac{\beta}{q}\mathbf{e}$, corresponds to the largest value of $\mathcal{S}(E)$. *vi*) It follows from (3.30) that for the system $\mathcal{G}$, graph of entropy versus energy is concave and smooth. *vii*) For a composite discrete-time large-scale dynamical system $\mathcal{G}_{\mathrm{C}}$ of two dynamical systems $\mathcal{G}_{\mathrm{A}}$ and $\mathcal{G}_{\mathrm{B}}$ the expression for the composite entropy $\mathcal{S}_{\mathrm{C}} = \mathcal{S}_{\mathrm{A}} + \mathcal{S}_{\mathrm{B}}$, where $\mathcal{S}_{\mathrm{A}}$ and $\mathcal{S}_{\mathrm{B}}$ are entropies of $\mathcal{G}_{\mathrm{A}}$ and $\mathcal{G}_{\mathrm{B}}$, respectively, is such that the expression for the equilibrium state where the composite maximum entropy is achieved is identical to those obtained for $\mathcal{G}_{\mathrm{A}}$ and $\mathcal{G}_{\mathrm{B}}$ individually. Specifically, if q_{A} and q_{B} denote the number of subsystems in $\mathcal{G}_{\mathrm{A}}$ and $\mathcal{G}_{\mathrm{B}}$, respectively, and β_{A} and β_{B} denote the total energies of $\mathcal{G}_{\mathrm{A}}$ and $\mathcal{G}_{\mathrm{B}}$, respectively, then the maximum entropy of $\mathcal{G}_{\mathrm{A}}$ and $\mathcal{G}_{\mathrm{B}}$ individually is achieved at $E_{\mathrm{A}}^* = \frac{\beta_{\mathrm{A}}}{q_{\mathrm{A}}}\mathbf{e}$ and $E_{\mathrm{B}}^* = \frac{\beta_{\mathrm{B}}}{q_{\mathrm{B}}}\mathbf{e}$, respectively, while the maximum entropy of the composite system $\mathcal{G}_{\mathrm{C}}$ is achieved at $E_{\mathrm{C}}^* = \frac{\beta_{\mathrm{A}}+\beta_{\mathrm{B}}}{q_{\mathrm{A}}+q_{\mathrm{B}}}\mathbf{e}$. *viii*) It follows from Theorem 3.3 that for a stable equilibrium state $E = \frac{\beta}{q}\mathbf{e}$, where $\beta \geq 0$ is the total energy of the system $\mathcal{G}$ and q is the number of subsystems of $\mathcal{G}$, the entropy is totally defined by β and q, that is, $\mathcal{S}(E) = q\log_e(c + \frac{\beta}{q}) - q\log_e c$. Dual criteria to the eight criteria outlined above can also be established for an analytic expression representing system ectropy.

3.8. Temperature Equipartition

The thermodynamic axioms introduced in Section 3.3 postulate that subsystem energies are synonymous to subsystem temperatures. In this section, we generalize the results of Section 3.3 to the case where the subsystem energies are proportional to the subsystem temperatures with the proportionality constants representing the subsystem *specific heats* or *thermal capacities*. In the case where the specific heats of all the subsystems are equal the results of this section specialize to those of Section 3.3. To include temperature notions in our discrete-time large-scale dynamical system model we replace Axioms $i) - iii)$ of Section

3.3 by the following conditions. Let $\beta_i > 0$, $i = 1, \ldots, q$, denote the reciprocal of the specific heat of the ith subsystem $\mathcal{G}_i$ so that the *absolute temperature* in ith subsystem is given by $\hat{T}_i = \beta_i E_i$.

Axiom *i*): For the connectivity matrix $\mathcal{C} \in \mathbb{R}^{q\times q}$ associated with the discrete-time large-scale dynamical system $\mathcal{G}$ defined by (3.13) and (3.14), rank $\mathcal{C} = q-1$ and for $\mathcal{C}_{(i,j)} = 1$, $i \neq j$, $\phi_{ij}(E) = 0$ if and only if $\beta_i E_i = \beta_j E_j$.

Axiom *ii*): For $i, j = 1, \ldots, q$, $(\beta_i E_i - \beta_j E_j)\phi_{ij}(E) \leq 0$, $E \in \overline{\mathbb{R}}_+^q$.

Axiom *iii*): For $i, j = 1, \ldots, q$, $\frac{\beta_i \Delta E_i - \beta_j \Delta E_j}{\beta_i E_i - \beta_j E_j} \geq -1$, $\beta_i E_i \neq \beta_j E_j$.

Axiom *i*) implies that if the temperatures in the connected subsystems $\mathcal{G}_i$ and $\mathcal{G}_j$ are equal, then heat exchange between these subsystems is not possible. This statement is consistent with the zeroth law of thermodynamics which postulates that temperature equality is a necessary and sufficient condition for *thermal equilibrium*. Axiom *ii*) implies that heat (energy) must flow in the direction of lower temperatures. This statement is consistent with the second law of thermodynamics which states that a transformation whose only final result is to transfer heat from a body at a given temperature to a body at a higher temperature is impossible. Axiom *iii*) implies that for any pair of connected subsystems $\mathcal{G}_i$ and $\mathcal{G}_j$, $i \neq j$, the temperature difference between consecutive time instants is monotonic, that is, $[\beta_i E_i(k+1) - \beta_j E_j(k+1)][\beta_i E_i(k) - \beta_j E_j(k)] \geq 0$ for all $\beta_i E_i \neq \beta_j E_j$, $k \geq k_0$, $i, j = 1, \ldots, q$. Next, in light of our modified conditions we give a generalized definition for the entropy and ectropy of $\mathcal{G}$. The following proposition is needed for the statement of the main results of this section.

Proposition 3.7. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2) and assume that Axioms *i*), *ii*), and *iii*) hold. Then for all $E_0 \in \overline{\mathbb{R}}_+^q$, $k_\mathrm{f} \geq k_0$, and $S(\cdot) \in \mathcal{U}$, such that $E(k_\mathrm{f}) = E(k_0) = E_0$,

$$\sum_{k=k_0}^{k_\mathrm{f}-1} \sum_{i=1}^{q} \frac{S_i(k) - \sigma_{ii}(E(k))}{c + \beta_i E_i(k+1)} = \sum_{k=k_0}^{k_\mathrm{f}-1} \sum_{i=1}^{q} \frac{Q_i(k)}{c + \beta_i E_i(k+1)} \leq 0, \tag{3.66}$$

$$\sum_{k=k_0}^{k_{\rm f}-1}\sum_{i=1}^{q}\beta_i E_i(k+1)[S_i(k)-\sigma_{ii}(E(k))] = \sum_{k=k_0}^{k_{\rm f}-1}\sum_{i=1}^{q}\beta_i E_i(k+1)Q_i(k) \geq 0, \qquad (3.67)$$

where $E(k)$, $k \geq k_0$, is the solution to (3.2) with initial condition $E(k_0) = E_0$. Furthermore, equalities hold in (3.66) and (3.67) if and only if $\Delta E_i = 0$ and $\beta_i E_i = \beta_j E_j$, $i, j = 1, \ldots, q$, $i \neq j$.

Proof. The proof is identical to the proofs of Propositions 3.2 and 3.4. $\square$

Note that with the modified Axiom $i)$ the isolated discrete-time large-scale dynamical system $\mathcal{G}$ has equilibrium energy states given by $E_{\rm e} = \alpha \boldsymbol{p}$, for $\alpha \geq 0$, where $\boldsymbol{p} \triangleq [1/\beta_1, \ldots, 1/\beta_q]^{\rm T}$. As in Section 3.3, we define an equilibrium process as a process where the trajectory of the system $\mathcal{G}$ stays at the equilibrium point of the isolated system $\mathcal{G}$ and a nonequilibrium process as a process that is not an equilibrium one. Thus, it follows from Axioms $i) - iii)$ that inequalities (3.66) and (3.67) are satisfied as equalities for an equilibrium process and as strict inequalities for a nonequilibrium process.

Definition 3.5. For the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2), a function $\mathcal{S} : \overline{\mathbb{R}}_+^q \to \mathbb{R}$ satisfying

$$\mathcal{S}(E(k_2)) \geq \mathcal{S}(E(k_1)) + \sum_{k=k_1}^{k_2-1}\sum_{i=1}^{q}\frac{S_i(k)-\sigma_{ii}(E(k))}{c+\beta_i E_i(k+1)}, \qquad (3.68)$$

for any $k_2 \geq k_1 \geq k_0$ and $S(\cdot) \in \mathcal{U}$, is called the *entropy* of $\mathcal{G}$.

Definition 3.6. For the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2), a function $\mathcal{E} : \overline{\mathbb{R}}_+^q \to \mathbb{R}$ satisfying

$$\mathcal{E}(E(k_2)) \leq \mathcal{E}(E(k_1)) + \sum_{k=k_1}^{k_2-1}\sum_{i=1}^{q}\beta_i E_i(k+1)[S_i(k)-\sigma_{ii}(E(k))], \qquad (3.69)$$

for any $k_2 \geq k_1 \geq k_0$ and $S(\cdot) \in \mathcal{U}$, is called the *ectropy* of $\mathcal{G}$.

For the next result define the available entropy and available ectropy of the large-scale dynamical system $\mathcal{G}$ by

$$\mathcal{S}_{\mathrm{a}}(E_0) \triangleq -\sup_{S(\cdot)\in\mathcal{U}_{\mathrm{c}},\, K\geq k_0} \sum_{k=k_0}^{K-1}\sum_{i=1}^{q} \frac{S_i(k)-\sigma_{ii}(E(k))}{c+\beta_i E_i(k+1)}, \tag{3.70}$$

$$\mathcal{E}_{\mathrm{a}}(E_0) \triangleq -\inf_{S(\cdot)\in\mathcal{U}_{\mathrm{c}},\, K\geq k_0} \sum_{k=k_0}^{K-1}\sum_{i=1}^{q} \beta_i E_i(k+1)[S_i(k)-\sigma_{ii}(E(k))], \tag{3.71}$$

where $E(k_0)=E_0\in\overline{\mathbb{R}}_+^q$ and $E(K)=0$, and define the required entropy supply and required ectropy supply of the large-scale dynamical system $\mathcal{G}$ by

$$\mathcal{S}_{\mathrm{r}}(E_0) \triangleq \sup_{S(\cdot)\in\mathcal{U}_{\mathrm{r}},\, K\geq -k_0+1} \sum_{k=-K}^{k_0-1}\sum_{i=1}^{q} \frac{S_i(k)-\sigma_{ii}(E(k))}{c+\beta_i E_i(k+1)}, \tag{3.72}$$

$$\mathcal{E}_{\mathrm{r}}(E_0) \triangleq \inf_{S(\cdot)\in\mathcal{U}_{\mathrm{r}},\, K\geq -k_0+1} \sum_{k=-K}^{k_0-1}\sum_{i=1}^{q} \beta_i E_i(k+1)[S_i(k)-\sigma_{ii}(E(k))], \tag{3.73}$$

where $E(-K)=0$ and $E(k_0)=E_0\in\overline{\mathbb{R}}_+^q$.

Theorem 3.4. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2) and assume that Axioms *ii*) and *iii*) hold. Then there exists an entropy and an ectropy function for $\mathcal{G}$. Moreover, $\mathcal{S}_{\mathrm{a}}(E)$, $E\in\overline{\mathbb{R}}_+^q$, and $\mathcal{S}_{\mathrm{r}}(E)$, $E\in\overline{\mathbb{R}}_+^q$, are possible entropy functions for $\mathcal{G}$ with $\mathcal{S}_{\mathrm{a}}(0)=\mathcal{S}_{\mathrm{r}}(0)=0$, and $\mathcal{E}_{\mathrm{a}}(E)$, $E\in\overline{\mathbb{R}}_+^q$, and $\mathcal{E}_{\mathrm{r}}(E)$, $E\in\overline{\mathbb{R}}_+^q$, are possible ectropy functions for $\mathcal{G}$ with $\mathcal{E}_{\mathrm{a}}(0)=\mathcal{E}_{\mathrm{r}}(0)=0$. Finally, all entropy functions $\mathcal{S}(E)$, $E\in\overline{\mathbb{R}}_+^q$, for $\mathcal{G}$ satisfy

$$\mathcal{S}_{\mathrm{r}}(E)\leq\mathcal{S}(E)-\mathcal{S}(0)\leq\mathcal{S}_{\mathrm{a}}(E),\quad E\in\overline{\mathbb{R}}_+^q, \tag{3.74}$$

and all ectropy functions $\mathcal{E}(E)$, $E\in\overline{\mathbb{R}}_+^q$, for $\mathcal{G}$ satisfy

$$\mathcal{E}_{\mathrm{a}}(E)\leq\mathcal{E}(E)-\mathcal{E}(0)\leq\mathcal{E}_{\mathrm{r}}(E),\quad E\in\overline{\mathbb{R}}_+^q. \tag{3.75}$$

Proof. The proof is identical to the proofs of Theorems 3.1 and 3.2. □

For the statement of the next result recall the definition of $\boldsymbol{p}=[1/\beta_1,\cdots,1/\beta_q]^{\mathrm{T}}$ and define $P\triangleq\mathrm{diag}[\beta_1,\cdots,\beta_q]$.

Proposition 3.8. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2) and assume that Axioms *i*), *ii*), and *iii*) hold. Then the function $\mathcal{S} : \overline{\mathbb{R}}_+^q \to \mathbb{R}$ given by

$$\mathcal{S}(E) = \boldsymbol{p}^{\mathrm{T}} \mathbf{log}_e(c\mathbf{e} + PE) - \mathbf{e}^{\mathrm{T}} \boldsymbol{p} \log_e c, \quad E \in \overline{\mathbb{R}}_+^q, \tag{3.76}$$

where $\mathbf{log}_e(c\mathbf{e}+PE)$ denotes the vector natural logarithm given by $[\log_e(c+\beta_1 E_1), \ldots, \log_e(c+\beta_q E_q)]^{\mathrm{T}}$, is an entropy function of $\mathcal{G}$. Furthermore, the function $\mathcal{E} : \overline{\mathbb{R}}_+^q \to \mathbb{R}$ given by

$$\mathcal{E}(E) = \tfrac{1}{2} E^{\mathrm{T}} P E, \quad E \in \overline{\mathbb{R}}_+^q, \tag{3.77}$$

is an ectropy function of $\mathcal{G}$.

Proof. The proof is identical to the proofs of Propositions 3.3 and 3.5. $\square$

Remark 3.5. As in Section 3.3, it can be shown that the entropy and ectropy functions for $\mathcal{G}$ defined by (3.76) and (3.77) satisfy, respectively, (3.68) and (3.69) as equalities for an equilibrium process and as strict inequalities for a nonequilibrium process.

Once again, inequality (3.68) is analogous to Clausius' inequality for reversible and irreversible thermodynamics, while inequality (3.69) is an anti Clausius inequality. Moreover, for the ectropy function given by (3.77) inequality (3.69) shows that a thermodynamically consistent large-scale dynamical system model is dissipative with respect to the supply rate $E^{\mathrm{T}} PS$ and with storage function corresponding to the system ectropy $\mathcal{E}(E)$. In addition, if we let $Q_i(k) = S_i(k) - \sigma_{ii}(E(k))$, $i = 1, \ldots, q$, denote the net amount of heat received or dissipated by the ith subsystem of $\mathcal{G}$ at a given time instant at the (shifted) *absolute* ith *subsystem temperature* $T_i \triangleq c + \beta_i E_i$, then it follows from (3.68) that the system entropy varies by an amount

$$\Delta \mathcal{S}(E(k)) \geq \sum_{i=1}^{q} \frac{Q_i(k)}{c + \beta_i E_i(k+1)}, \quad k \geq k_0. \tag{3.78}$$

Finally, note that the nonconservation of entropy and ectropy equations (3.49) and (3.50), respectively, for isolated discrete-time large-scale dynamical systems also hold for the more general definitions of entropy and ectropy given in Definitions 3.5 and 3.6. The following theorem is a generalization of Theorem 3.3.

Theorem 3.5. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2) with $S(k) \equiv 0$ and $d(E) \equiv 0$, and assume that Axioms $i) - iii)$ hold. Then for every $\alpha \geq 0$, $\alpha \boldsymbol{p}$ is a semistable equilibrium state of (3.2). Furthermore, $E(k) \to \frac{1}{\mathbf{e}^{\mathrm{T}}\boldsymbol{p}}\boldsymbol{p}\mathbf{e}^{\mathrm{T}}E(k_0)$ as $k \to \infty$ and $\frac{1}{\mathbf{e}^{\mathrm{T}}\boldsymbol{p}}\boldsymbol{p}\mathbf{e}^{\mathrm{T}}E(k_0)$ is a semistable equilibrium state. Finally, if for some $m \in \{1, \ldots, q\}$, $\sigma_{mm}(E) \geq 0$ and $\sigma_{mm}(E) = 0$ if and only if $E_m = 0$, then the zero solution $E(k) \equiv 0$ to (3.2) is a globally asymptotically stable equilibrium state of (3.2).

Proof. It follows from Axiom $i)$ that $\alpha \boldsymbol{p} \in \overline{\mathbb{R}}_+^q$, $\alpha \geq 0$, is an equilibrium state for (3.2). To show Lyapunov stability of the equilibrium state $\alpha \boldsymbol{p}$ consider the system shifted ectropy $\mathcal{E}_{\mathrm{s}}(E) = \frac{1}{2}(E-\alpha\boldsymbol{p})^{\mathrm{T}}P(E-\alpha\boldsymbol{p})$ as a Lyapunov function candidate. Now, the proof follows as in the proof of Theorem 3.3 by invoking Axioms $i)-iii)$ and noting that $\phi_{ij}(E) = -\phi_{ji}(E)$, $E \in \overline{\mathbb{R}}_+^q$, $i \neq j$, $i, j = 1, \ldots, q$, $P\boldsymbol{p} = \mathbf{e}$, and $\mathbf{e}^{\mathrm{T}}w(E) = \mathbf{e}^{\mathrm{T}}E$, $E \in \overline{\mathbb{R}}_+^q$. Alternatively, in the case where for some $m \in \{1, \ldots, q\}$, $\sigma_{mm}(E) \geq 0$ and $\sigma_{mm}(E) = 0$ if and only if $E_m = 0$, global asymptotic stability of the zero solution $E(k) \equiv 0$ to (3.2) follows from standard Lyapunov arguments using the system ectropy $\mathcal{E}(E) = \frac{1}{2}E^{\mathrm{T}}PE$ as a candidate Lyapunov function. $\square$

It follows from Theorem 3.5 that the steady-state value of the energy in each subsystem $\mathcal{G}_i$ of the isolated discrete-time large-scale dynamical system $\mathcal{G}$ is given by $E_\infty = \frac{1}{\mathbf{e}^{\mathrm{T}}\boldsymbol{p}}\boldsymbol{p}\mathbf{e}^{\mathrm{T}}E(k_0)$ which implies that $E_{i\,\infty} = \frac{1}{\beta_i\mathbf{e}^{\mathrm{T}}\boldsymbol{p}}\mathbf{e}^{\mathrm{T}}E(k_0)$ or, equivalently, $\hat{T}_{i\,\infty} = \beta_i E_{i\,\infty} = \frac{1}{\mathbf{e}^{\mathrm{T}}\boldsymbol{p}}\mathbf{e}^{\mathrm{T}}E(k_0)$. Hence, the steady state temperature of the isolated discrete-time large-scale dynamical system $\mathcal{G}$ given by $\hat{T}_\infty = \frac{1}{\mathbf{e}^{\mathrm{T}}\boldsymbol{p}}\mathbf{e}^{\mathrm{T}}E(k_0)\mathbf{e}$ is uniformly distributed over all the subsystems of $\mathcal{G}$. This phenomenon is known as *temperature equipartition* in which all the system energy is eventually transformed into heat at a uniform temperature and hence all system motion would cease.

Proposition 3.9. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation (3.2), let $\mathcal{E} : \overline{\mathbb{R}}_+^q \to \overline{\mathbb{R}}_+$ and $\mathcal{S} : \overline{\mathbb{R}}_+^q \to \mathbb{R}$ denote the ectropy and entropy of $\mathcal{G}$ and be given by (3.77) and (3.76), respectively, and define $\mathcal{D}_{\rm c} \triangleq \{E \in \overline{\mathbb{R}}_+^q : \mathbf{e}^{\rm T} E = \beta\}$, where $\beta \geq 0$. Then,

$$\arg\min_{E\in\mathcal{D}_{\rm c}}(\mathcal{E}(E)) = \arg\max_{E\in\mathcal{D}_{\rm c}}(\mathcal{S}(E)) = E^* = \frac{\beta}{\mathbf{e}^{\rm T}\boldsymbol{p}}\boldsymbol{p}. \tag{3.79}$$

Furthermore, $\mathcal{E}_{\min} \triangleq \mathcal{E}(E^*) = \frac{1}{2}\frac{\beta^2}{\mathbf{e}^{\rm T}\boldsymbol{p}}$ and $\mathcal{S}_{\max} \triangleq \mathcal{S}(E^*) = \mathbf{e}^{\rm T}\boldsymbol{p}\log_e(c + \frac{\beta}{\mathbf{e}^{\rm T}\boldsymbol{p}}) - \mathbf{e}^{\rm T}\boldsymbol{p}\log_e c$.

Proof. The proof is identical to the proof of Proposition 3.6 and hence is omitted. □

Proposition 3.9 shows that when all the energy of a discrete-time large-scale dynamical system is transformed into heat at a uniform temperature, entropy is a maximum and ectropy is a minimum.

Next, we provide an interpretation of the (steady-state) expressions for entropy and ectropy presented in this section that is consistent with kinetic theory. Specifically, we assume that each subsystem $\mathcal{G}_i$ of the discrete-time large-scale dynamical system $\mathcal{G}$ is a simple system consisting of an ideal gas with rigid walls. Furthermore, we assume that all subsystems $\mathcal{G}_i$ are divided by *diathermal walls* (i.e., walls that permit energy flow) and the overall dynamical system is a closed system, that is, the system is separated from the environment by a rigid adiabatic wall. In this case, $\beta_i = k/n_i$, $i = 1, \ldots, q$, where n_i, $i = 1, \ldots, q$, is the number of molecules in the ith subsystem and $k > 0$ is the *Boltzmann constant* (i.e., gas constant per molecule). Without loss of generality and for simplicity of exposition let $k = 1$. In addition, we assume that the molecules in the ideal gas are hard elastic spheres; that is, there are no forces between the molecules except during collisions and the molecules are not deformed by collisions. Thus, there is no internal potential energy and the system internal energy of the ideal gas is entirely kinetic. Hence, in this case, the temperature of each subsystem $\mathcal{G}_i$ is the average translational kinetic energy per molecule which is consistent with the kinetic theory of ideal gases.

Definition 3.7. For a given isolated discrete-time large-scale dynamical system $\mathcal{G}$ in *thermal equilibrium* define the *equilibrium entropy* of $\mathcal{G}$ by $\mathcal{S}_{\mathrm{e}} = n\log_e(c + \frac{\mathbf{e}^{\mathrm{T}}E_\infty}{n}) - n\log_e c$ and the *equilibrium ectropy* of $\mathcal{G}$ by $\mathcal{E}_{\mathrm{e}} = \frac{1}{2}\frac{(\mathbf{e}^{\mathrm{T}}E_\infty)^2}{n}$, where $\mathbf{e}^{\mathrm{T}}E_\infty$ denotes the total steady-state energy of the discrete-time large-scale dynamical system $\mathcal{G}$ and n denotes the number of molecules in $\mathcal{G}$.

Note that the equilibrium entropy and ectropy in Definition 3.7 is entirely consistent with the equilibrium (maximum) entropy and equilibrium (minimum) ectropy given by Proposition 3.9. Next, assume that each subsystem $\mathcal{G}_i$ is initially in thermal equilibrium. Furthermore, for each subsystem, let E_i and n_i, $i = 1, \ldots, q$, denote the total internal energy and the number of molecules, respectively, in the ith subsystem. Hence, the entropy and ectropy of the ith subsystem are given by $\mathcal{S}_i = n_i\log_e(c + E_i/n_i) - n_i\log_e c$ and $\mathcal{E}_i = \frac{1}{2}\frac{E_i^2}{n_i}$, respectively. Next, note that the entropy and the ectropy of the overall system (after reaching a thermal equilibrium) are given by $\mathcal{S}_{\mathrm{e}} = n\log_e(c + \frac{\mathbf{e}^{\mathrm{T}}E_\infty}{n}) - n\log_e c$ and $\mathcal{E}_{\mathrm{e}} = \frac{1}{2}\frac{(\mathbf{e}^{\mathrm{T}}E_\infty)^2}{n}$. Now, it follows from the convexity of $-\log_e(\cdot)$ and conservation of energy that the entropy of $\mathcal{G}$ at thermal equilibrium is given by

$$\begin{aligned}
\mathcal{S}_{\mathrm{e}} &= n\log_e\left(c + \frac{\mathbf{e}^{\mathrm{T}}E_\infty}{n}\right) - n\log_e c \\
&= n\log_e\left[\sum_{i=1}^{q}\frac{n_i}{n}\left(c + \frac{E_i}{n_i}\right)\right] - \sum_{i=1}^{q} n_i\log_e c \\
&\geq n\sum_{i=1}^{q}\frac{n_i}{n}\log_e\left(c + \frac{E_i}{n_i}\right) - \sum_{i=1}^{q} n_i\log_e c \\
&= \sum_{i=1}^{q}\mathcal{S}_i.
\end{aligned} \tag{3.80}$$

Furthermore, the ectropy of $\mathcal{G}$ at thermal equilibrium is given by

$$\mathcal{E}_{\mathrm{e}} = \frac{1}{2}\frac{(\mathbf{e}^{\mathrm{T}}E_\infty)^2}{n}$$

$$
\begin{aligned}
&= \sum_{i=1}^{q} \frac{1}{2}\frac{E_i^2}{n_i} - \frac{1}{2n}\sum_{i=1}^{q-1}\sum_{j=i+1}^{q} \frac{(n_jE_i - n_iE_j)^2}{n_in_j} \\
&\le \sum_{i=1}^{q} \frac{1}{2}\frac{E_i^2}{n_i} \\
&= \sum_{i=1}^{q} \mathcal{E}_i. \qquad (3.81)
\end{aligned}
$$

It follows from (3.80) (respectively, (3.81)) that the equilibrium entropy (respectively, ectropy) of the system (gas) $\mathcal{G}$ is always greater (respectively, less) than or equal to the sum of entropies (respectively, ectropies) of the individual subsystems $\mathcal{G}_i$. Hence, the entropy (respectively, ectropy) of the gas increases (respectively, decreases) as a more evenly distributed (disordered) state is reached. Finally, note that it follows from (3.80) and (3.81) that $\mathcal{S}_{\mathrm{e}} = \sum_{i=1}^{q} \mathcal{S}_i$ and $\mathcal{E}_{\mathrm{e}} = \sum_{i=1}^{q} \mathcal{E}_i$ if and only if $\frac{E_i}{n_i} = \frac{E_j}{n_j}$, $i \neq j$, $i, j = 1, \ldots, q$; that is, the initial temperatures of all subsystems are equal.

3.9. Thermodynamic Models with Linear Energy Exchange

In this section, we specialize the results of Section 3.3 to the case of large-scale dynamical systems with linear energy exchange between subsystems, that is, $w(E) = WE$ and $d(E) = DE$, where $W \in \mathbb{R}^{q\times q}$ and $D \in \mathbb{R}^{q\times q}$. In this case, the vector form of the energy balance equation (3.2), with $k_0 = 0$, is given by

$$
E(k+1) = WE(k) - DE(k) + S(k), \quad E(0) = E_0, \quad k \ge 0. \qquad (3.82)
$$

Next, let the net energy exchange from the jth subsystem $\mathcal{G}_j$ to the ith subsystem $\mathcal{G}_i$ be parameterized as $\phi_{ij}(E) = \Phi_{ij}^{\mathrm{T}} E$, where $\Phi_{ij} \in \mathbb{R}^q$ and $E \in \overline{\mathbb{R}}_+^q$. In this case, since $w_i(E) = E_i + \sum_{i=1, j\neq i}^{q} \phi_{ij}(E)$, it follows that

$$
W = I_q + \left[\sum_{j=2}^{q} \Phi_{1j}, \ldots, \sum_{j=1, j\neq i}^{q} \Phi_{ij}, \ldots, \sum_{j=1}^{q-1} \Phi_{qj}\right]^{\mathrm{T}}. \qquad (3.83)
$$

Since $\phi_{ij}(E) = -\phi_{ji}(E)$, $i, j = 1, \ldots, q$, $i \neq j$, $E \in \overline{\mathbb{R}}_+^q$, it follows that $\Phi_{ij} = -\Phi_{ji}$, $i \neq j$, $i, j = 1, \ldots, q$. The following proposition considers the special case where W is symmetric.

Proposition 3.10. Consider the large-scale dynamical system $\mathcal{G}$ with energy balance equation given by (3.82) and with $D = 0$. Then Axioms i) and ii) hold if and only if $W = W^{\mathrm{T}}$, $(W - I_q)\mathbf{e} = 0$, rank $(W - I_q) = q - 1$, and W is nonnegative. In addition, if $S = 0$ and Axiom iii) holds, then rank $(W + I_q) = q$ and rank $(W^2 - I_q) = q - 1$.

Proof. Assume Axioms i) and ii) hold. Since, by Axiom ii), $(E_i - E_j)\phi_{ij}(E) \le 0$, $E \in \overline{\mathbb{R}}_+^q$, it follows that $E^{\mathrm{T}}\Phi_{ij}\mathbf{e}_{ij}^{\mathrm{T}}E \le 0$, $i, j = 1, \ldots, q$, $i \ne j$, where $E \in \overline{\mathbb{R}}_+^q$ and $\mathbf{e}_{ij} \in \mathbb{R}^q$ is a vector whose ith entry is 1, jth entry is -1, and remaining entries are zero. Next, it can be shown that $E^{\mathrm{T}}\Phi_{ij}\mathbf{e}_{ij}^{\mathrm{T}}E \le 0$, $E \in \overline{\mathbb{R}}_+^q$, $i \ne j$, $i, j = 1, \ldots, q$, if and only if $\Phi_{ij} \in \mathbb{R}^q$ is such that its ith entry is $-\sigma_{ij}$, its jth entry is σ_{ij}, and its remaining entries are zero, where $\sigma_{ij} \ge 0$. Furthermore, since $\Phi_{ij} = -\Phi_{ji}$, $i \ne j$, $i, j = 1, \ldots, q$, it follows that $\sigma_{ij} = \sigma_{ji}$, $i \ne j$, $i, j = 1, \ldots, q$. Hence, W is given by

$$W_{(i,j)} = \begin{cases} 1 - \sum_{k=1, k\ne j}^{q} \sigma_{kj}, & i = j, \\ \sigma_{ij}, & i \ne j, \end{cases} \tag{3.84}$$

which implies that W is symmetric (since $\sigma_{ij} = \sigma_{ji}$) and $(W - I_q)\mathbf{e} = 0$. Note that since at any given instant of time energy can only be transported or stored but not created and the maximum amount of energy that can be transported cannot exceed the energy in a compartment, it follows that $1 \ge \sum_{k=1, k\ne j}^{q} \sigma_{kj}$. Thus, W is a nonnegative matrix. Now, since by Axiom i), $\phi_{ij}(E) = 0$ if and only if $E_i = E_j$ for all $i, j = 1, \ldots, q$, $i \ne j$, such that $\mathcal{C}_{(i,j)} = 1$, it follows that $\sigma_{ij} > 0$ for all $i, j = 1, \ldots, q$, $i \ne j$, such that $\mathcal{C}_{(i,j)} = 1$. Hence, rank $(W - I_q) =$rank $\mathcal{C} = q - 1$. The converse is immediate and, hence, is omitted.

Next, assume Axiom iii) holds. Since, by Axiom iii), $(E_i(k+1) - E_j(k+1))(E_i(k) - E_j(k)) \ge 0$, $i, j = 1, \ldots, q$, $i \ne j$, $k \ge k_0$, it follows that $E^{\mathrm{T}}(k+1)\mathbf{e}_{ij}\mathbf{e}_{ij}^{\mathrm{T}}E(k) \ge 0$ or, equivalently, $E^{\mathrm{T}}(k)W^{\mathrm{T}}\mathbf{e}_{ij}\mathbf{e}_{ij}^{\mathrm{T}}E(k) \ge 0$, $i, j = 1, \ldots, q$, $i \ne j$, $k \ge k_0$, where $E \in \overline{\mathbb{R}}_+^q$. Next, we show that $I_q + W$ is strictly diagonally dominant. Suppose, *ad absurdum*, that $1 + W_{(i,i)} \le \sum_{l=1, l\ne i}^{q} W_{(i,l)}$ for some i, $1 \le i \le q$. Let $E(k_0) = \mathbf{e}_i$, $i = 1, \ldots, q$, where $\mathbf{e}_i \in \overline{\mathbb{R}}_+^q$ is a vector whose ith entry is 1 and remaining entries are zero. Then,

$$E^{\mathrm{T}}(k_0)W^{\mathrm{T}}\mathbf{e}_{ij}\mathbf{e}_{ij}^{\mathrm{T}}E(k_0) = \mathbf{e}_i^{\mathrm{T}}W^{\mathrm{T}}\mathbf{e}_{ij}\mathbf{e}_{ij}^{\mathrm{T}}\mathbf{e}_i$$

$$
\begin{aligned}
&= W_{(i,i)} - W_{(i,j)} \\
&= 1 - \sum_{k=1, k\neq j}^{q} \sigma_{kj} - \sigma_{ij} \\
&\geq 0, \quad i,j = 1,\ldots,q, \quad i \neq j.
\end{aligned} \tag{3.85}
$$

Now, it follows from (3.85) that

$$
1 + W_{(i,j)} \leq 1 + W_{(i,i)} \leq \sum_{l=1, l\neq i}^{q} W_{(i,l)}, \quad j = 1,\ldots,q, \quad j \neq i, \quad 1 \leq i \leq q, \tag{3.86}
$$

or, equivalently,

$$
1 \leq \sum_{l=1, l\neq i, l\neq j}^{q} W_{(i,l)}, \quad j = 1,\ldots,q, \quad j \neq i, \quad 1 \leq i \leq q. \tag{3.87}
$$

However, since W is compartmental and symmetric, it follows that

$$
\sum_{l=1, l\neq i}^{q} W_{(i,l)} = \sum_{l=1, l\neq i}^{q} W_{(l,i)} = \sum_{l=1, l\neq i}^{q} \sigma_{l,i} \leq 1, \quad i = 1,\ldots,q. \tag{3.88}
$$

Now, since $W_{(i,j)} = \sigma_{ij} > 0$ for all $i,j = 1,\ldots,q$, $i \neq j$, it follows that

$$
\sum_{l=1, l\neq i, l\neq j}^{q} W_{(i,l)} < \sum_{l=1, l\neq i}^{q} W_{(i,l)} \leq 1, \quad i = 1,\ldots,q, \tag{3.89}
$$

which contradicts (3.87).

Next, since $I_q + W$ is strictly diagonally dominant it follows from Theorem 6.1.10 of [122] that rank $(I_q + W) = q$. Furthermore, since rank $(W^2 - I_q) =$ rank $(W + I_q)(W - I_q)$, it follows from Sylvester's inequality that

$$
\begin{aligned}
\operatorname{rank}(W + I_q) + \operatorname{rank}(W - I_q) - q &\leq \operatorname{rank}(W^2 - I_q) \\
&\leq \min\{\operatorname{rank}(W + I_q), \operatorname{rank}(W - I_q)\}.
\end{aligned} \tag{3.90}
$$

Now, rank $(W^2 - I_q) = q - 1$ follows from (3.90) by noting that rank $(W - I_q) = q - 1$ and rank $(W + I_q) = q$. □

Next, we specialize the energy balance equation (3.82) to the case where $D = \mathrm{diag}[\sigma_{11}, \sigma_{22}, \ldots, \sigma_{qq}]$. In this case, the vector form of the energy balance equation (3.2), with $k_0 = 0$, is given by

$$E(k+1) = AE(k) + S(k), \quad E(0) = E_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{3.91}$$

where $A \triangleq W - D$ is such that

$$A_{(i,j)} = \begin{cases} 1 - \sum_{k=1}^{q} \sigma_{kj}, & i = j, \\ \sigma_{ij}, & i \neq j. \end{cases} \tag{3.92}$$

Note that (3.92) implies $\sum_{i=1}^{q} A_{(i,j)} = 1 - \sigma_{ii} \leq 1$, $j = 1, \ldots, q$, and hence, A is a Lyapunov stable compartmental matrix. If $\sigma_{ii} > 0$, $i = 1, \ldots, q$, then A is an asymptotically stable compartmental matrix.

An important special case of (3.91) is the case where A is symmetric or, equivalently, $\sigma_{ij} = \sigma_{ji}$, $i \neq j$, $i, j = 1, \ldots, q$. In this case, it follows from (3.91) that for each subsystem the energy balance equation satisfies

$$\Delta E_i(k) + \sigma_{ii} E_i(k) + \sum_{j=1,\, j\neq i}^{q} \sigma_{ij}[E_i(k) - E_j(k)] = S_i(k), \quad k \in \overline{\mathbb{Z}}_+. \tag{3.93}$$

Note that $\phi_i(E) \triangleq \sum_{j=1,\, j\neq i}^{q} \sigma_{ij}(E_i - E_j)$, $i = 1, \ldots, q$, represents the energy exchange from the ith subsystem to all other subsystems and is given by the sum of the individual energy exchanges from the ith subsystem to the jth subsystem. Furthermore, these energy exchanges are proportional to the energy differences of the subsystems, that is, $E_i - E_j$. Hence, (3.93) is an energy balance equation that governs the energy exchange among coupled subsystems and is completely analogous to the equations of thermal transfer with subsystem energies playing the role of temperatures. Furthermore, note that since $\sigma_{ij} \geq 0$, $i, j = 1, \ldots, q$, energy is exchanged from more energetic subsystems to less energetic subsystems, which is consistent with the second law of thermodynamics which requires that heat (energy) *must* flow in the direction of lower temperatures.

The next lemma and proposition are needed for developing expressions for steady-state energy distributions of the discrete-time large-scale dynamical system $\mathcal{G}$ with linear energy balance equation (3.91).

Lemma 3.1. Let $A \in \mathbb{R}^{q\times q}$ be compartmental and let $S \in \mathbb{R}^q$. Then the following properties hold:

i) $I_q - A$ is an M-matrix.

ii) $|\lambda| \leq 1$, $\lambda \in \operatorname{spec}(A)$.

iii) If A is semistable and $\lambda \in \operatorname{spec}(A)$, then either $|\lambda| < 1$ or $\lambda = 1$ and $\lambda = 1$ is semisimple.

iv) $\operatorname{ind}(I_q - A) \leq 1$ and $\operatorname{ind}(A) \leq 1$.

v) If A is semistable, then $\lim_{k\to\infty} A^k = I_q - (A - I_q)(A - I_q)^{\#} \geq\geq 0$.

vi) $\mathcal{R}(A - I_q) = \mathcal{N}(I_q - (A - I_q)(A - I_q)^{\#})$ and $\mathcal{N}(A - I_q) = \mathcal{R}(I_q - (A - I_q)(A - I_q)^{\#})$.

vii) $\sum_{i=0}^{k} A^i = (A - I_q)^{\#}(A^{k+1} - I_q) + (k + 1)[I_q - (A - I_q)(A - I_q)^{\#}]$, $k \in \overline{\mathbb{Z}}_+$.

viii) If A is semistable, then $\sum_{i=0}^{\infty} A^i S$ exists if and only if $S \in \mathcal{R}(A - I_q)$, where $S \in \mathbb{R}^q$.

ix) If A is semistable and $S \in \mathcal{R}(A - I_q)$, then $\sum_{i=0}^{\infty} A^i S = -(A - I_q)^{\#}S$.

x) If A is semistable, $S \in \mathcal{R}(A - I_q)$, and $S \geq\geq 0$, then $-(A - I_q)^{\#}S \geq\geq 0$.

xi) $A - I_q$ is nonsingular if and only if $I_q - A$ is a nonsingular M-matrix.

xii) If A is semistable and $A - I_q$ is nonsingular, then A is asymptotically stable and $(I_q - A)^{-1} \geq\geq 0$.

Proof. *i)* Note that

$$A^{\mathrm{T}}\mathbf{e} = \left[-(1 - \sum_{i=1}^{q} A_{(i,1)}), -(1 - \sum_{i=1}^{q} A_{(i,2)}), \ldots, -(1 - \sum_{i=1}^{q} A_{(i,q)})\right]^{\mathrm{T}} + \mathbf{e}. \qquad (3.94)$$

Then $(I_q - A)^{\mathrm{T}}\mathbf{e} \geq\geq 0$ and $I_q - A$ is a Z-matrix. It follows from Theorem 1 of [20] that $(I_q - A)^{\mathrm{T}}$, and hence, $I_q - A$ is an M-matrix.

ii) The result follows from *i*) and Lemma 1 of [97].

iii) The result follows from Theorem 2 of [97].

iv) Since $(I_q - A)^{\mathrm{T}}\mathbf{e} \geq\geq 0$ it follows that $I_q - A$ is an M-matrix and has "property c" (See [19]). Hence, it follows from Lemma 4.11 of [19] that $I_q - A$ has "property c" if and only if $\mathrm{ind}(I_q - A) \leq 1$. Next, since $\mathrm{ind}(I_q - A) \leq 1$, it follows from the real Jordan decomposition that there exist invertible matrices $J \in \mathbb{R}^{r\times r}$, where $r = \mathrm{rank}(I_q - A)$, and $U \in \mathbb{R}^{q\times q}$ such that J is diagonal and

$$I_q - A = U \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} U^{-1}, \tag{3.95}$$

which implies

$$A = U \begin{bmatrix} I_r - J & 0 \\ 0 & I_{q-r} \end{bmatrix} U^{-1}. \tag{3.96}$$

Hence, $\mathrm{ind}(A) \leq 1$.

v) The result follows from Theorem 2 of [97].

vi) Let $x \in \mathcal{R}(A - I_q)$, that is, there exists $y \in \mathbb{R}^q$ such that $x = (A - I_q)y$. Now, $(I_q - (A-I_q)(A-I_q)^{\#})x = x - (A-I_q)(A-I_q)^{\#}(A-I_q)y = x - (A-I_q)y = 0$, which implies that $\mathcal{R}(A-I_q) \subseteq \mathcal{N}(I_q - (A-I_q)(A-I_q)^{\#})$. Conversely, let $x \in \mathcal{N}(I_q - (A-I_q)(A-I_q)^{\#})$. Hence, $(I_q - (A-I_q)(A-I_q)^{\#})x = 0$, or, equivalently, $x = (A-I_q)(A-I_q)^{\#}x$, which implies that $x \in \mathcal{R}(A - I_q)$, and hence, $\mathcal{R}(A - I_q) = \mathcal{N}(I_q - (A - I_q)(A - I_q)^{\#})$. The equality $\mathcal{N}(A - I_q) = \mathcal{R}(I_q - (A - I_q)(A - I_q)^{\#})$ can be proved in an analogous manner.

vii) Note since $A = U \begin{bmatrix} I_r - J & 0 \\ 0 & I_{q-r} \end{bmatrix} U^{-1}$ and J is invertible it follows that

$$\begin{aligned}
\sum_{i=0}^{k} A^i &= \sum_{i=0}^{k} U \begin{bmatrix} (I_r - J)^i & 0 \\ 0 & I_{q-r} \end{bmatrix} U^{-1} \\
&= U \begin{bmatrix} \sum_{i=0}^{k}(I_r - J)^i & 0 \\ 0 & (k+1)I_{q-r} \end{bmatrix} U^{-1} \\
&= U \begin{bmatrix} -J^{-1}[(I_r - J)^{k+1} - I_r] & 0 \\ 0 & (k+1)I_{q-r} \end{bmatrix} U^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= U\begin{bmatrix} -J^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^{-1}U\begin{bmatrix} (I_r-J)^{k+1}-I_r & 0 \\ 0 & 0 \end{bmatrix}U^{-1} + U\begin{bmatrix} 0 & 0 \\ 0 & (k+1)I_{q-r} \end{bmatrix}U^{-1} \\
&= (A-I_q)^{\#}(A^{k+1}-I_q) \\
&\quad +(k+1)\Big(I_q - U\begin{bmatrix} J-I_r & 0 \\ 0 & 0 \end{bmatrix}U^{-1}U\begin{bmatrix} (J-I_r)^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^{-1}\Big) \\
&= (A-I_q)^{\#}(A^{k+1}-I_q) + (k+1)[I_q-(A-I_q)(A-I_q)^{\#}], \quad k \in \overline{\mathbb{Z}}_+.
\end{aligned} \tag{3.97}
$$

viii) The result is a direct consequence of *v*)–*vii*).

ix) The result follows from *v*) and *vii*).

x) The result follows from *ix*).

xi) The result follows from *i*).

xii) Asymptotic stability of A is a direct consequence of *iii*). $(I_q - A)^{-1} \geq\geq 0$ follows from Lemma 1 of [97]. □

Proposition 3.11 [97]. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation given by (3.91). Suppose $E_0 \geq\geq 0$, and $S(k) \geq\geq 0$, $k \in \overline{\mathbb{Z}}_+$. Then the solution $E(k)$, $k \in \overline{\mathbb{Z}}_+$, to (3.91) is nonnegative for all $k \in \overline{\mathbb{Z}}_+$ if and only if A is nonnegative.

Next, we develop expressions for the steady-state energy distribution for a discrete-time large-scale linear dynamical system $\mathcal{G}$ for the cases where A is semistable, and the supplied system energy $S(k)$ is a periodic function with period $\tau \in \overline{\mathbb{Z}}_+$, $\tau > 0$, that is, $S(k+\tau) = S(k)$, $k \in \overline{\mathbb{Z}}_+$, and $S(k)$ is constant, that is, $S(k) \equiv S$. Define $e(k) \triangleq E(k) - E(k+\tau)$, $k \in \overline{\mathbb{Z}}_+$, and note that

$$
e(k+1) = Ae(k), \quad e(0) = E(0) - E(\tau), \quad k \in \overline{\mathbb{Z}}_+. \tag{3.98}
$$

Hence, since

$$
e(k) = A^k[E(0) - E(\tau)], \quad k \in \overline{\mathbb{Z}}_+, \tag{3.99}
$$

and A is semistable, it follows from v) of Lemma 3.1 that

$$\lim_{k\to\infty} e(k) = \lim_{k\to\infty} [E(k) - E(k+\tau)] = [I_q - (A-I_q)(A-I_q)^{\#}][E(0) - E(\tau)], \quad (3.100)$$

which represents a constant offset to the steady-state error energy distribution in the discrete-time large-scale nonlinear dynamical system $\mathcal{G}$. For the case where $S(k) \equiv S$, $\tau \to \infty$ and hence the following result is immediate.

Proposition 3.12. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation given by (3.91). Suppose that A is semistable, $E_0 \geq\geq 0$, and $S(k) \equiv S \geq\geq 0$. Then $E_\infty \triangleq \lim_{k\to\infty} E(k)$ exists if and only if $S \in \mathcal{R}(A - I_q)$. In this case,

$$E_\infty = [I_q - (A-I_q)(A-I_q)^{\#}]E_0 - (A-I_q)^{\#}S \quad (3.101)$$

and $E_\infty \geq\geq 0$. If, in addition, $A - I_q$ is nonsingular, then E_∞ exists for all $S \geq\geq 0$ and is given by

$$E_\infty = (I_q - A)^{-1}S. \quad (3.102)$$

Proof. Note that it follows from Lagrange's formula that the solution $E(k)$, $k \in \overline{\mathbb{Z}}_+$, to (3.91) is given by

$$E(k) = A^k E_0 + \sum_{i=0}^{k-1} A^{(k-1-i)} S(i), \quad k \in \overline{\mathbb{Z}}_+. \quad (3.103)$$

Now, the result is a direct consequence of Proposition 3.11 and v), $viii$), ix), and x) of Lemma 3.1. $\square$

Next, we specialize the result of Proposition 3.12 to the case where there is no energy dissipation from each subsystem $\mathcal{G}_i$ of $\mathcal{G}$, that is, $\sigma_{ii} = 0$, $i = 1, \ldots, q$. Note that in this case $\mathbf{e}^{\mathrm{T}}(A - I_q) = 0$, and hence, $\operatorname{rank}(A - I_q) \leq q - 1$. Furthermore, if $S = 0$ it follows from (3.91) that $\mathbf{e}^{\mathrm{T}}\Delta E(k) = \mathbf{e}^{\mathrm{T}}(A - I_q)E(k) = 0$, $k \in \overline{\mathbb{Z}}_+$, and hence, the total energy of the isolated discrete-time large-scale nonlinear dynamical system $\mathcal{G}$ is conserved.

Proposition 3.13. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation given by (3.91). Assume $\operatorname{rank}(A - I_q) = \operatorname{rank}(A^2 - I_q) = q - 1$, $\sigma_{ii} = 0$, $i = 1, \ldots, q$, and $A = A^{\mathrm{T}}$. If $E_0 \geq\geq 0$, and $S = 0$, then the equilibrium state $\alpha\mathbf{e}$, $\alpha \geq 0$, of the isolated system $\mathcal{G}$ is semistable and the steady-state energy distribution E_∞ of the isolated discrete-time large-scale dynamical system $\mathcal{G}$ is given by

$$E_\infty = \left[\frac{1}{q}\sum_{i=1}^{q} E_{i0}\right]\mathbf{e}. \tag{3.104}$$

If, in addition, for some $m \in \{1, \ldots, q\}$, $\sigma_{mm} > 0$, then the zero solution $E(k) \equiv 0$ to (3.91) is globally asymptotically stable.

Proof. Note that since $\mathbf{e}^{\mathrm{T}}(A - I_q) = 0$ it follows from (3.91) with $S(k) \equiv 0$ that $\mathbf{e}^{\mathrm{T}}\Delta E(k) = 0$, $k \geq 0$, and hence $\mathbf{e}^{\mathrm{T}}E(k) = \mathbf{e}^{\mathrm{T}}E_0$, $k \geq 0$. Furthermore, since by Proposition 3.11 the solution $E(k)$, $k \geq k_0$, to (3.91) is nonnegative, it follows that $0 \leq E_i(k) \leq \mathbf{e}^{\mathrm{T}}E(k) = \mathbf{e}^{\mathrm{T}}E_0$, $k \geq 0$, $i = 1, ..., q$. Hence, the solution $E(k)$, $k \geq 0$, to (3.91) is bounded for all $E_0 \in \overline{\mathbb{R}}_+^q$. Next, note that $\phi_{ij}(E) = \sigma_{ij}(E_j - E_i)$ and $(E_i - E_j)\phi_{ij}(E) = -\sigma_{ij}(E_i - E_j)^2 \leq 0$, $E \in \overline{\mathbb{R}}_+^q$, $i \neq j$, $i, j = 1, ..., q$, which implies that Axioms i) and ii) are satisfied. Thus, $E = \alpha\mathbf{e}$, $\alpha \geq 0$, is the equilibrium state of the isolated large-scale dynamical system $\mathcal{G}$. Furthermore, define the Lyapunov function candidate $\mathcal{E}_{\mathrm{s}}(E) = \frac{1}{2}(E - \alpha\mathbf{e})^{\mathrm{T}}(E - \alpha\mathbf{e})$, $E \in \overline{\mathbb{R}}_+^q$. Since A is compartmental and symmetric, it follows from ii) of Lemma 3.1 that

$$\begin{aligned}
\Delta\mathcal{E}_{\mathrm{s}}(E) &= \tfrac{1}{2}(AE - \alpha\mathbf{e})^{\mathrm{T}}(AE - \alpha\mathbf{e}) - \tfrac{1}{2}(E - \alpha\mathbf{e})^{\mathrm{T}}(E - \alpha\mathbf{e}) \\
&= \tfrac{1}{2}E^{\mathrm{T}}(A^2 - I_q)E \\
&\leq 0,
\end{aligned} \tag{3.105}$$

which implies Lyapunov stability of the equilibrium state $\alpha\mathbf{e}$, $\alpha \geq 0$.

Next, consider the set $\mathcal{R} \triangleq \{E \in \overline{\mathbb{R}}_+^q : \Delta\mathcal{E}_{\mathrm{s}}(E) = 0\} = \{E \in \overline{\mathbb{R}}_+^q : E^{\mathrm{T}}(A^2 - I_q)E = 0\}$. Since A is compartmental and symmetric it follows from ii) of Lemma 3.1 that $A^2 - I_q$ is a negative semi-definite matrix, and hence, $E^{\mathrm{T}}(A^2 - I_q)E = 0$ if and only if $(A^2 - I_q)E = 0$.

Furthermore, since, by assumption, $\operatorname{rank}(A - I_q) = \operatorname{rank}(A^2 - I_q) = q - 1$, it follows that there exists one and only one linearly independent solution to $(A^2 - I_q)E = 0$ given by $E = \mathbf{e}$. Hence, $\mathcal{R} = \{E \in \overline{\mathbb{R}}_+^q : E = \alpha\mathbf{e}, \alpha \geq 0\}$. Since $\mathcal{R}$ consists of only equilibrium states of (3.91) it follows that $\mathcal{M} = \mathcal{R}$, where $\mathcal{M}$ is the largest invariant set contained in $\mathcal{R}$. Hence, for every $E_0 \in \overline{\mathbb{R}}_+^q$, it follows from the Krasovskii-LaSalle invariant set theorem that $E(k) \to \alpha\mathbf{e}$ as $k \to \infty$ for some $\alpha \geq 0$ and, hence, $\alpha\mathbf{e}$, $\alpha \geq 0$, is a semistable equilibrium state of (3.91). Furthermore, since the energy is conserved in the isolated large-scale dynamical system $\mathcal{G}$ it follows that $q\alpha = \mathbf{e}^{\mathrm{T}}E_0$. Thus, $\alpha = \frac{1}{q}\sum_{i=1}^{q} E_{i0}$, which implies (3.104).

Finally, to show that in case where $\sigma_{mm} > 0$ for some $m \in \{1, \ldots, q\}$, the zero solution $E(k) \equiv 0$ to (3.91) is globally asymptotically stable, consider the system ectropy $\mathcal{E}(E) = \frac{1}{2}E^{\mathrm{T}}E$, $E \in \overline{\mathbb{R}}_+^q$, as a candidate Lyapunov function. Note that Lyapunov stability of the zero equilibrium state follows from the previous analysis with $\alpha = 0$. Next, note that

$$\begin{aligned}
\Delta\mathcal{E}(E) &= \tfrac{1}{2}E^{\mathrm{T}}(A^2 - I_q)E \\
&= \tfrac{1}{2}E^{\mathrm{T}}[(W - D)^2 - I_q]E \\
&= \tfrac{1}{2}E^{\mathrm{T}}(W^2 - I_q)E - \tfrac{1}{2}E^{\mathrm{T}}(WD + DW - D^2)E \\
&= \tfrac{1}{2}E^{\mathrm{T}}(W^2 - I_q)E - \textstyle\sum_{i=1, i\neq m}^{q} \sigma_{mm}\sigma_{mi}E_mE_i \\
&\quad -\sigma_{mm}(W_{(m,m)} - \sigma_{mm})E_m^2 - \tfrac{1}{2}\sigma_{mm}^2E_m^2, \quad E \in \overline{\mathbb{R}}_+^q.
\end{aligned} \tag{3.106}$$

Consider the set $\mathcal{R} \triangleq \{E \in \overline{\mathbb{R}}_+^q : \Delta\mathcal{E}(E) = 0\} = \{E \in \overline{\mathbb{R}}_+^q : E_1 = \cdots = E_q\} \cap \{E \in \overline{\mathbb{R}}_+^q : E_m = 0, m \in \{1, \ldots, q\}\} = \{0\}$. Hence, the largest invariant set contained in $\mathcal{R}$ is given by $\mathcal{M} = \mathcal{R} = \{0\}$, and thus, it follows from the Krasovskii-LaSalle invariant set theorem that the zero solution $E(k) \equiv 0$ to (3.91) is globally asymptotically stable. $\square$

Finally, we examine the steady-state energy distribution for large-scale nonlinear dynamical systems $\mathcal{G}$ in case of strong coupling between subsystems, that is, $\sigma_{ij} \to \infty$, $i \neq j$. For this analysis we assume that A given by (3.91) is symmetric, that is, $\sigma_{ij} = \sigma_{ji}$, $i \neq j$, $i, j = 1, \ldots, q$, and $\sigma_{ii} > 0$, $i = 1, \ldots, q$. Thus, $I_q - A$ is a nonsingular M-matrix for all values of

σ_{ij}, $i \neq j$, $i, j = 1, \ldots, q$. Moreover, in this case it follows that if $\frac{\sigma_{ij}}{\sigma_{kl}} \to 1$ as $\sigma_{ij} \to \infty$, $i \neq j$, and $\sigma_{kl} \to \infty$, $k \neq l$, then

$$\lim_{\sigma_{ij} \to \infty, i \neq j} (I_q - A)^{-1} = \lim_{\sigma \to \infty} [D - \sigma(-qI_q + \mathbf{e}\mathbf{e}^{\mathrm{T}})]^{-1}, \tag{3.107}$$

where $D = \mathrm{diag}[\sigma_{11}, \ldots, \sigma_{qq}] > 0$. The following lemmas are needed for the next result.

Lemma 3.2. Let $Y \in \mathbb{R}^{q \times q}$ be such that $\mathrm{ind}\,(Y) \leq 1$. Then $\lim_{\sigma \to \infty}(I_q - \sigma Y)^{-1} = I_q - Y^{\#}Y$.

Proof. Note that

$$\begin{aligned} (I_q - \sigma Y)^{-1} &= I_q + \sigma(I_q - \sigma Y)^{-1}Y \\ &= I_q + \left(\frac{1}{\sigma}I_q - Y\right)^{-1} Y \\ &= I_q - \left(Y - \frac{1}{\sigma}I_q\right)^{-1} Y. \end{aligned} \tag{3.108}$$

Now, using the fact that if $A \in \mathbb{R}^{q \times q}$ and $\mathrm{ind}\,A \leq 1$, then

$$\lim_{\alpha \to 0} (A + \alpha I)^{-1} A = AA^{\#} = A^{\#}A, \tag{3.109}$$

it follows that

$$\lim_{\sigma \to \infty} (I_q - \sigma Y)^{-1} = I_q - \lim_{\frac{1}{\sigma} \to 0} \left(Y - \frac{1}{\sigma}I_q\right)^{-1} Y = I_q - Y^{\#}Y, \tag{3.110}$$

which proves the result. □

Lemma 3.3. Let $D \in \mathbb{R}^{q \times q}$ and $X \in \mathbb{R}^{q \times q}$ be such that $D > 0$ and $X = -qI_q + \mathbf{e}\mathbf{e}^{\mathrm{T}}$. Then

$$I_q - Y^{\#}Y = \frac{D^{\frac{1}{2}}\mathbf{e}\mathbf{e}^{\mathrm{T}}D^{\frac{1}{2}}}{\mathbf{e}^{\mathrm{T}}D\mathbf{e}}, \tag{3.111}$$

where $Y \triangleq D^{-\frac{1}{2}}XD^{-\frac{1}{2}}$.

Proof. Note that

$$Y = D^{-\frac{1}{2}}(-qI_q + \mathbf{e}\mathbf{e}^{\mathrm{T}})D^{-\frac{1}{2}} = -qD^{-1} + D^{-\frac{1}{2}}\mathbf{e}\mathbf{e}^{\mathrm{T}}D^{-\frac{1}{2}}. \tag{3.112}$$

Now, using the fact that if $N \in \mathbb{R}^{q\times q}$ is nonsingular and symmetric and $b \in \mathbb{R}^q$ is a nonzero vector, then

$$(N + bb^{\mathrm{T}})^{+} = \left(I - \frac{1}{b^{\mathrm{T}}N^{-2}b}N^{-1}bb^{\mathrm{T}}N^{-1}\right)N^{-1}\left(I - \frac{1}{b^{\mathrm{T}}N^{-2}b}N^{-1}bb^{\mathrm{T}}N^{-1}\right), \tag{3.113}$$

it follows that

$$-Y^{\#} = \frac{1}{q}\left(I_q - \frac{D^{\frac{1}{2}}\mathbf{e}\mathbf{e}^{\mathrm{T}}D^{\frac{1}{2}}}{\mathbf{e}^{\mathrm{T}}D\mathbf{e}}\right)D\left(I_q - \frac{D^{\frac{1}{2}}\mathbf{e}\mathbf{e}^{\mathrm{T}}D^{\frac{1}{2}}}{\mathbf{e}^{\mathrm{T}}D\mathbf{e}}\right). \tag{3.114}$$

Hence,

$$\begin{aligned}
-Y^{\#}Y &= -\left(I_q - \frac{D^{\frac{1}{2}}\mathbf{e}\mathbf{e}^{\mathrm{T}}D^{\frac{1}{2}}}{\mathbf{e}^{\mathrm{T}}D\mathbf{e}}\right)D\left(I_q - \frac{D^{\frac{1}{2}}\mathbf{e}\mathbf{e}^{\mathrm{T}}D^{\frac{1}{2}}}{\mathbf{e}^{\mathrm{T}}D\mathbf{e}}\right)\left(D^{-1} - \frac{1}{q}D^{-\frac{1}{2}}\mathbf{e}\mathbf{e}^{\mathrm{T}}D^{-\frac{1}{2}}\right) \\
&= -\left(I_q - \frac{D^{\frac{1}{2}}\mathbf{e}\mathbf{e}^{\mathrm{T}}D^{\frac{1}{2}}}{\mathbf{e}^{\mathrm{T}}D\mathbf{e}}\right).
\end{aligned} \tag{3.115}$$

Thus, $I_q - Y^{\#}Y = \frac{D^{\frac{1}{2}}\mathbf{e}\mathbf{e}^{\mathrm{T}}D^{\frac{1}{2}}}{\mathbf{e}^{\mathrm{T}}D\mathbf{e}}$. □

Proposition 3.14. Consider the discrete-time large-scale dynamical system $\mathcal{G}$ with energy balance equation given by (3.91). Let $S(k) \equiv S$, $S \in \mathbb{R}^{q\times q}$, $A \in \mathbb{R}^{q\times q}$ be compartmental and assume A is symmetric, $\sigma_{ii} > 0$, $i = 1, \ldots, q$, and $\frac{\sigma_{ij}}{\sigma_{kl}} \to 1$ as $\sigma_{ij} \to \infty$, $i \neq j$, and $\sigma_{kl} \to \infty$, $k \neq l$. Then the steady-state energy distribution E_∞ of the discrete-time large-scale dynamical system $\mathcal{G}$ is given by

$$E_\infty = \left[\frac{\mathbf{e}^{\mathrm{T}}S}{\sum_{i=1}^{q}\sigma_{ii}}\right]\mathbf{e}. \tag{3.116}$$

Proof. Note that in the case where $\frac{\sigma_{ij}}{\sigma_{kl}} \to 1$ as $\sigma_{ij} \to \infty$, $i \neq j$, and $\sigma_{kl} \to \infty$, $k \neq l$, it follows that the corresponding limit of $(I_q - A)^{-1}$ can be equivalently taken as in (3.107). Next, with $D = \mathrm{diag}[\sigma_{11}, \ldots, \sigma_{qq}]$ and $X = -qI_q + \mathbf{e}\mathbf{e}^{\mathrm{T}}$, it follows that $I_q - A = D - \sigma X = D^{\frac{1}{2}}(I_q - \sigma D^{-\frac{1}{2}}XD^{-\frac{1}{2}})D^{\frac{1}{2}}$. Now, it follows from Lemmas 3.2 and 3.3 that

$$E_\infty = \lim_{\sigma_{ij}\to\infty,\, i\neq j}(I_q - A)^{-1}S = \frac{\mathbf{e}\mathbf{e}^{\mathrm{T}}}{\mathbf{e}^{\mathrm{T}}D\mathbf{e}}S = \left[\frac{\mathbf{e}^{\mathrm{T}}S}{\sum_{i=1}^{q}\sigma_{ii}}\right]\mathbf{e}, \tag{3.117}$$

which proves the result. $\square$

Proposition 3.14 shows that in the limit of strong coupling the steady-state energy distribution E_∞ given by (3.102) becomes

$$E_\infty = \lim_{\sigma_{ij}\to\infty,\, i\neq j} (I_q - A)^{-1} S = \left[\frac{\mathbf{e}^{\mathrm{T}} S}{\sum_{i=1}^{q} \sigma_{ii}}\right] \mathbf{e}, \tag{3.118}$$

which implies energy equipartition.

Chapter 4

Vector Dissipativity Theory for Large-Scale Impulsive Dynamical Systems

4.1. Introduction

Recent technological demands have required the analysis and control design of increasingly complex, large-scale nonlinear dynamical systems. The complexity of modern controlled large-scale dynamical systems is further exacerbated by the use of hierarchial embedded control subsystems within the feedback control system; that is, abstract decision-making units performing logical checks that identity system mode operation and specify the continuous-variable subcontroller to be activated. Such systems typically possess a multiechelon hierarchical *hybrid* decentralized control architecture characterized by continuous-time dynamics at the lower levels of the hierarchy and discrete-time dynamics at the higher levels of the hierarchy (see [5, 179] and the numerous references therein). The lower-level units directly interact with the dynamical system to be controlled while the higher-level units receive information from the lower-level units as inputs and provide (possibly discrete) output commands which serve to coordinate and reconcile the (sometimes competing) actions of the lower-level units. The hierarchical controller organization reduces processor cost and controller complexity by breaking up the processing task into relatively small pieces and decomposing the fast and slow control functions. Typically, the higher-level units perform logical checks that determine system mode operation, while the lower-level units execute continuous-variable commands for a given system mode of operation.

In analyzing hybrid large-scale dynamical systems it is often desirable to treat the overall system as a collection of interconnected subsystems. The behavior of the composite hybrid large-scale system can then be predicted from the behaviors of the individual subsystems and their interconnections. The mathematical description of many of these systems can be characterized by impulsive differential equations [98, 147]. In particular, general hybrid dynamical systems involve an abstract axiomatic definition of a dynamical system involving left-continuous (or right-continuous) flows defined on a completely ordered time set as a mapping between vector spaces satisfying an appropriate set of axioms and include hybrid inputs and hybrid outputs that take their values in appropriate vector spaces [91, 173, 242]. In contrast, impulsive dynamical systems are a subclass of hybrid dynamical systems and consist of three elements; namely, a continuous-time differential equation, which governs the motion of the dynamical system between impulsive events; a difference equation, which governs the way that the system states are instantaneously changed when an impulsive event occurs; and a criterion for determining when the states are to be reset [98, 147].

An approach to analyzing large-scale dynamical systems was introduced by the pioneering work of Šiljak [50] and involves the notion of *connective stability*. In particular, the large-scale dynamical system is decomposed into a collection of subsystems with local dynamics and uncertain interactions. Then, each subsystem is considered independently so that the stability of each subsystem is combined with the interconnection constraints to obtain a *vector Lyapunov function* for the composite large-scale dynamical system guaranteeing connective stability for the overall system. Vector Lyapunov functions were first introduced by Bellman [17] and Matrosov [171] and further developed in [51, 86, 148, 162, 167–169, 174], with [50, 51, 86, 162] exploiting their utility for analyzing large-scale systems. Extensions of vector Lyapunov function theory that include matrix-valued Lyapunov functions for stability analysis of large-scale dynamical systems appear in the monographs by Martynyuk [168, 169]. As noted in Chapter 2, the use of vector Lyapunov functions in large-scale system analysis offers a very flexible framework since each component of the vector Lyapunov function can satisfy less rigid requirements as compared to a single scalar Lyapunov function. Weakening the hypothesis on the Lyapunov function enlarges the class of Lyapunov functions that can be used for analyzing the stability of large-scale dynamical systems. In particular, each component of a vector Lyapunov function need not be positive definite with a negative or even negative-semidefinite derivative. The time derivative of the vector Lyapunov function need only satisfy an element-by-element vector inequality involving a vector field of a certain comparison system.

In light of the fact that energy flow modeling arises naturally in large-scale dynamical systems and vector Lyapunov functions provide a powerful stability analysis framework for these systems, it seems natural that hybrid dissipativity theory [91, 98, 99], on the subsystem level, should play a key role in analyzing large-scale impulsive dynamical systems. Specifically, hybrid dissipativity theory provides a fundamental framework for the analysis and design of impulsive dynamical systems using an input-output description based on system energy[4] related considerations [91, 98]. The hybrid dissipation hypothesis on impulsive dynamical systems results in a fundamental constraint on their dynamic behavior wherein a dissipative impulsive dynamical system can only deliver a fraction of its energy to its surroundings and can only store a fraction of the work done to it. Such conservation laws are prevalent in large-scale impulsive dynamical systems such as aerospace systems, power systems, network systems, telecommunication systems, and transportation systems. Since these systems have numerous input-output properties related to conservation, dissipation, and transport of energy, extending hybrid dissipativity theory to capture conservation and dissipation notions on the subsystem level would provide a natural energy flow model for large-scale impulsive dynamical systems. Aggregating the dissipativity properties of each of the impulsive subsystems by appropriate storage functions and hybrid supply rates would allow us to study the dissipativity properties of the composite large-scale impulsive system using *vector storage functions* and *vector hybrid supply rates*. Furthermore, since vector Lyapunov functions can be viewed as generalizations of composite energy functions for all of the impulsive subsystems, a generalized notion of hybrid dissipativity, namely, *vector hybrid*

[4]Here the notion of energy refers to abstract energy for which a physical system energy interpretation is not necessary.

dissipativity, with appropriate vector storage functions and vector hybrid supply rates, can be used to construct vector Lyapunov functions for nonlinear feedback large-scale impulsive systems by appropriately combining vector storage functions for the forward and feedback large-scale impulsive systems. Finally, as in classical dynamical system theory, vector dissipativity theory can play a fundamental role in addressing robustness, disturbance rejection, stability of feedback interconnections, and optimality for large-scale impulsive dynamical systems.

In this chapter, we develop vector dissipativity notions for large-scale nonlinear impulsive dynamical systems; a notion not previously considered in the literature. In particular, we introduce a generalized definition of dissipativity for large-scale nonlinear impulsive dynamical systems in terms of a *hybrid vector inequality* involving a vector hybrid supply rate, a vector storage function, and an essentially nonnegative, semistable dissipation matrix. Generalized notions of vector available storage and vector required supply are also defined and shown to be element-by-element ordered, nonnegative, and finite. On the impulsive subsystem level, the proposed approach provides an energy flow balance over the continuous-time dynamics and the resetting events in terms of the stored subsystem energy, the supplied subsystem energy, the subsystem energy gained from all other subsystems independent of the subsystem coupling strengths, and the subsystem energy dissipated. Furthermore, for large-scale impulsive dynamical systems decomposed into interconnected impulsive subsystems, dissipativity of the composite impulsive system is shown to be determined from the dissipativity properties of the individual impulsive subsystems and the nature of the interconnections. In addition, we develop extended Kalman-Yakubovich-Popov conditions, in terms of the local impulsive subsystem dynamics and the interconnection constraints, for characterizing vector dissipativeness via vector storage functions for large-scale impulsive dynamical systems. Using the concepts of vector dissipativity and vector storage functions as candidate vector Lyapunov functions, we develop feedback interconnection stability results of large-scale impulsive nonlinear dynamical systems. General stability criteria are given for Lyapunov and asymptotic stability of feedback large-scale impulsive dynamical systems. In the case of vector quadratic supply rates involving net subsystem powers and input-output subsystem energies, these results provide a positivity and small gain theorem for large-scale impulsive systems predicated on vector Lyapunov functions. Finally, it is important to note that vector dissipativity notions were first addressed in [102] in the context of continuous-time, large-scale dynamical systems. However, the results of [102] predominately concentrate on connections between thermodynamic models and large-scale dynamical systems. Kalman-Yakubovich-Popov conditions characterizing vector dissipativeness via vector system storage functions and feedback interconnection stability result for large-scale systems are not addressed in [102].

4.2. Notation and Mathematical Preliminaries

In this section, we introduce notation, several definitions, and some key results needed for analyzing large-scale impulsive dynamical systems. We write $V'(x)$ for the Fréchet derivative of V at x. The following definition introduces the notion of essentially nonnegative matrices.

Definition 4.1 [19, 26, 96]. Let $W \in \mathbb{R}^{q\times q}$. W is *essentially nonnegative* if $W_{(i,j)} \geq 0$, $i, j = 1, \ldots, q$, $i \neq j$, where $W_{(i,j)}$ denotes the (i, j)th entry of W.

The following definition introduces the notion of class $\mathcal{W}$ functions involving *quasimonotone increasing functions*.

Definition 4.2 [50]. A function $w = [w_1, ..., w_q]^{\mathrm{T}} : \mathbb{R}^q \to \mathbb{R}^q$ is of *class* $\mathcal{W}$ if $w_i(r') \leq w_i(r'')$, $i = 1, ..., q$, for all $r', r'' \in \mathbb{R}^q$ such that $r'_j \leq r''_j$, $r'_i = r''_i$, $j = 1, ..., q$, $i \neq j$, where r_i denotes the ith component of r.

If $w(\cdot) \in \mathcal{W}$ we say that w satisfies the *Kamke condition*. Note that if $w(r) = Wr$, where $W \in \mathbb{R}^{q\times q}$, then the function $w(\cdot)$ is of class $\mathcal{W}$ if and only if W is essentially nonnegative.

Furthermore, note that it follows from Definition 4.2 that any scalar ($q = 1$) function $w(r)$ is of class $\mathcal{W}$. The following definition introduces the notion of essentially nonnegative functions [24, 96].

Definition 4.3. Let $w = [w_1, \cdots, w_q]^{\mathrm{T}} : \mathcal{V} \to \mathbb{R}^q$, where $\mathcal{V}$ is an open subset of $\mathbb{R}^q$ that contains $\overline{\mathbb{R}}_+^q$. Then w is *essentially nonnegative* if $w_i(r) \geq 0$ for all $i = 1, \ldots, q$ and $r \in \overline{\mathbb{R}}_+^q$ such that $r_i = 0$.

Note that if $w : \mathbb{R}^q \to \mathbb{R}^q$ is such that $w(\cdot) \in \mathcal{W}$ and $w(0) \geq\geq 0$, then w is essentially nonnegative; the converse however is not generally true. However, if $w(r) = Wr$, where $W \in \mathbb{R}^{q \times q}$ is essentially nonnegative, then $w(\cdot)$ is essentially nonnegative and $w(\cdot) \in \mathcal{W}$.

Proposition 4.1 [24, 96]. Suppose $\overline{\mathbb{R}}_+^q \subset \mathcal{V}$. Then $\overline{\mathbb{R}}_+^q$ is an invariant set with respect to

$$\dot{r}(t) = w(r(t)), \quad r(0) = r_0, \quad t \geq t_0, \tag{4.1}$$

where $r_0 \in \overline{\mathbb{R}}_+^q$, if and only if $w : \mathcal{V} \to \mathbb{R}^q$ is essentially nonnegative.

The following corollary to Proposition 4.1 is immediate.

Corollary 4.1. Let $W \in \mathbb{R}^{q \times q}$. Then W is essentially nonnegative if and only if e^{Wt} is nonnegative for all $t \geq 0$.

It follows from Proposition 4.1 that if $r_0 \geq\geq 0$, then $r(t) \geq\geq 0$, $t \geq t_0$, if and only if $w(\cdot)$ is essentially nonnegative. In this case, the usual stability definitions for the equilibrium solution $r(t) \equiv r_{\mathrm{e}}$ to (4.1) are not valid. In particular, stability notions need to be defined with respect to relatively open subsets of $\overline{\mathbb{R}}_+^q$ containing r_{e} [100, 102]. The following lemma is needed for developing several of the results in later sections. For the statement of this lemma recall that a matrix $W \in \mathbb{R}^{q \times q}$ is *semistable* if and only if $\lim_{t\to\infty} e^{Wt}$ exists [26, 96] while W is *asymptotically stable* if and only if $\lim_{t\to\infty} e^{Wt} = 0$.

Lemma 4.1 [100]. Suppose $W \in \mathbb{R}^{q\times q}$ is essentially nonnegative. If W is semistable (respectively, asymptotically stable), then there exist a scalar $\alpha \geq 0$ (respectively, $\alpha > 0$) and a nonnegative vector $p \in \overline{\mathbb{R}}_+^q$, $p \neq 0$, (respectively, positive vector $p \in \mathbb{R}_+^q$) such that

$$W^{\mathrm{T}}p + \alpha p = 0. \tag{4.2}$$

Next, we present a stability result for large-scale impulsive dynamical systems using vector Lyapunov functions. In particular, we consider state-dependent impulsive dynamical systems of the form

$$\dot{x}(t) = F_{\mathrm{c}}(x(t)), \quad x(t_0) = x_0, \quad x(t) \notin \mathcal{Z}_x, \quad t \geq t_0, \tag{4.3}$$

$$\Delta x(t) = F_{\mathrm{d}}(x(t)), \quad x(t) \in \mathcal{Z}_x, \tag{4.4}$$

where $x(t) \in \mathcal{D}$, $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$, $\Delta x(t) \triangleq x(t^+) - x(t)$, $F_{\mathrm{c}} : \mathcal{D} \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $F_{\mathrm{c}}(0) = 0$, $F_{\mathrm{d}} : \mathcal{D} \to \mathbb{R}^n$ is continuous, and $\mathcal{Z}_x \subset \mathcal{D} \subseteq \mathbb{R}^n$ is a resetting set. Here, we assume that (4.3) and (4.4) characterize a large-scale impulsive dynamical system composed of q interconnected subsystems such that, for all $i = 1, ..., q$, each element of $F_{\mathrm{c}}(x)$ and $F_{\mathrm{d}}(x)$ is given by $F_{\mathrm{c}i}(x) = f_{\mathrm{c}i}(x_i) + \mathcal{I}_{\mathrm{c}i}(x)$ and $F_{\mathrm{d}i}(x) = f_{\mathrm{d}i}(x_i) + \mathcal{I}_{\mathrm{d}i}(x)$, respectively, where $f_{\mathrm{c}i} : \mathcal{D}_i \subseteq \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ and $f_{\mathrm{d}i} : \mathcal{D}_i \subseteq \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ define the vector fields of each isolated impulsive subsystem of (4.3) and (4.4), $\mathcal{I}_{\mathrm{c}i} : \mathcal{D} \to \mathbb{R}^{n_i}$ and $\mathcal{I}_{\mathrm{d}i} : \mathcal{D} \to \mathbb{R}^{n_i}$ define the structure of interconnection dynamics of the ith impulsive subsystem with all other impulsive subsystems, $x_i \in \mathcal{D}_i \subseteq \mathbb{R}^{n_i}$, $f_{\mathrm{c}i}(0) = 0$, $\mathcal{I}_{\mathrm{c}i}(0) = 0$, and $\sum_{i=1}^q n_i = n$. For the large-scale impulsive dynamical system (4.3), (4.4) we note that the subsystem states $x_i(t)$, $t \geq t_0$, for all $i = 1, ..., q$, belong to $\mathcal{D}_i \subseteq \mathbb{R}^{n_i}$ as long as $x(t) \triangleq [x_1^{\mathrm{T}}(t), ..., x_q^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathcal{D}$, $t \geq t_0$. We make the following additional assumptions:

A1. If $x(t) \in \overline{\mathcal{Z}}_x \setminus \mathcal{Z}_x$, then there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$, $x(t+\delta) \notin \mathcal{Z}_x$.

A2. If $x \in \mathcal{Z}_x$, then $x + F_{\mathrm{d}}(x) \notin \mathcal{Z}_x$.

Assumption A1 ensures that if a trajectory reaches the closure of $\mathcal{Z}_x$ at a point that does not belong to $\mathcal{Z}_x$, then the trajectory must be directed away from $\mathcal{Z}_x$, that is, a trajectory cannot enter $\mathcal{Z}_x$ through a point that belongs to the closure of $\mathcal{Z}_x$ but not to $\mathcal{Z}_x$. Furthermore, A2 ensures that when a trajectory intersects the resetting set $\mathcal{Z}_x$, it instantaneously exits $\mathcal{Z}_x$. Finally, we note that if $x_0 \in \mathcal{Z}_x$, then the system initially resets to $x_0^+ = x_0 + F_{\mathrm{d}}(x_0) \notin \mathcal{Z}_x$ which serves as the initial condition for the continuous dynamics (4.3). It follows from A1 and A2 that $\partial\mathcal{Z}_x \cap \mathcal{Z}_x$ is closed, and hence, the resetting times $\tau_k(x_0)$ are well defined and distinct. Furthermore, it follows from A2 that if $x^* \in \mathbb{R}^n$ satisfies $F_{\mathrm{d}}(x^*) = 0$, then $x^* \notin \mathcal{Z}_x$. To see this, suppose $x^* \in \mathcal{Z}_x$. Then $x^* + F_{\mathrm{d}}(x^*) = x^* \in \mathcal{Z}_x$, contradicting A2. In particular, we note that $0 \notin \mathcal{Z}_x$. For further insights on Assumptions A1 and A2 the interested reader is referred to [91, 98].

The next theorem presents a stability result for (4.3) and (4.4) via vector Lyapunov functions by relating the stability properties of a *comparison system* to the stability properties of the large-scale impulsive dynamical system.

Theorem 4.1 [147, 175]. Consider the large-scale impulsive dynamical system given by (4.3), (4.4). Suppose there exist a continuously differentiable vector function $V : \mathcal{D} \to \overline{\mathbb{R}}_+^q$ and a positive vector $p \in \mathbb{R}_+^q$ such that $V(0) = 0$, the scalar function $v : \mathcal{D} \to \overline{\mathbb{R}}_+$ defined by $v(x) = p^{\mathrm{T}}V(x)$, $x \in \mathcal{D}$, is such that $v(0) = 0$, $v(x) > 0$, $x \neq 0$, and

$$V'(x)F_{\mathrm{c}}(x) \leq\leq w_{\mathrm{c}}(V(x)), \quad x \notin \mathcal{Z}_x, \tag{4.5}$$

$$V(x + F_{\mathrm{d}}(x)) \leq\leq V(x), \quad x \in \mathcal{Z}_x, \tag{4.6}$$

where $w_{\mathrm{c}} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ is a class $\mathcal{W}$ function such that $w_{\mathrm{c}}(0) = 0$. Then the stability properties of the zero solution $r(t) \equiv 0$ to

$$\dot{r}(t) = w_{\mathrm{c}}(r(t)), \quad r(t_0) = r_0, \quad t \geq t_0, \tag{4.7}$$

imply the corresponding stability properties of the zero solution $x(t) \equiv 0$ to (4.3), (4.4). That is, if the zero solution $r(t) \equiv 0$ to (4.7) is Lyapunov (respectively, asymptotically) stable, then the zero solution $x(t) \equiv 0$ to (4.3), (4.4) is Lyapunov (respectively, asymptotically) stable. If, in addition, $\mathcal{D} = \mathbb{R}^n$ and $V(x) \to \infty$ as $\|x\| \to \infty$, then global asymptotic stability of the zero solution $r(t) \equiv 0$ to (4.7) implies global asymptotic stability of the zero solution $x(t) \equiv 0$ to (4.3), (4.4).

If $V : \mathcal{D} \to \overline{\mathbb{R}}_+^q$ satisfies the conditions of Theorem 4.1 we say that $V(x)$, $x \in \mathcal{D}$, is a *vector Lyapunov function* for the large-scale impulsive dynamical system (4.3) and (4.4). Finally, we recall the standard notions of dissipativity and exponential dissipativity [91, 98] for input/state-dependent impulsive dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f_{\rm c}(x(t)) + G_{\rm c}(x(t))u_{\rm c}(t), \quad x(t_0) = x_0, \quad (x(t), u_{\rm c}(t)) \notin \mathcal{Z}, \tag{4.8}$$

$$\Delta x(t) = f_{\rm d}(x(t)) + G_{\rm d}(x(t))u_{\rm d}(t), \quad (x(t), u_{\rm c}(t)) \in \mathcal{Z}, \tag{4.9}$$

$$y_{\rm c}(t) = h_{\rm c}(x(t)) + J_{\rm c}(x(t))u_{\rm c}(t), \quad (x(t), u_{\rm c}(t)) \notin \mathcal{Z}, \tag{4.10}$$

$$y_{\rm d}(t) = h_{\rm d}(x(t)) + J_{\rm d}(x(t))u_{\rm d}(t), \quad (x(t), u_{\rm c}(t)) \in \mathcal{Z}, \tag{4.11}$$

where $t \geq t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $u_{\rm c}(t) \in \mathcal{U}_{\rm c} \subseteq \mathbb{R}^{m_{\rm c}}$, $u_{\rm d}(t_k) \in \mathcal{U}_{\rm d} \subseteq \mathbb{R}^{m_{\rm d}}$, t_k denotes the kth instant of time at which $(x(t), u_{\rm c}(t))$ intersects $\mathcal{Z} \subset \mathcal{D} \times \mathcal{U}_{\rm c}$ for a particular trajectory $x(t)$ and input $u_{\rm c}(t)$, $y_{\rm c}(t) \in \mathcal{Y}_{\rm c} \subseteq \mathbb{R}^{l_{\rm c}}$, $y_{\rm d}(t_k) \in \mathcal{Y}_{\rm d} \subseteq \mathbb{R}^{l_{\rm d}}$, $f_{\rm c} : \mathcal{D} \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $f_{\rm c}(0) = 0$, $G_{\rm c} : \mathcal{D} \to \mathbb{R}^{n \times m_{\rm c}}$, $f_{\rm d} : \mathcal{D} \to \mathbb{R}^n$ is continuous, $G_{\rm d} : \mathcal{D} \to \mathbb{R}^{n \times m_{\rm d}}$, $h_{\rm c} : \mathcal{D} \to \mathbb{R}^{l_{\rm c}}$ satisfies $h_{\rm c}(0) = 0$, $J_{\rm c} : \mathcal{D} \to \mathbb{R}^{l_{\rm c} \times m_{\rm c}}$, $h_{\rm d} : \mathcal{D} \to \mathbb{R}^{l_{\rm d}}$, and $J_{\rm d} : \mathcal{D} \to \mathbb{R}^{l_{\rm d} \times m_{\rm d}}$. For the impulsive dynamical system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied, that is, $u_{\rm c}(\cdot)$ satisfies sufficient regularity conditions such that (4.8) has a unique solution forward in time.

For the impulsive dynamical system $\mathcal{G}$ given by (4.8)–(4.11) a function $(s_{\rm c}(u_{\rm c}, y_{\rm c}), s_{\rm d}(u_{\rm d}, y_{\rm d}))$, where $s_{\rm c} : \mathcal{U}_{\rm c} \times \mathcal{Y}_{\rm c} \to \mathbb{R}$ and $s_{\rm d} : \mathcal{U}_{\rm d} \times \mathcal{Y}_{\rm d} \to \mathbb{R}$ are such that $s_{\rm c}(0, 0) = 0$ and $s_{\rm d}(0, 0) = 0$, is called a *hybrid supply rate* [91, 98] if it is locally integrable for all input-output pairs satisfying (4.8) and (4.10), that is, for all input-output pairs $u_{\rm c} \in \mathcal{U}_{\rm c}$, $y_{\rm c} \in \mathcal{Y}_{\rm c}$ satisfying (4.8) and (4.10), $s_{\rm c}(\cdot, \cdot)$ satisfies $\int_t^{\hat{t}} |s_{\rm c}(u_{\rm c}(\sigma), y_{\rm c}(\sigma))| {\rm d}\sigma < \infty$, $t, \hat{t} \geq 0$. Note that since all input-output pairs $u_{\rm d}(t_k) \in \mathcal{U}_{\rm d}$, $y_{\rm d}(t_k) \in \mathcal{Y}_{\rm d}$ satisfying (4.9) and (4.11) are defined for discrete instants, $s_{\rm d}(\cdot, \cdot)$ satisfies $\sum_{k \in \mathbb{Z}_{[t, \hat{t})}} |s_{\rm d}(u_{\rm d}(t_k), y_{\rm d}(t_k))| < \infty$, where $\mathbb{Z}_{[t, \hat{t})} \triangleq \{k : t \leq t_k < \hat{t}\}$.

Definition 4.4 [98]. The impulsive dynamical system $\mathcal{G}$ given by (4.8)–(4.11) is *exponentially dissipative* (respectively, *dissipative*) *with respect to the hybrid supply rate* $(s_{\mathrm{c}}, s_{\mathrm{d}})$ if there exist a continuous, nonnegative-definite function $v_{\mathrm{s}} : \mathcal{D} \to \mathbb{R}$ and a scalar $\varepsilon > 0$ (respectively, $\varepsilon = 0$) such that $v_{\mathrm{s}}(0) = 0$, called a *storage function*, and the *hybrid dissipation inequality*

$$\begin{aligned} e^{\varepsilon T} v_{\mathrm{s}}(x(T)) \leq e^{\varepsilon t_0} v_{\mathrm{s}}\,(x(t_0)) &+ \int_{t_0}^{T} e^{\varepsilon t} s_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))\mathrm{d}t \\ &+ \sum_{k \in \mathbb{Z}_{[t_0, T)}} e^{\varepsilon t_k} s_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)), \quad T \geq t_0, \end{aligned} \tag{4.12}$$

is satisfied for all $T \geq t_0$. The impulsive dynamical system $\mathcal{G}$ given by (4.8)–(4.11) is *lossless with respect to the hybrid supply rate* $(s_{\mathrm{c}}, s_{\mathrm{d}})$ if the hybrid dissipation inequality is satisfied as an equality with $\varepsilon = 0$ for all $T \geq t_0$.

The following result gives necessary and sufficient conditions for dissipativity over an interval $t \in (t_k, t_{k+1}]$ involving the consecutive resetting times t_k and t_{k+1}. First, however, the following definition is required.

Definition 4.5 [98]. A large-scale impulsive dynamical system $\mathcal{G}$ given by (4.8)–(4.11) is *completely reachable* if for all $(t_0, x_{\mathrm{i}}) \in \mathbb{R} \times \mathcal{D}$, there exist a finite time $t_{\mathrm{i}} < t_0$, a square integrable input $u_{\mathrm{c}}(t)$ defined on $[t_{\mathrm{i}}, t_0]$, and inputs $u_{\mathrm{d}}(t_k)$ defined on $k \in \mathbb{Z}_{[t_{\mathrm{i}},\, t_0)}$, such that the state $x(t)$, $t \geq t_{\mathrm{i}}$, can be driven from $x(t_{\mathrm{i}}) = 0$ to $x(t_0) = x_{\mathrm{i}}$.

Theorem 4.2 [98]. Assume $\mathcal{G}$ is completely reachable. Then $\mathcal{G}$ is exponentially dissipative (respectively, dissipative) with respect to the hybrid supply rate $(s_{\mathrm{c}}, s_{\mathrm{d}})$ if and only if there exist a continuous nonnegative-definite function $v_{\mathrm{s}} : \mathcal{D} \to \mathbb{R}$ and a scalar $\varepsilon > 0$ (respectively, $\varepsilon = 0$) such that $v_{\mathrm{s}}(0) = 0$ and for all $k \in \overline{\mathbb{Z}}_+$,

$$e^{\varepsilon \hat{t}} v_{\mathrm{s}}(x(\hat{t})) \leq e^{\varepsilon t} v_{\mathrm{s}}(x(t)) + \textstyle\int_t^{\hat{t}} e^{\varepsilon s} s_{\mathrm{c}}(u_{\mathrm{c}}(s), y_{\mathrm{c}}(s))\mathrm{d}s, \quad t_k < t \leq \hat{t} \leq t_{k+1}, \tag{4.13}$$

$$v_{\mathrm{s}}(x(t_k) + f_{\mathrm{d}}(x(t_k)) + G_{\mathrm{d}}(x(t_k))u_{\mathrm{d}}(t_k)) \leq v_{\mathrm{s}}(x(t_k)) + s_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)). \tag{4.14}$$

Finally, $\mathcal{G}$ given by (4.8)–(4.11) is lossless with respect to the hybrid supply rate $(s_{\rm c}, s_{\rm d})$ if and only if (4.13) and (4.14) are satisfied as equalities with $\varepsilon = 0$ for all $k \in \overline{\mathbb{Z}}_+$.

4.3. Vector Dissipativity Theory for Large-Scale Impulsive Dynamical Systems

In this section, we extend the notion of dissipative impulsive dynamical systems to develop the generalized notion of vector dissipativity for large-scale impulsive dynamical systems. We begin by considering input/state-dependent impulsive dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = F_{\rm c}(x(t), u_{\rm c}(t)), \quad x(t_0) = x_0, \quad (x(t), u_{\rm c}(t)) \notin \mathcal{Z}, \quad t \geq t_0, \tag{4.15}$$

$$\Delta x(t) = F_{\rm d}(x(t), u_{\rm d}(t)), \quad (x(t), u_{\rm c}(t)) \in \mathcal{Z}, \tag{4.16}$$

$$y_{\rm c}(t) = H_{\rm c}(x(t), u_{\rm c}(t)), \quad (x(t), u_{\rm c}(t)) \notin \mathcal{Z}, \tag{4.17}$$

$$y_{\rm d}(t) = H_{\rm d}(x(t), u_{\rm d}(t)), \quad (x(t), u_{\rm c}(t)) \in \mathcal{Z}, \tag{4.18}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \geq t_0$, $u_{\rm c} \in \mathcal{U}_{\rm c} \subseteq \mathbb{R}^{m_{\rm c}}$, $u_{\rm d} \in \mathcal{U}_{\rm d} \subseteq \mathbb{R}^{m_{\rm d}}$, $y_{\rm c} \in \mathcal{Y}_{\rm c} \subseteq \mathbb{R}^{l_{\rm c}}$, $y_{\rm d} \in \mathcal{Y}_{\rm d} \subseteq \mathbb{R}^{l_{\rm d}}$, $F_{\rm c} : \mathcal{D} \times \mathcal{U}_{\rm c} \to \mathbb{R}^n$, $F_{\rm d} : \mathcal{D} \times \mathcal{U}_{\rm d} \to \mathbb{R}^n$, $H_{\rm c} : \mathcal{D} \times \mathcal{U}_{\rm c} \to \mathcal{Y}_{\rm c}$, $H_{\rm d} : \mathcal{D} \times \mathcal{U}_{\rm d} \to \mathcal{Y}_{\rm d}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $\mathcal{Z} \subset \mathcal{D} \times \mathcal{U}_{\rm c}$, and $F_{\rm c}(0, 0) = 0$.

Here, we assume that $\mathcal{G}$ represents a large-scale impulsive dynamical system composed of q interconnected controlled impulsive subsystems $\mathcal{G}_i$ such that, for all $i = 1, ..., q$,

$$F_{{\rm c}i}(x, u_{{\rm c}i}) = f_{{\rm c}i}(x_i) + \mathcal{I}_{{\rm c}i}(x) + G_{{\rm c}i}(x_i) u_{{\rm c}i}, \tag{4.19}$$

$$F_{{\rm d}i}(x, u_{{\rm d}i}) = f_{{\rm d}i}(x_i) + \mathcal{I}_{{\rm d}i}(x) + G_{{\rm d}i}(x_i) u_{{\rm d}i}, \tag{4.20}$$

$$H_{{\rm c}i}(x_i, u_{{\rm c}i}) = h_{{\rm c}i}(x_i) + J_{{\rm c}i}(x_i) u_{{\rm c}i}, \tag{4.21}$$

$$H_{{\rm d}i}(x_i, u_{{\rm d}i}) = h_{{\rm d}i}(x_i) + J_{{\rm d}i}(x_i) u_{{\rm d}i}, \tag{4.22}$$

where $x_i \in \mathcal{D}_i \subseteq \mathbb{R}^{n_i}$, $u_{{\rm c}i} \in \mathcal{U}_{{\rm c}i} \subseteq \mathbb{R}^{m_{{\rm c}i}}$, $u_{{\rm d}i} \in \mathcal{U}_{{\rm d}i} \subseteq \mathbb{R}^{m_{{\rm d}i}}$, $y_{{\rm c}i} \triangleq H_{{\rm c}i}(x_i, u_{{\rm c}i}) \in \mathcal{Y}_{{\rm c}i} \subseteq \mathbb{R}^{l_{{\rm c}i}}$, $y_{{\rm d}i} \triangleq H_{{\rm d}i}(x_i, u_{{\rm d}i}) \in \mathcal{Y}_{{\rm d}i} \subseteq \mathbb{R}^{l_{{\rm d}i}}$, $((u_{{\rm c}i}, u_{{\rm d}i}), (y_{{\rm c}i}, y_{{\rm d}i}))$ is the hybrid input-output pair for the ith subsystem, $f_{{\rm c}i} : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ and $\mathcal{I}_{{\rm c}i} : \mathcal{D} \to \mathbb{R}^{n_i}$ are Lipschitz continuous and satisfy $f_{{\rm c}i}(0) = 0$

and $\mathcal{I}_{\mathrm{c}i}(0) = 0$, $f_{\mathrm{d}i} : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ and $\mathcal{I}_{\mathrm{d}i} : \mathcal{D} \to \mathbb{R}^{n_i}$ are continuous, $G_{\mathrm{c}i} : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i \times m_{\mathrm{c}i}}$ and $G_{\mathrm{d}i} : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i \times m_{\mathrm{d}i}}$ are continuous, $h_{\mathrm{c}i} : \mathbb{R}^{n_i} \to \mathbb{R}^{l_{\mathrm{c}i}}$ and satisfies $h_{\mathrm{c}i}(0) = 0$, $h_{\mathrm{d}i} : \mathbb{R}^{n_i} \to \mathbb{R}^{l_{\mathrm{d}i}}$, $J_{\mathrm{c}i} : \mathbb{R}^{n_i} \to \mathbb{R}^{l_{\mathrm{c}i} \times m_{\mathrm{c}i}}$, $J_{\mathrm{d}i} : \mathbb{R}^{n_i} \to \mathbb{R}^{l_{\mathrm{d}i} \times m_{\mathrm{d}i}}$, $\sum_{i=1}^{q} n_i = n$, $\sum_{i=1}^{q} m_{\mathrm{c}i} = m_{\mathrm{c}}$, $\sum_{i=1}^{q} m_{\mathrm{d}i} = m_{\mathrm{d}}$, $\sum_{i=1}^{q} l_{\mathrm{c}i} = l_{\mathrm{c}}$, and $\sum_{i=1}^{q} l_{\mathrm{d}i} = l_{\mathrm{d}}$. Furthermore, for the large-scale impulsive dynamical system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, for each $i \in \{1, ..., q\}$, $u_{\mathrm{c}i}(\cdot)$ satisfies sufficient regularity conditions such that the system (4.15), (4.16) has a unique solution forward in time. We define the composite input and composite output for the large-scale impulsive dynamical system $\mathcal{G}$ as $u_{\mathrm{c}} \triangleq [u_{\mathrm{c}1}^{\mathrm{T}}, ..., u_{\mathrm{c}q}^{\mathrm{T}}]^{\mathrm{T}}$, $u_{\mathrm{d}} \triangleq [u_{\mathrm{d}1}^{\mathrm{T}}, ..., u_{\mathrm{d}q}^{\mathrm{T}}]^{\mathrm{T}}$, $y_{\mathrm{c}} \triangleq [y_{\mathrm{c}1}^{\mathrm{T}}, ..., y_{\mathrm{c}q}^{\mathrm{T}}]^{\mathrm{T}}$, and $y_{\mathrm{d}} \triangleq [y_{\mathrm{d}1}^{\mathrm{T}}, ..., y_{\mathrm{d}q}^{\mathrm{T}}]^{\mathrm{T}}$, respectively.

Definition 4.6. For the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) a function $(S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$, where $S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}) \triangleq [s_{\mathrm{c}1}(u_{\mathrm{c}1}, y_{\mathrm{c}1}), ..., s_{\mathrm{c}q}(u_{\mathrm{c}q}, y_{\mathrm{c}q})]^{\mathrm{T}}$, $S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}) \triangleq [s_{\mathrm{d}1}(u_{\mathrm{d}1}, y_{\mathrm{d}1}), ..., s_{\mathrm{d}q}(u_{\mathrm{d}q}, y_{\mathrm{d}q})]^{\mathrm{T}}$, $s_{\mathrm{c}i} : \mathcal{U}_{\mathrm{c}i} \times \mathcal{Y}_{\mathrm{c}i} \to \mathbb{R}$, and $s_{\mathrm{d}i} : \mathcal{U}_{\mathrm{d}i} \times \mathcal{Y}_{\mathrm{d}i} \to \mathbb{R}$, $i = 1, ..., q$, such that $S_{\mathrm{c}}(0, 0) = 0$ and $S_{\mathrm{d}}(0, 0) = 0$, is called a *vector hybrid supply rate* if it is locally componentwise integrable for all input-output pairs satisfying (4.15)–(4.18); that is, for every $i \in \{1, ..., q\}$ and for all input-output pairs $u_{\mathrm{c}i} \in \mathcal{U}_{\mathrm{c}i}$, $y_{\mathrm{c}i} \in \mathcal{Y}_{\mathrm{c}i}$ satisfying (4.15)–(4.18), $s_{\mathrm{c}i}(\cdot, \cdot)$ satisfies $\int_t^{\hat{t}} |s_{\mathrm{c}i}(u_{\mathrm{c}i}(s), y_{\mathrm{c}i}(s))| \mathrm{d}s < \infty$, $t, \hat{t} \geq t_0$.

Note that since all input-output pairs $u_{\mathrm{d}i}(t_k) \in \mathcal{U}_{\mathrm{d}i}$, $y_{\mathrm{d}i}(t_k) \in \mathcal{Y}_{\mathrm{d}i}$ are defined for discrete instants, $s_{\mathrm{d}i}(\cdot, \cdot)$ in Definition 4.6 satisfies $\sum_{k \in \mathbb{Z}_{[t, \hat{t})}} |s_{\mathrm{d}i}(u_{\mathrm{d}i}(t_k), y_{\mathrm{d}i}(t_k))| < \infty$.

Definition 4.7. The large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) is *vector dissipative* (respectively, *exponentially vector dissipative*) *with respect to the vector hybrid supply rate* $(S_{\mathrm{c}}, S_{\mathrm{d}})$ if there exist a continuous, nonnegative definite vector function $V_{\mathrm{s}} = [v_{\mathrm{s}1}, ..., v_{\mathrm{s}q}]^{\mathrm{T}} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$, called a *vector storage function*, and an essentially nonnegative *dissipation matrix* $W \in \mathbb{R}^{q \times q}$ such that $V_{\mathrm{s}}(0) = 0$, W is semistable (respectively, asymptotically stable), and the *vector hybrid dissipation inequality*

$$V_{\mathrm{s}}(x(T)) \leq\leq e^{W(T - t_0)} V_{\mathrm{s}}(x(t_0)) + \int_{t_0}^{T} e^{W(T-t)} S_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t)) \mathrm{d}t$$

$$+\sum_{k\in\mathbb{Z}_{[t_0,T)}} e^{W(T-t_k)}S_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k)),\quad T\geq t_0, \tag{4.23}$$

is satisfied, where $x(t)$, $t \geq t_0$, is the solution to (4.15)–(4.18) with $(u_{\rm c}(t), u_{\rm d}(t_k)) \in \mathcal{U}_{\rm c} \times \mathcal{U}_{\rm d}$ and $x(t_0) = x_0$. The large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) is *vector lossless with respect to the vector hybrid supply rate* $(S_{\rm c}, S_{\rm d})$ if the vector hybrid dissipation inequality is satisfied as an equality with W semistable.

Note that if the subsystems $\mathcal{G}_i$ of $\mathcal{G}$ are *disconnected*; that is, $\mathcal{I}_{{\rm c}i}(x) \equiv 0$ and $\mathcal{I}_{{\rm d}i}(x) \equiv 0$ for all $i = 1, ..., q$, and $-W \in \mathbb{R}^{q\times q}$ is diagonal and nonnegative definite, then it follows from Definition 4.7 that each of disconnected subsystems $\mathcal{G}_i$ is dissipative or exponentially dissipative in the sense of Definition 4.4. A similar remark holds in the case where $q = 1$.

Next, define the *vector available storage* of the large-scale impulsive dynamical system $\mathcal{G}$ by

$$V_{\rm a}(x_0) \triangleq -\inf_{T\geq t_0,\,(u_{\rm c}(\cdot),\,u_{\rm d}(\cdot))}\Big[\int_{t_0}^{T} e^{-W(t-t_0)}S_{\rm c}(u_{\rm c}(t),y_{\rm c}(t)){\rm d}t$$
$$+\sum_{k\in\mathbb{Z}_{[t_0,T)}} e^{-W(t_k-t_0)}S_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\Big], \tag{4.24}$$

where $x(t)$, $t \geq t_0$, is the solution to (4.15)–(4.18) with $x(t_0) = x_0$ and admissible inputs $(u_{\rm c}, u_{\rm d}) \in \mathcal{U}_{\rm c} \times \mathcal{U}_{\rm d}$. The infimum in (4.24) is taken componentwise which implies that for different elements of $V_{\rm a}(\cdot)$ the infimum is calculated separately. Note that $V_{\rm a}(x_0) \geq\geq 0$, $x_0 \in \mathcal{D}$, since $V_{\rm a}(x_0)$ is the infimum over a set of vectors containing the zero vector ($T = t_0$).

Theorem 4.3. Consider the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) and assume that $\mathcal{G}$ is completely reachable. Then $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector hybrid supply rate $(S_{\rm c}, S_{\rm d})$ if and only if there exist a continuous, nonnegative-definite vector function $V_{\rm s} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$ and an essentially nonnegative dissipation matrix $W \in \mathbb{R}^{q\times q}$ such that $V_{\rm s}(0) = 0$, W is semistable (respectively, asymptotically stable), and for all $k \in \overline{\mathbb{Z}}_+$,

$$V_{\rm s}(x(\hat t)) \leq\leq e^{W(\hat t-t)}V_{\rm s}(x(t)) + \textstyle\int_t^{\hat t} e^{W(\hat t-s)}S_{\rm c}(u_{\rm c}(s),y_{\rm c}(s)){\rm d}s,\quad t_k < t \leq \hat t \leq t_{k+1}, \tag{4.25}$$

$$V_{\rm s}(x(t_k)+F_{\rm d}(x(t_k),u_{\rm d}(t_k))) \leq\leq V_{\rm s}(x(t_k))+S_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k)). \tag{4.26}$$

Alternatively, $\mathcal{G}$ is vector lossless with respect to the vector hybrid supply rate $(S_{\rm c}, S_{\rm d})$ if and only if (4.25) and (4.26) are satisfied as equalities with W semistable.

Proof. Let $k \in \overline{\mathbb{Z}}_+$ and suppose $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector hybrid supply rate $(S_{\rm c}, S_{\rm d})$. Then, there exist a continuous nonnegative-definite vector function $V_{\rm s} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$ and an essentially nonnegative matrix $W \in \mathbb{R}^{q\times q}$ such that (4.23) holds. Now, since for $t_k < t \leq \hat{t} \leq t_{k+1}$, $\mathbb{Z}_{[t,\hat{t})} = \emptyset$, (4.25) is immediate. Next, it follows from (4.23) that

$$\begin{aligned} V_{\rm s}(x(t_k^+)) \leq\leq\ & e^{W(t_k^+-t_k)}V_{\rm s}(x(t_k)) + \int_{t_k}^{t_k^+} e^{W(t_k^+-s)}S_{\rm c}(u_{\rm c}(s),y_{\rm c}(s)){\rm d}s \\ & + \sum_{k\in\mathbb{Z}_{[t_k,t_k^+)}} e^{W(t_k^+-t_k)}S_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k)) \end{aligned} \tag{4.27}$$

which, since $\mathbb{Z}_{[t_k,t_k^+)} = k$, implies (4.26).

Conversely, suppose (4.25) and (4.26) hold and let $\hat{t} \geq t \geq t_0$ and $\mathbb{Z}_{[t,\hat{t})} = \{i, i+1, ..., j\}$. (Note that if $\mathbb{Z}_{[t,\hat{t})} = \emptyset$ the converse result is a direct consequence of (4.25).) If $\mathbb{Z}_{[t,\hat{t})} \neq \emptyset$, it follows from (4.25) and (4.26) that

$$\begin{aligned} V_{\rm s}(x(\hat{t})) - e^{W(\hat{t}-t)}V_{\rm s}(x(t)) \ =\ & V_{\rm s}(x(\hat{t})) - e^{W(\hat{t}-t_j^+)}V_{\rm s}(x(t_j^+)) \\ & + e^{W(\hat{t}-t_j^+)}V_{\rm s}(x(t_j^+)) - e^{W(\hat{t}-t_{j-1}^+)}V_{\rm s}(x(t_{j-1}^+)) \\ & + e^{W(\hat{t}-t_{j-1}^+)}V_{\rm s}(x(t_{j-1}^+)) - \cdots - e^{W(\hat{t}-t_i^+)}V_{\rm s}(x(t_i^+)) \\ & + e^{W(\hat{t}-t_i^+)}V_{\rm s}(x(t_i^+)) - e^{W(\hat{t}-t)}V_{\rm s}(x(t)) \\ =\ & V_{\rm s}(x(\hat{t})) - e^{W(\hat{t}-t_j)}V_{\rm s}(x(t_j^+)) \\ & + e^{W(\hat{t}-t_j)}V_{\rm s}(x(t_j)+F_{\rm d}(x(t_j),u_{\rm d}(t_j))) - e^{W(\hat{t}-t_j)}V_{\rm s}(x(t_j)) \\ & + e^{W(\hat{t}-t_j)}V_{\rm s}(x(t_j)) - e^{W(\hat{t}-t_{j-1}^+)}V_{\rm s}(x(t_{j-1}^+)) + \cdots \\ & + e^{W(\hat{t}-t_i)}V_{\rm s}(x(t_i)+F_{\rm d}(x(t_i),u_{\rm d}(t_i))) - e^{W(\hat{t}-t_i)}V_{\rm s}(x(t_i)) \\ & + e^{W(\hat{t}-t_i)}V_{\rm s}(x(t_i)) - e^{W(\hat{t}-t)}V_{\rm s}(x(t)) \end{aligned}$$

$$
\begin{aligned}
&= V_{\mathrm{s}}(x(\hat{t})) - e^{W(\hat{t}-t_j)} V_{\mathrm{s}}(x(t_j^+)) \\
&\quad + e^{W(\hat{t}-t_j)} [V_{\mathrm{s}}(x(t_j) + F_{\mathrm{d}}(x(t_j), u_{\mathrm{d}}(t_j))) - V_{\mathrm{s}}(x(t_j))] \\
&\quad + e^{W(\hat{t}-t_j)} [V_{\mathrm{s}}(x(t_j)) - e^{W(t_j - t_{j-1})} V_{\mathrm{s}}(x(t_{j-1}^+))] + \cdots \\
&\quad + e^{W(\hat{t}-t_i)} [V_{\mathrm{s}}(x(t_i) + F_{\mathrm{d}}(x(t_i), u_{\mathrm{d}}(t_i))) - V_{\mathrm{s}}(x(t_i))] \\
&\quad + e^{W(\hat{t}-t_i)} [V_{\mathrm{s}}(x(t_i)) - e^{W(t_i - t)} V_{\mathrm{s}}(x(t))] \\
&\leq\leq \int_{t_j}^{\hat{t}} e^{W(\hat{t}-s)} S_{\mathrm{c}}(u_{\mathrm{c}}(s), y_{\mathrm{c}}(s)) \mathrm{d}s + e^{W(\hat{t}-t_j)} S_{\mathrm{d}}(u_{\mathrm{d}}(t_j), y_{\mathrm{d}}(t_j)) \\
&\quad + e^{W(\hat{t}-t_j)} \int_{t_{j-1}}^{t_j} e^{W(t_j - s)} S_{\mathrm{c}}(u_{\mathrm{c}}(s), y_{\mathrm{c}}(s)) \mathrm{d}s + \cdots \\
&\quad + e^{W(\hat{t}-t_i)} S_{\mathrm{d}}(u_{\mathrm{d}}(t_i), y_{\mathrm{d}}(t_i)) \\
&\quad + e^{W(\hat{t}-t_i)} \int_{t}^{t_i} e^{W(t_i - s)} S_{\mathrm{c}}(u_{\mathrm{c}}(s), y_{\mathrm{c}}(s)) \mathrm{d}s \\
&= \int_{t}^{\hat{t}} e^{W(\hat{t}-s)} S_{\mathrm{c}}(u_{\mathrm{c}}(s), y_{\mathrm{c}}(s)) \mathrm{d}s \\
&\quad + \sum_{k \in \mathbb{Z}_{[t,\hat{t})}} e^{W(\hat{t}-t_k)} S_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)), \qquad (4.28)
\end{aligned}
$$

which implies that $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector hybrid supply rate $(S_{\mathrm{c}}, S_{\mathrm{d}})$. Finally, similar constructions show that $\mathcal{G}$ is vector lossless with respect to the vector hybrid supply rate $(S_{\mathrm{c}}, S_{\mathrm{d}})$ if and only if (4.25) and (4.26) are satisfied as equalities with W semistable. $\square$

Theorem 4.4. Consider the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) and assume that $\mathcal{G}$ is completely reachable. Let $W \in \mathbb{R}^{q\times q}$ be essentially nonnegative and semistable (respectively, asymptotically stable). Then

$$
\int_{t_0}^{T} e^{-W(t-t_0)} S_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t)) \mathrm{d}t + \sum_{k \in \mathbb{Z}_{[t_0,T)}} e^{-W(t_k - t_0)} S_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)) \geq\geq 0, \quad T \geq t_0, \tag{4.29}
$$

for $x(t_0) = 0$ and $(u_{\mathrm{c}}, u_{\mathrm{d}}) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}$ if and only if $V_{\mathrm{a}}(0) = 0$ and $V_{\mathrm{a}}(x)$ is finite for all $x \in \mathcal{D}$. Moreover, if (4.29) holds, then $V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$, and hence, $\mathcal{G}$

is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector hybrid supply rate $(S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$.

Proof. Suppose $V_{\mathrm{a}}(0) = 0$ and $V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, is finite. Then

$$
\begin{aligned}
0 = V_{\mathrm{a}}(0) = -\inf_{T\geq t_0,\,(u_{\mathrm{c}}(\cdot),\,u_{\mathrm{d}}(\cdot))} \Big[&\int_{t_0}^{T} e^{-W(t-t_0)} S_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))\mathrm{d}t \\
&+ \sum_{k\in\mathbb{Z}_{[t_0,T)}} e^{-W(t_k-t_0)} S_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k))\Big],
\end{aligned} \tag{4.30}
$$

which implies (4.29).

Next, suppose (4.29) holds. Then for $x(t_0) = 0$,

$$
\begin{aligned}
-\inf_{T\geq t_0,\,(u_{\mathrm{c}}(\cdot),\,u_{\mathrm{d}}(\cdot))} \Big[&\int_{t_0}^{T} e^{-W(t-t_0)} S_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))\mathrm{d}t \\
&+ \sum_{k\in\mathbb{Z}_{[t_0,T)}} e^{-W(t_k-t_0)} S_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k))\Big] \leq\leq 0,
\end{aligned} \tag{4.31}
$$

which implies that $V_{\mathrm{a}}(0) \leq\leq 0$. However, since $V_{\mathrm{a}}(x_0) \geq\geq 0$, $x_0 \in \mathcal{D}$, it follows that $V_{\mathrm{a}}(0) = 0$. Moreover, since $\mathcal{G}$ is completely reachable it follows that for every $x_0 \in \mathcal{D}$ there exists $\hat{t} > t_0$ and an admissible input $u(\cdot)$ defined on $[t_0, \hat{t}]$ such that $x(\hat{t}) = x_0$. Now, since (4.29) holds for $x(t_0) = 0$ it follows that for all admissible $(u_{\mathrm{c}}, y_{\mathrm{c}}) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{Y}_{\mathrm{c}}$ and $(u_{\mathrm{d}}, y_{\mathrm{d}}) \in \mathcal{U}_{\mathrm{d}} \times \mathcal{Y}_{\mathrm{d}}$,

$$
\int_{t_0}^{T} e^{-W(t-t_0)} S_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))\mathrm{d}t + \sum_{k\in\mathbb{Z}_{[t_0,T)}} e^{-W(t_k-t_0)} S_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)) \geq\geq 0, \quad T \geq \hat{t}, \tag{4.32}
$$

or, equivalently, multiplying (4.32) by the nonnegative matrix $e^{W(\hat{t}-t_0)}$, $\hat{t} \geq t_0$, yields

$$
\begin{aligned}
-\int_{\hat{t}}^{T} e^{-W(t-\hat{t})} S_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))\mathrm{d}t &- \sum_{k\in\mathbb{Z}_{[\hat{t},T)}} e^{-W(t_k-\hat{t})} S_{\mathrm{d}}(u_{\mathrm{d}}(t_k), u_{\mathrm{d}}(t_k)) \\
&\leq\leq \int_{t_0}^{\hat{t}} e^{-W(t-\hat{t})} S_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))\mathrm{d}t + \sum_{k\in\mathbb{Z}_{[t_0,\hat{t})}} e^{-W(t_k-\hat{t})} S_{\mathrm{d}}(u_{\mathrm{d}}(t_k), u_{\mathrm{d}}(t_k)) \\
&\leq\leq Q(x_0) \\
&<< \infty, \quad T \geq \hat{t}, \quad (u_{\mathrm{c}}, u_{\mathrm{d}}) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}},
\end{aligned} \tag{4.33}
$$

where $Q : \mathcal{D} \to \mathbb{R}^q$. Hence,

$$V_{\rm a}(x_0) = -\inf_{T\geq \hat{t},\,(u_{\rm c}(\cdot),\,u_{\rm d}(\cdot))} \Big[\int_{\hat{t}}^{T} e^{-W(t-\hat{t})} S_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)){\rm d}t + \sum_{k\in\mathbb{Z}_{[\hat{t},T)}} e^{-W(t_k-\hat{t})} S_{\rm d}(u_{\rm d}(t_k), u_{\rm d}(t_k))\Big] \leq\leq Q(x_0) << \infty, \quad x_0 \in \mathcal{D}, \quad (4.34)$$

which implies that $V_{\rm a}(x_0)$, $x_0 \in \mathcal{D}$, is finite.

Finally, since (4.29) implies that $V_{\rm a}(0) = 0$ and $V_{\rm a}(x)$, $x \in \mathcal{D}$, is finite it follows from the definition of the vector available storage that

$$\begin{aligned} -V_{\rm a}(x_0) &\leq\leq \int_{t_0}^{T} e^{-W(t-t_0)} S_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)){\rm d}t + \sum_{k\in\mathbb{Z}_{[t_0,T)}} e^{-W(t_k-t_0)} S_{\rm d}(u_{\rm d}(t_k), u_{\rm d}(t_k)) \\ &= \int_{t_0}^{t_{\rm f}} e^{-W(t-t_0)} S_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)){\rm d}t + \sum_{k\in\mathbb{Z}_{[t_0,t_{\rm f})}} e^{-W(t_k-t_0)} S_{\rm d}(u_{\rm d}(t_k), u_{\rm d}(t_k)) \\ &\quad + \int_{t_{\rm f}}^{T} e^{-W(t-t_0)} S_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)){\rm d}t + \sum_{k\in\mathbb{Z}_{[t_{\rm f},T)}} e^{-W(t_k-t_0)} S_{\rm d}(u_{\rm d}(t_k), u_{\rm d}(t_k)), \\ &\qquad T \geq t_0. \quad (4.35) \end{aligned}$$

Now, multiplying (4.35) by the nonnegative matrix $e^{W(t_{\rm f}-t_0)}$, $t_{\rm f} \geq t_0$, it follows that

$$\begin{aligned} & e^{W(t_{\rm f}-t_0)} V_{\rm a}(x_0) + \int_{t_0}^{t_{\rm f}} e^{W(t_{\rm f}-t)} S_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)){\rm d}t + \sum_{k\in\mathbb{Z}_{[t_0,t_{\rm f})}} e^{W(t_{\rm f}-t_k)} S_{\rm d}(u_{\rm d}(t_k), u_{\rm d}(t_k)) \\ & \geq\geq -\inf_{T\geq t_{\rm f},\,(u_{\rm c}(\cdot),\,u_{\rm d}(\cdot))} \Big[\int_{t_{\rm f}}^{T} e^{-W(t-t_{\rm f})} S_{\rm c}(u_{\rm c}(t), y_{\rm d}(t)){\rm d}t + \sum_{k\in\mathbb{Z}_{[t_{\rm f},T)}} e^{-W(t_k-t_{\rm f})} S_{\rm d}(u_{\rm d}(t_k), u_{\rm d}(t_k))\Big] \\ & = V_{\rm a}(x(t_{\rm f})), \quad (4.36) \end{aligned}$$

which implies that $V_{\rm a}(x)$, $x \in \mathcal{D}$, is a vector storage function, and hence, $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector hybrid supply rate $(S_{\rm c}(u_{\rm c}, y_{\rm c}), S_{\rm d}(u_{\rm d}, y_{\rm d}))$. □

It follows from Lemma 4.1 that if $W \in \mathbb{R}^{q\times q}$ is essentially nonnegative and semistable (respectively, asymptotically stable), then there exist a scalar $\alpha \geq 0$ (respectively, $\alpha > 0$) and a nonnegative vector $p \in \overline{\mathbb{R}}_+^q$, $p \neq 0$, (respectively, $p \in \mathbb{R}_+^q$) such that (4.2) holds. In this case,

$$p^{\mathrm{T}}e^{Wt} = p^{\mathrm{T}}[I_q + Wt + \tfrac{1}{2}W^2t^2 + \cdots] = p^{\mathrm{T}}[I_q - \alpha t I_q + \tfrac{1}{2}\alpha^2 t^2 I_q + \cdots] = e^{-\alpha t}p^{\mathrm{T}}, \quad t \in \mathbb{R}. \tag{4.37}$$

Using (4.37), we define the (scalar) *available storage* for the large-scale impulsive dynamical system $\mathcal{G}$ by

$$\begin{aligned} v_{\mathrm{a}}(x_0) &\triangleq -\inf_{T \geq t_0,\, (u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot))} \Big[\int_{t_0}^{T} p^{\mathrm{T}} e^{-W(t-t_0)} S_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t)) \mathrm{d}t \\ &\quad + \sum_{k \in \mathbb{Z}_{[t_0,T)}} p^{\mathrm{T}} e^{-W(t_k - t_0)} S_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)) \Big] \\ &= -\inf_{T \geq t_0,\, (u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot))} \Big[\int_{t_0}^{T} e^{\alpha(t-t_0)} s_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t)) \mathrm{d}t \\ &\quad + \sum_{k \in \mathbb{Z}_{[t_0,T)}} e^{\alpha(t_k - t_0)} s_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)) \Big], \end{aligned} \tag{4.38}$$

where $s_{\mathrm{c}} : \mathcal{U}_{\mathrm{c}} \times \mathcal{Y}_{\mathrm{c}} \to \mathbb{R}$ and $s_{\mathrm{d}} : \mathcal{U}_{\mathrm{d}} \times \mathcal{Y}_{\mathrm{d}} \to \mathbb{R}$ defined as $s_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}) \triangleq p^{\mathrm{T}} S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}})$ and $s_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}) \triangleq p^{\mathrm{T}} S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}})$ form the (scalar) hybrid supply rate $(s_{\mathrm{c}}, s_{\mathrm{d}})$ for the large-scale impulsive dynamical system $\mathcal{G}$. Clearly, $v_{\mathrm{a}}(x) \geq 0$ for all $x \in \mathcal{D}$. As in standard hybrid dissipativity theory [98], the available storage $v_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, denotes the maximum amount of (scaled) energy that can be extracted from the large-scale impulsive dynamical system $\mathcal{G}$ at any time T.

The following theorem relates vector storage functions and vector hybrid supply rates to scalar storage functions and scalar hybrid supply rates of large-scale impulsive dynamical systems.

Theorem 4.5. Consider the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18). Suppose $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector hybrid supply rate $(S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}})) : (\mathcal{U}_{\mathrm{c}} \times \mathcal{Y}_{\mathrm{c}}, \mathcal{U}_{\mathrm{d}} \times \mathcal{Y}_{\mathrm{d}}) \to \mathbb{R}^q \times \mathbb{R}^q$ and with vector storage function $V_{\mathrm{s}} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$. Then there exists $p \in \overline{\mathbb{R}}_+^q$, $p \neq 0$, (respectively, $p \in \mathbb{R}_+^q$) such that $\mathcal{G}$ is dissipative (respectively, exponentially dissipative) with respect to the scalar hybrid supply rate $(s_{\rm c}(u_{\rm c},y_{\rm c}),s_{\rm d}(u_{\rm d},y_{\rm d}))=(p^{\rm T}S_{\rm c}(u_{\rm c},y_{\rm c}),p^{\rm T}S_{\rm d}(u_{\rm d},y_{\rm d}))$ and with storage function $v_{\rm s}(x)=p^{\rm T}V_{\rm s}(x)$, $x\in\mathcal{D}$. Moreover, in this case $v_{\rm a}(x)$, $x\in\mathcal{D}$, is a storage function for $\mathcal{G}$ and

$$0\leq v_{\rm a}(x)\leq v_{\rm s}(x),\quad x\in\mathcal{D}. \tag{4.39}$$

Proof. Suppose $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector hybrid supply rate $(S_{\rm c}(u_{\rm c},y_{\rm c}),S_{\rm d}(u_{\rm d},y_{\rm d}))$. Then there exist an essentially nonnegative, semistable (respectively, asymptotically stable) dissipation matrix W and a vector storage function $V_{\rm s}:\mathcal{D}\to\overline{\mathbb{R}}_+^q$ such that the dissipation inequality (4.23) holds. Furthermore, it follows from Lemma 4.1 that there exist $\alpha\geq 0$ (respectively, $\alpha>0$) and a nonzero vector $p\in\overline{\mathbb{R}}_+^q$ (respectively, $p\in\mathbb{R}_+^q$) satisfying (4.2). Hence, premultiplying (4.23) by $p^{\rm T}$ and using (4.37) it follows that

$$e^{\alpha T}v_{\rm s}(x(T))\leq e^{\alpha t_0}v_{\rm s}(x(t_0))+\int_{t_0}^{T}e^{\alpha t}s_{\rm c}(u_{\rm c}(t),y_{\rm c}(t)){\rm d}t+\sum_{k\in\mathbb{Z}_{[t_0,T)}}e^{\alpha t_k}s_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k)),$$
$$T\geq t_0,\quad (u_{\rm c},u_{\rm d})\in\mathcal{U}_{\rm c}\times\mathcal{U}_{\rm d}, \tag{4.40}$$

where $v_{\rm s}(x)=p^{\rm T}V_{\rm s}(x)$, $x\in\mathcal{D}$, which implies dissipativity (respectively, exponential dissipativity) of $\mathcal{G}$ with respect to the scalar hybrid supply rate $(s_{\rm c}(u_{\rm c},y_{\rm c}),s_{\rm d}(u_{\rm d},y_{\rm d}))$ and with storage function $v_{\rm s}(x)$, $x\in\mathcal{D}$. Moreover, since $v_{\rm s}(0)=0$, it follows from (4.40) that for $x(t_0)=0$,

$$\int_{t_0}^{T}e^{\alpha(t-t_0)}s_{\rm c}(u_{\rm c}(t),y_{\rm c}(t)){\rm d}t+\sum_{k\in\mathbb{Z}_{[t_0,T)}}e^{\alpha(t_k-t_0)}s_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\geq 0,$$
$$T\geq t_0,\quad (u_{\rm c},u_{\rm d})\in\mathcal{U}_{\rm c}\times\mathcal{U}_{\rm d}, \tag{4.41}$$

which, using (4.38), implies that $v_{\rm a}(0)=0$. Now, it can be easily shown that $v_{\rm a}(x)$, $x\in\mathcal{D}$, satisfies (4.40), and hence, the available storage defined by (4.38) is a storage function for $\mathcal{G}$.

Finally, it follows from (4.40) that

$$v_{\rm s}(x(t_0))\geq e^{\alpha(T-t_0)}v_{\rm s}(x(T))-\int_{t_0}^{T}e^{\alpha(t-t_0)}s_{\rm c}(u_{\rm c}(t),y_{\rm c}(t)){\rm d}t$$

$$
\begin{aligned}
&-\sum_{k\in\mathbb{Z}_{[t_0,T)}} e^{\alpha(t_k-t_0)} s_{\mathrm{d}}(u_{\mathrm{d}}(t_k),y_{\mathrm{d}}(t_k)) \\
&\geq -\int_{t_0}^{T} e^{\alpha(t-t_0)} s_{\mathrm{c}}(u(t),y(t))\mathrm{d}t - \sum_{k\in\mathbb{Z}_{[t_0,T)}} e^{\alpha(t_k-t_0)} s_{\mathrm{d}}(u_{\mathrm{d}}(t_k),y_{\mathrm{d}}(t_k)), \\
&\qquad\qquad T\geq t_0, \quad (u_{\mathrm{c}},u_{\mathrm{d}})\in\mathcal{U}_{\mathrm{c}}\times\mathcal{U}_{\mathrm{d}}, \qquad (4.42)
\end{aligned}
$$

which implies

$$
\begin{aligned}
v_{\mathrm{s}}(x(t_0)) &\geq -\inf_{T\geq t_0,\,(u_{\mathrm{c}}(\cdot),u_{\mathrm{d}}(\cdot))}\Big[\int_{t_0}^{T} e^{\alpha(t-t_0)} s_{\mathrm{c}}(u_{\mathrm{c}}(t),y_{\mathrm{c}}(t))\mathrm{d}t \\
&\quad + \sum_{k\in\mathbb{Z}_{[t_0,T)}} e^{\alpha(t_k-t_0)} s_{\mathrm{d}}(u_{\mathrm{d}}(t_k),y_{\mathrm{d}}(t_k))\Big] \\
&= v_{\mathrm{a}}(x(t_0)), \qquad (4.43)
\end{aligned}
$$

and hence, (4.39) holds. □

Remark 4.1. It follows from Theorem 4.4 that if (4.29) holds for $x(t_0)=0$, then the vector available storage $V_{\mathrm{a}}(x)$, $x\in\mathcal{D}$, is a vector storage function for $\mathcal{G}$. In this case, it follows from Theorem 4.5 that there exists $p\in\overline{\mathbb{R}}_+^q$, $p\neq 0$, such that $v_{\mathrm{s}}(x)\triangleq p^{\mathrm{T}}V_{\mathrm{a}}(x)$ is a storage function for $\mathcal{G}$ that satisfies (4.40), and hence, by (4.39), $v_{\mathrm{a}}(x)\leq p^{\mathrm{T}}V_{\mathrm{a}}(x)$, $x\in\mathcal{D}$.

Remark 4.2. It is important to note that it follows from Theorem 4.5 that if $\mathcal{G}$ is vector dissipative, then $\mathcal{G}$ can either be (scalar) dissipative or (scalar) exponentially dissipative.

The following theorem provides sufficient conditions guaranteeing that all scalar storage functions defined in terms of vector storage functions, that is, $v_{\mathrm{s}}(x)=p^{\mathrm{T}}V_{\mathrm{s}}(x)$, of a given vector dissipative large-scale impulsive nonlinear dynamical system are positive definite. To state this result the following definition is needed.

Definition 4.8 [98]. A large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) is *zero-state observable* if $(u_{\mathrm{c}}(t),u_{\mathrm{d}}(t_k))\equiv(0,0)$ and $(y_{\mathrm{c}}(t),y_{\mathrm{d}}(t_k))\equiv(0,0)$ imply $x(t)\equiv 0$.

Theorem 4.6. Consider the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) and assume that $\mathcal{G}$ is zero-state observable. Furthermore, assume that $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector hybrid supply rate $(S_{\rm c}(u_{\rm c}, y_{\rm c}), S_{\rm d}(u_{\rm d}, y_{\rm d}))$ and there exist $\alpha \geq 0$ and $p \in \mathbb{R}^q_+$ such that (4.2) holds. In addition, assume that there exist functions $\kappa_{{\rm c}i} : \mathcal{Y}_{{\rm c}i} \to \mathcal{U}_{{\rm c}i}$ and $\kappa_{{\rm d}i} : \mathcal{Y}_{{\rm d}i} \to \mathcal{U}_{{\rm d}i}$ such that $\kappa_{{\rm c}i}(0) = 0$, $\kappa_{{\rm d}i}(0) = 0$, $s_{{\rm c}i}(\kappa_{{\rm c}i}(y_{{\rm c}i}), y_{{\rm c}i}) < 0$, $y_{{\rm c}i} \neq 0$, and $s_{{\rm d}i}(\kappa_{{\rm d}i}(y_{{\rm d}i}), y_{{\rm d}i}) < 0$, $y_{{\rm d}i} \neq 0$, for all $i = 1, ..., q$. Then for all vector storage functions $V_{\rm s} : \mathcal{D} \to \overline{\mathbb{R}}^q_+$ the storage function $v_{\rm s}(x) \triangleq p^{\rm T} V_{\rm s}(x)$, $x \in \mathcal{D}$, is positive definite, that is, $v_{\rm s}(0) = 0$ and $v_{\rm s}(x) > 0$, $x \in \mathcal{D}$, $x \neq 0$.

Proof. The proof is similar to the proof of Theorem 3.3 of [102]. $\square$

Next, we introduce the concept of *vector required supply* of a large-scale impulsive dynamical system. Specifically, define the vector required supply of the large-scale impulsive dynamical system $\mathcal{G}$ by

$$V_{\rm r}(x_0) \triangleq \inf_{T \leq t_0,\, (u_{\rm c}(\cdot), u_{\rm d}(\cdot))} \Big[\int_T^{t_0} e^{-W(t-t_0)} S_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)) {\rm d}t + \sum_{k \in \mathbb{Z}_{[T,t_0)}} e^{-W(t_k - t_0)} S_{\rm d}(u_{\rm d}(t_k), y_{\rm d}(t_k)) \Big], \tag{4.44}$$

where $x(t)$, $t \geq T$, is the solution to (4.15)–(4.18) with $x(T) = 0$ and $x(t_0) = x_0$. Note that since, with $x(t_0) = 0$, the infimum in (4.44) is the zero vector it follows that $V_{\rm r}(0) = 0$. Moreover, since $\mathcal{G}$ is completely reachable it follows that $V_{\rm r}(x) << \infty$, $x \in \mathcal{D}$. Using the notion of the vector required supply we present necessary and sufficient conditions for vector dissipativity of a large-scale impulsive dynamical system with respect to a vector hybrid supply rate.

Theorem 4.7. Consider the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) and assume that $\mathcal{G}$ is completely reachable. Then $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector hybrid supply rate $(S_{\rm c}(u_{\rm c}, y_{\rm c}), S_{\rm d}$

$(u_{\rm d}, y_{\rm d}))$ if and only if

$$0 \leq\leq V_{\rm r}(x) << \infty, \quad x \in \mathcal{D}. \tag{4.45}$$

Moreover, if (4.45) holds, then $V_{\rm r}(x)$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$. Finally, if the vector available storage $V_{\rm a}(x)$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$, then

$$0 \leq\leq V_{\rm a}(x) \leq\leq V_{\rm r}(x) << \infty, \quad x \in \mathcal{D}. \tag{4.46}$$

Proof. Suppose (4.45) holds and let $x(t)$, $t \in \mathbb{R}$, satisfy (4.15)–(4.18) with admissible inputs $(u_{\rm c}(t), u_{\rm d}(t)) \in \mathcal{U}_{\rm c} \times \mathcal{U}_{\rm d}$, $t \in \mathbb{R}$, and $x(t_0) = x_0$. Then it follows from the definition of $V_{\rm r}(\cdot)$ that for $T \leq t_{\rm f} \leq t_0$, $u_{\rm c}(\cdot) \in \mathcal{U}_{\rm c}$, and $u_{\rm d}(\cdot) \in \mathcal{U}_{\rm d}$,

$$\begin{aligned}
V_{\rm r}(x_0) \leq\leq & \int_T^{t_0} e^{-W(t-t_0)} S_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)){\rm d}t + \sum_{k \in \mathbb{Z}_{[T,t_0)}} e^{-W(t_k-t_0)} S_{\rm d}(u_{\rm d}(t_k), y_{\rm d}(t_k)) \\
= & \int_T^{t_{\rm f}} e^{-W(t-t_0)} S_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)){\rm d}t + \sum_{k \in \mathbb{Z}_{[T,t_{\rm f})}} e^{-W(t_k-t_0)} S_{\rm d}(u_{\rm d}(t_k), y_{\rm d}(t_k)) \\
& + \int_{t_{\rm f}}^{t_0} e^{-W(t-t_0)} S_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)){\rm d}t + \sum_{k \in \mathbb{Z}_{[t_{\rm f},t_0)}} e^{-W(t_k-t_0)} S_{\rm d}(u_{\rm d}(t_k), y_{\rm d}(t_k)),
\end{aligned}$$

and hence,

$$\begin{aligned}
V_{\rm r}(x_0) \leq\leq & \ e^{W(t_0-t_{\rm f})} \inf_{T \leq t_{\rm f},\, (u_{\rm c}(\cdot), u_{\rm d}(\cdot))} \Big[\int_T^{t_{\rm f}} e^{-W(t-t_{\rm f})} S_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)){\rm d}t \\
& + \sum_{k \in \mathbb{Z}_{[T,t_{\rm f})}} e^{-W(t_k-t_{\rm f})} S_{\rm d}(u_{\rm d}(t_k), y_{\rm d}(t_k)) \Big] \\
& + \int_{t_{\rm f}}^{t_0} e^{-W(t-t_0)} S_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)){\rm d}t + \sum_{k \in \mathbb{Z}_{[t_{\rm f},t_0)}} e^{-W(t_k-t_0)} S_{\rm d}(u_{\rm d}(t_k), y_{\rm d}(t_k)) \\
= & \ e^{W(t_0-t_{\rm f})} V_{\rm r}(x(t_{\rm f})) + \int_{t_{\rm f}}^{t_0} e^{-W(t-t_0)} S_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)){\rm d}t \\
& + \sum_{k \in \mathbb{Z}_{[t_{\rm f},t_0)}} e^{-W(t_k-t_0)} S_{\rm d}(u_{\rm d}(t_k), y_{\rm d}(t_k)),
\end{aligned} \tag{4.47}$$

which shows that $V_{\rm r}(x)$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$, and hence, $\mathcal{G}$ is vector dissipative with respect to the vector hybrid supply rate $(S_{\rm c}(u_{\rm c}, y_{\rm c}), S_{\rm d}(u_{\rm d}, y_{\rm d}))$.

Conversely, suppose that $\mathcal{G}$ is vector dissipative with respect to the vector hybrid supply rate $(S_{\rm c}(u_{\rm c},y_{\rm c}),S_{\rm d}(u_{\rm d},y_{\rm d}))$. Then there exists a nonnegative vector storage function $V_{\rm s}(x)$, $x\in\mathcal{D}$, such that $V_{\rm s}(0)=0$. Since $\mathcal{G}$ is completely reachable it follows that for $x(t_0)=x_0$ there exist $T<t_0$ and $u(t)$, $t\in[T,t_0]$, such that $x(T)=0$. Hence, it follows from the vector hybrid dissipation inequality (4.23) that

$$\begin{aligned}0\leq\leq V_{\rm s}(x(t_0))\leq\leq e^{W(t_0-T)}V_{\rm s}(x(T))+\int_T^{t_0}e^{W(t_0-t)}S_{\rm c}(u_{\rm c}(t),y_{\rm c}(t)){\rm d}t\\+\sum_{k\in\mathbb{Z}_{[T,t_0)}}e^{W(t_0-t_k)}S_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k)),\end{aligned}\tag{4.48}$$

which implies that for all $T\leq t_0$, $u_{\rm c}\in\mathcal{U}_{\rm c}$, and $u_{\rm d}\in\mathcal{U}_{\rm d}$,

$$0\leq\leq\int_T^{t_0}e^{W(t_0-t)}S_{\rm c}(u_{\rm c}(t),y_{\rm c}(t)){\rm d}t+\sum_{k\in\mathbb{Z}_{[T,t_0)}}e^{W(t_0-t_k)}S_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\tag{4.49}$$

or, equivalently,

$$\begin{aligned}0\leq\leq&\inf_{T\leq t_0,\,(u_{\rm c}(\cdot),u_{\rm d}(\cdot))}\Big[\int_T^{t_0}e^{W(t_0-t)}S_{\rm c}(u_{\rm c}(t),y_{\rm c}(t)){\rm d}t\\&+\sum_{k\in\mathbb{Z}_{[T,t_0)}}e^{W(t_0-t_k)}S_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\Big]\\=&\ V_{\rm r}(x_0).\end{aligned}\tag{4.50}$$

Since, by complete reachability, $V_{\rm r}(x)<<\infty$, $x\in\mathcal{D}$, it follows that (4.45) holds.

Finally, suppose that $V_{\rm a}(x)$, $x\in\mathcal{D}$, is a vector storage function. Then for $x(T)=0$, $x(t_0)=x_0$, $u_{\rm c}\in\mathcal{U}_{\rm c}$, and $u_{\rm d}\in\mathcal{U}_{\rm d}$, it follows that

$$\begin{aligned}V_{\rm a}(x(t_0))\leq\leq e^{W(t_0-T)}V_{\rm a}(x(T))+\int_T^{t_0}e^{W(t_0-t)}S_{\rm c}(u_{\rm c}(t),y_{\rm c}(t)){\rm d}t\\+\sum_{k\in\mathbb{Z}_{[T,t_0)}}e^{W(t_0-t_k)}S_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k)),\end{aligned}\tag{4.51}$$

which implies that

$$\begin{aligned}0\leq\leq V_{\rm a}(x(t_0))\leq\leq&\inf_{T\leq t_0,\,(u_{\rm c}(\cdot),u_{\rm d}(\cdot))}\Big[\int_T^{t_0}e^{W(t_0-t)}S_{\rm c}(u_{\rm c}(t),y_{\rm c}(t)){\rm d}t\\&+\sum_{k\in\mathbb{Z}_{[T,t_0)}}e^{W(t_0-t_k)}S_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\Big]\\=&\ V_{\rm r}(x(t_0)),\quad x\in\mathcal{D}.\end{aligned}\tag{4.52}$$

Since $x(t_0) = x_0 \in \mathcal{D}$ is arbitrary and, by complete reachability, $V_{\mathrm{r}}(x) << \infty$, $x \in \mathcal{D}$, (4.52) implies (4.46). $\square$

The next result is a direct consequence of Theorems 4.4 and 4.7.

Proposition 4.2. Consider the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18). Let $M = \mathrm{diag}\,[\mu_1, ..., \mu_q]$ be such that $0 \leq \mu_i \leq 1$, $i = 1, ..., q$. If $V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, and $V_{\mathrm{r}}(x)$, $x \in \mathcal{D}$, are vector storage functions for $\mathcal{G}$, then

$$V_{\mathrm{s}}(x) = MV_{\mathrm{a}}(x) + (I_q - M)V_{\mathrm{r}}(x), \quad x \in \mathcal{D}, \tag{4.53}$$

is a vector storage function for $\mathcal{G}$.

Next, recall that if $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative), then there exist $p \in \overline{\mathbb{R}}_+^q$, $p \neq 0$, and $\alpha \geq 0$ (respectively, $p \in \mathbb{R}_+^q$ and $\alpha > 0$) such that (4.2) and (4.37) hold. Now, define the (scalar) *required supply* for the large-scale impulsive dynamical system $\mathcal{G}$ by

$$\begin{aligned} v_{\mathrm{r}}(x_0) \triangleq &\inf_{T \leq t_0,\, (u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot))} \Big[\int_T^{t_0} p^{\mathrm{T}} e^{-W(t-t_0)} S_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))\mathrm{d}t \\ &+ \sum_{k \in \mathbb{Z}_{[T,t_0)}} e^{-W(t_k-t_0)} S_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k))\Big] \\ = &\inf_{T \leq t_0,\, (u_{\mathrm{c}}(\cdot), u_{\mathrm{d}}(\cdot))} \Big[\int_T^{t_0} e^{\alpha(t-t_0)} s_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))\mathrm{d}t \\ &+ \sum_{k \in \mathbb{Z}_{[T,t_0)}} e^{\alpha(t_k-t_0)} s_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k))\Big], \quad x_0 \in \mathcal{D}, \end{aligned} \tag{4.54}$$

where $s_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}) = p^{\mathrm{T}} S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}})$, $s_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}) = p^{\mathrm{T}} S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}})$, and $x(t)$, $t \geq T$, is the solution to (4.15)–(4.18) with $x(T) = 0$ and $x(t_0) = x_0$. It follows from (4.54) that the required supply of a large-scale impulsive dynamical system is the minimum amount of generalized energy which can be delivered to the large-scale system in order to transfer it from an initial state $x(T) = 0$ to a given state $x(t_0) = x_0$. Using the same arguments as in case of the vector required supply, it follows that $v_{\mathrm{r}}(0) = 0$ and $v_{\mathrm{r}}(x) < \infty$, $x \in \mathcal{D}$.

Next, using the notion of required supply, we show that all storage functions of the form $v_{\mathrm{s}}(x) = p^{\mathrm{T}}V_{\mathrm{s}}(x)$, where $p \in \overline{\mathbb{R}}_{+}^{q}$, $p \neq 0$, are bounded from above by the required supply and bounded from below by the available storage. Hence, a dissipative large-scale impulsive dynamical system can only deliver to its surroundings a fraction of all of its stored subsystem energies and can only store a fraction of the work done to all of its subsystems.

Corollary 4.2. Consider the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18). Assume that $\mathcal{G}$ is vector dissipative with respect to the vector hybrid supply rate $(S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$ and with vector storage function $V_{\mathrm{s}} : \mathcal{D} \to \overline{\mathbb{R}}_{+}^{q}$. Then $v_{\mathrm{r}}(x)$, $x \in \mathcal{D}$, is a storage function for $\mathcal{G}$. Moreover, if $v_{\mathrm{s}}(x) \triangleq p^{\mathrm{T}}V_{\mathrm{s}}(x)$, $x \in \mathcal{D}$, where $p \in \overline{\mathbb{R}}_{+}^{q}$, $p \neq 0$, then

$$0 \leq v_{\mathrm{a}}(x) \leq v_{\mathrm{s}}(x) \leq v_{\mathrm{r}}(x) < \infty, \quad x \in \mathcal{D}. \tag{4.55}$$

Proof. It follows from Theorem 4.5 that if $\mathcal{G}$ is vector dissipative with respect to the vector hybrid supply rate $(S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$ and with a vector storage function $V_{\mathrm{s}} : \mathcal{D} \to \overline{\mathbb{R}}_{+}^{q}$, then there exists $p \in \overline{\mathbb{R}}_{+}^{q}$, $p \neq 0$, such that $\mathcal{G}$ is dissipative with respect to the hybrid supply rate $(s_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), s_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}})) = (p^{\mathrm{T}}S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), p^{\mathrm{T}}S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$ and with storage function $v_{\mathrm{s}}(x) = p^{\mathrm{T}}V_{\mathrm{s}}(x)$, $x \in \mathcal{D}$. Hence, it follows from (4.40), with $x(T) = 0$ and $x(t_0) = x_0$, that

$$\int_T^{t_0} e^{\alpha(t-t_0)} s_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))\mathrm{d}t + \sum_{k\in\mathbb{Z}_{[T,t_0)}} e^{\alpha(t_k-t_0)} s_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)) \geq 0,$$
$$T \leq t_0, \quad (u_{\mathrm{c}}, u_{\mathrm{d}}) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}, \tag{4.56}$$

which implies that $v_{\mathrm{r}}(x_0) \geq 0$, $x_0 \in \mathcal{D}$. Furthermore, it is easy to see from the definition of a required supply that $v_{\mathrm{r}}(x)$, $x \in \mathcal{D}$, satisfies the dissipation inequality (4.40). Hence, $v_{\mathrm{r}}(x)$, $x \in \mathcal{D}$, is a storage function for $\mathcal{G}$. Moreover, it follows from the dissipation inequality (4.40), with $x(T) = 0$, $x(t_0) = x_0$, $u_{\mathrm{c}} \in \mathcal{U}_{\mathrm{c}}$, and $u_{\mathrm{d}} \in \mathcal{U}_{\mathrm{d}}$ that

$$\begin{aligned} e^{\alpha t_0} v_{\mathrm{s}}(x(t_0)) &\leq e^{\alpha T} v_{\mathrm{s}}(x(T)) + \int_T^{t_0} e^{\alpha t} s_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))\mathrm{d}t \\ &\quad + \sum_{k\in\mathbb{Z}_{[T,t_0)}} e^{\alpha t_k} s_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)) \\ &= \int_T^{t_0} e^{\alpha t} s_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))\mathrm{d}t + \sum_{k\in\mathbb{Z}_{[T,t_0)}} e^{\alpha t_k} s_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)), \end{aligned} \tag{4.57}$$

which implies that

$$\begin{aligned} v_{\rm s}(x(t_0)) &\le \inf_{T\le t_0,\,(u_{\rm c}(\cdot),u_{\rm d}(\cdot))} \Big[\int_T^{t_0} e^{\alpha(t-t_0)} s_{\rm c}(u_{\rm c}(t),y_{\rm c}(t)){\rm d}t \\ &\quad + \sum_{k\in\mathbb{Z}_{[T,t_0)}} e^{\alpha(t_k-t_0)} s_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k))\Big] \\ &= v_{\rm r}(x(t_0)). \end{aligned} \tag{4.58}$$

Finally, it follows from Theorem 4.5 that $v_{\rm a}(x)$, $x\in\mathcal{D}$, is a storage function for $\mathcal{G}$, and hence, using (4.39) and (4.58), (4.55) holds. $\square$

Remark 4.3. It follows from Theorem 4.7 that if $\mathcal{G}$ is vector dissipative with respect to the vector hybrid supply rate $(S_{\rm c}(u_{\rm c},y_{\rm c}),S_{\rm d}(u_{\rm d},y_{\rm d}))$, then $V_{\rm r}(x)$, $x\in\mathcal{D}$, is a vector storage function for $\mathcal{G}$ and, by Theorem 4.5, there exists $p\in\overline{\mathbb{R}}_+^q$, $p\neq 0$, such that $v_{\rm s}(x)\triangleq p^{\rm T}V_{\rm r}(x)$, $x\in\mathcal{D}$, is a storage function for $\mathcal{G}$ satisfying (4.40). Hence, it follows from Corollary 4.2 that $p^{\rm T}V_{\rm r}(x)\le v_{\rm r}(x)$, $x\in\mathcal{D}$.

The next result relates vector (respectively, scalar) available storage and vector (respectively, scalar) required supply for vector lossless large-scale impulsive dynamical systems.

Theorem 4.8. Consider the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18). Assume that $\mathcal{G}$ is completely reachable to and from the origin. If $\mathcal{G}$ is vector lossless with respect to the vector hybrid supply rate $(S_{\rm c}(u_{\rm c},y_{\rm c}),S_{\rm d}(u_{\rm d},y_{\rm d}))$ and $V_{\rm a}(x)$, $x\in\mathcal{D}$, is a vector storage function, then $V_{\rm a}(x)=V_{\rm r}(x)$, $x\in\mathcal{D}$. Moreover, if $V_{\rm s}(x)$, $x\in\mathcal{D}$, is a vector storage function, then all (scalar) storage functions of the form $v_{\rm s}(x)=p^{\rm T}V_{\rm s}(x)$, $x\in\mathcal{D}$, where $p\in\overline{\mathbb{R}}_+^q$, $p\neq 0$, are given by

$$\begin{aligned} v_{\rm s}(x_0)=v_{\rm a}(x_0)=v_{\rm r}(x_0) &= -\int_{t_0}^{T} e^{\alpha(t-t_0)} s_{\rm c}(u_{\rm c}(t),y_{\rm c}(t)){\rm d}t \\ &\quad - \sum_{k\in\mathbb{Z}_{[t_0,T)}} e^{\alpha(t_k-t_0)} s_{\rm d}(u_{\rm d}(t_k),y_{\rm d}(t_k)) \\ &= \int_{T'}^{t_0} e^{\alpha(t-t_0)} s_{\rm c}(u_{\rm c}(t),y_{\rm c}(t)){\rm d}t \end{aligned}$$

$$+\sum_{k\in\mathbb{Z}_{[T',t_0)}} e^{\alpha(t_k-t_0)} s_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)), \tag{4.59}$$

where $x(t)$, $t \geq t_0$, is the solution to (4.15)–(4.18) with $u_{\mathrm{c}} \in \mathcal{U}_{\mathrm{c}}$, $u_{\mathrm{d}} \in \mathcal{U}_{\mathrm{d}}$, $x(T') = 0$, $x(T) = 0$, $x(t_0) = x_0 \in \mathcal{D}$, $s_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}) = p^{\mathrm{T}} S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}})$, and $s_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}) = p^{\mathrm{T}} S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}})$.

Proof. The proof is similar to the proof of Theorem 3.5 of [102]. □

The next proposition presents a characterization for vector dissipativity of large-scale impulsive dynamical systems in the case where $V_{\mathrm{s}}(\cdot)$ is continuously differentiable.

Proposition 4.3. Consider the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) and assume $V_{\mathrm{s}} = [v_{\mathrm{s}1}, ..., v_{\mathrm{s}q}]^{\mathrm{T}} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$ is a continuously differentiable vector storage function for $\mathcal{G}$ and $\mathcal{G}$ is completely reachable. Then $\mathcal{G}$ is vector dissipative with respect to the vector hybrid supply rate $(S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$ if and only if

$$\dot{V}_{\mathrm{s}}(x(t)) \leq\leq WV_{\mathrm{s}}(x(t)) + S_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t)), \quad t_k < t \leq t_{k+1}, \tag{4.60}$$

$$V_{\mathrm{s}}(x(t_k) + F_{\mathrm{d}}(x(t_k), u_{\mathrm{d}}(t_k))) \leq\leq V_{\mathrm{s}}(x(t_k)) + S_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)), \quad k \in \overline{\mathbb{Z}}_+, \tag{4.61}$$

where $\dot{V}_{\mathrm{s}}(x(t))$ denotes the total time derivative of each component of $V_{\mathrm{s}}(\cdot)$ along the state trajectories $x(t), t_k < t \leq t_{k+1}$, of $\mathcal{G}$.

Proof. The proof is similar to the proof of Proposition 3.2 of [102]. □

Recall that if a disconnected subsystem $\mathcal{G}_i$ (i.e., $\mathcal{I}_{\mathrm{c}i}(x) \equiv 0$ and $\mathcal{I}_{\mathrm{d}i}(x) \equiv 0, i \in \{1, ..., q\}$) of a large-scale impulsive dynamical system $\mathcal{G}$ is exponentially dissipative (respectively, dissipative) with respect to a hybrid supply rate $(s_{\mathrm{c}i}(u_{\mathrm{c}i}, y_{\mathrm{c}i}), s_{\mathrm{d}i}(u_{\mathrm{d}i}, y_{\mathrm{d}i}))$, then there exist a storage function $v_{\mathrm{s}i} : \mathbb{R}^{n_i} \to \overline{\mathbb{R}}_+$ and a constant $\varepsilon_i > 0$ (respectively, $\varepsilon_i = 0$) such that the dissipation inequality (4.12) holds. In the case where $v_{\mathrm{s}i} : \mathbb{R}^{n_i} \to \overline{\mathbb{R}}_+$ is continuously differentiable and $\mathcal{G}$ is completely reachable, (4.12) yields

$$v'_{\mathrm{s}i}(x_i)(f_{\mathrm{c}i}(x_i) + G_{\mathrm{c}i}(x_i)u_{\mathrm{c}i}) \leq -\varepsilon_i v_{\mathrm{s}i}(x_i) + s_{\mathrm{c}i}(u_{\mathrm{c}i}, y_{\mathrm{c}i}), \quad x \notin \mathcal{Z}_i, \quad u_{\mathrm{c}i} \in \mathcal{U}_{\mathrm{c}i}, \tag{4.62}$$

$$v_{\mathrm{s}i}(x_i + f_{\mathrm{d}i}(x_i) + G_{\mathrm{d}i}(x_i)u_{\mathrm{d}i}) \leq v_{\mathrm{s}i}(x_i) + s_{\mathrm{d}i}(u_{\mathrm{d}i}, y_{\mathrm{d}i}), \quad x \in \mathcal{Z}_i, \quad u_{\mathrm{d}i} \in \mathcal{U}_{\mathrm{d}i}, \tag{4.63}$$

where $\mathcal{Z}_i \triangleq \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_{i-1}} \times \mathcal{Z}_{x_i} \times \mathbb{R}^{n_{i+1}} \times \cdots \times \mathbb{R}^{q} \subset \mathbb{R}^n$ and $\mathcal{Z}_{x_i} \subset \mathbb{R}^{n_i}$, $i = 1, ..., q$. The next result relates exponential dissipativity with respect to a scalar hybrid supply rate of each disconnected subsystem $\mathcal{G}_i$ of $\mathcal{G}$ with vector dissipativity (or, possibly, exponential vector dissipativity) of $\mathcal{G}$ with respect to a hybrid vector supply rate.

Proposition 4.4. Consider the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) with $\mathcal{Z}_x = \cup_{i=1}^{q} \mathcal{Z}_i$. Assume that $\mathcal{G}$ is completely reachable and each disconnected subsystem $\mathcal{G}_i$ of $\mathcal{G}$ is exponentially dissipative with respect to the hybrid supply rate $(s_{\mathrm{c}i}(u_{\mathrm{c}i}, y_{\mathrm{c}i}), s_{\mathrm{d}i}(u_{\mathrm{d}i}, y_{\mathrm{d}i}))$ and with a continuously differentiable storage function $v_{\mathrm{s}i} : \mathbb{R}^{n_i} \to \overline{\mathbb{R}}_+$, $i = 1, ..., q$. Furthermore, assume that interconnection functions $\mathcal{I}_{\mathrm{c}i} : \mathcal{D} \to \mathbb{R}^{n_i}$ and $\mathcal{I}_{\mathrm{d}i} : \mathcal{D} \to \mathbb{R}^{n_i}$, $i = 1, ..., q$, of $\mathcal{G}$ are such that

$$v'_{\mathrm{s}i}(x_i)\mathcal{I}_{\mathrm{c}i}(x) \le \sum_{j=1}^{q} \xi_{ij}(x) v_{\mathrm{s}j}(x_j), \quad x \notin \mathcal{Z}_x, \tag{4.64}$$

$$v_{\mathrm{s}i}(x_i + f_{\mathrm{d}i}(x_i) + \mathcal{I}_{\mathrm{d}i}(x) + G_{\mathrm{d}i}(x_i)u_{\mathrm{d}i}) \le v_{\mathrm{s}i}(x_i + f_{\mathrm{d}i}(x_i) + G_{\mathrm{d}i}(x_i)u_{\mathrm{d}i}), \tag{4.65}$$

$$x \in \mathcal{Z}_x, \quad u_{\mathrm{d}i} \in \mathcal{U}_{\mathrm{d}i}, \quad i = 1, ..., q,$$

where $\xi_{ij} : \mathcal{D} \to \mathbb{R}$, $i, j = 1, ..., q$, are given bounded functions. If $W \in \mathbb{R}^{q \times q}$ is semistable (respectively, asymptotically stable), with

$$W_{(i,j)} = \begin{cases} -\varepsilon_i + \alpha_{ii}, & i = j, \\ \alpha_{ij}, & i \ne j, \end{cases} \tag{4.66}$$

where $\varepsilon_i > 0$ and $\alpha_{ij} \triangleq \max\{0, \sup_{x \in \mathcal{D}} \xi_{ij}(x)\}, i, j = 1, ..., q$, then $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector hybrid supply rate $(S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}})) \triangleq \left([s_{\mathrm{c}1}(u_{\mathrm{c}1}, y_{\mathrm{c}1}), ..., s_{\mathrm{c}q}(u_{\mathrm{c}q}, y_{\mathrm{c}q})]^{\mathrm{T}}, [s_{\mathrm{d}1}(u_{\mathrm{d}1}, y_{\mathrm{d}1}), ..., s_{\mathrm{d}q}(u_{\mathrm{d}q}, y_{\mathrm{d}q})]^{\mathrm{T}}\right)$ and with vector storage function $V_{\mathrm{s}}(x) \triangleq [v_{\mathrm{s}1}(x_1), ..., v_{\mathrm{s}q}(x_q)]^{\mathrm{T}}$, $x \in \mathcal{D}$.

Proof. Since each disconnected impulsive subsystem $\mathcal{G}_i$ of $\mathcal{G}$ is exponentially dissipative with respect to the hybrid supply rate $s_{\mathrm{c}i}(u_{\mathrm{c}i}, y_{\mathrm{c}i})$, $i = 1, ..., q$, it follows from (4.62)–(4.65) that, for all $u_{\mathrm{c}i} \in \mathcal{U}_{\mathrm{c}i}$ and $i = 1, ..., q$,

$$\dot{v}_{\mathrm{s}i}(x_i(t)) = v'_{\mathrm{s}i}(x_i(t))[f_{\mathrm{c}i}(x_i(t)) + \mathcal{I}_{\mathrm{c}i}(x(t)) + G_{\mathrm{c}i}(x_i(t))u_{\mathrm{c}i}(t)]$$

$$\leq -\varepsilon_i v_{\mathrm{s}i}(x_i(t)) + s_{\mathrm{c}i}(u_{\mathrm{c}i}(t), y_{\mathrm{c}i}(t)) + \sum_{j=1}^{q} \xi_{ij}(x(t)) v_{\mathrm{s}j}(x_j(t))$$

$$\leq -\varepsilon_i v_{\mathrm{s}i}(x_i(t)) + s_{\mathrm{c}i}(u_{\mathrm{c}i}(t), y_{\mathrm{c}i}(t)) + \sum_{j=1}^{q} \alpha_{ij} v_{\mathrm{s}j}(x_j(t)), \quad t_k < t \leq t_{k+1} \tag{4.67}$$

and

$$\begin{aligned} v_{\mathrm{s}i}(x_i(t_k) + f_{\mathrm{d}i}(x_i(t_k)) + \mathcal{I}_{\mathrm{d}i}(x(t_k)) + G_{\mathrm{d}i}(x_i(t_k)) u_{\mathrm{d}i}(t_k)) & \\ \leq v_{\mathrm{s}i}(x_i(t_k) + f_{\mathrm{d}i}(x_i(t_k)) + G_{\mathrm{d}i}(x_i(t_k)) u_{\mathrm{d}i}(t_k)) & \\ \leq v_{\mathrm{s}i}(x_i(t_k)) + s_{\mathrm{d}i}(u_{\mathrm{d}i}(t_k), y_{\mathrm{d}i}(t_k)), \quad k \in \overline{\mathbb{Z}}_+. & \end{aligned} \tag{4.68}$$

Now, the result follows from Proposition 4.3 by noting that for all subsystems $\mathcal{G}_i$ of $\mathcal{G}$,

$$\dot{V}_{\mathrm{s}}(x(t)) \leq\leq W V_{\mathrm{s}}(x(t)) + S_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t)), \quad t_k < t \leq t_{k+1}, \quad u_{\mathrm{c}} \in \mathcal{U}_{\mathrm{c}}, \tag{4.69}$$

$$V_{\mathrm{s}}(x(t_k) + F_{\mathrm{d}}(x(t_k), u_{\mathrm{d}}(t_k))) \leq\leq V_{\mathrm{s}}(x(t_k)) + S_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)), \; k \in \overline{\mathbb{Z}}_+, \quad u_{\mathrm{d}} \in \mathcal{U}_{\mathrm{d}}, \tag{4.70}$$

where W is essentially nonnegative and, by assumption, semistable (respectively, asymptotically stable), $V_{\mathrm{s}}(x) \triangleq [v_{\mathrm{s}1}(x_1), \ldots, v_{\mathrm{s}q}(x_q)]^{\mathrm{T}}$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$. □

4.4. Extended Kalman-Yakubovich-Popov Conditions for Large-Scale Impulsive Dynamical Systems

In this section, we show that vector dissipativeness (respectively, exponential vector dissipativeness) of a large-scale impulsive dynamical system $\mathcal{G}$ of the form (4.15)–(4.18) can be characterized in terms of the local subsystem functions $f_{\mathrm{c}i}(\cdot)$, $G_{\mathrm{c}i}(\cdot)$, $h_{\mathrm{c}i}(\cdot)$, $J_{\mathrm{c}i}(\cdot)$, $f_{\mathrm{d}i}(\cdot)$, $G_{\mathrm{d}i}(\cdot)$, $h_{\mathrm{d}i}(\cdot)$, and $J_{\mathrm{d}i}(\cdot)$, along with the interconnection structures $\mathcal{I}_{\mathrm{c}i}(\cdot)$ and $\mathcal{I}_{\mathrm{d}i}(\cdot)$ for $i = 1, \ldots, q$. For the results in this section we consider the special case of dissipative systems with quadratic vector hybrid supply rates and set $\mathcal{D} = \mathbb{R}^n$, $\mathcal{U}_{\mathrm{c}i} = \mathbb{R}^{m_{\mathrm{c}i}}$, $\mathcal{U}_{\mathrm{d}i} = \mathbb{R}^{m_{\mathrm{d}i}}$, $\mathcal{Y}_{\mathrm{c}i} = \mathbb{R}^{l_{\mathrm{c}i}}$, and $\mathcal{Y}_{\mathrm{d}i} = \mathbb{R}^{l_{\mathrm{d}i}}$. Furthermore, we assume that $\mathcal{Z} = \mathcal{Z}_x \times \mathbb{R}^{m_{\mathrm{c}}}$, where $\mathcal{Z}_x \subset \mathcal{D}$, so that resetting occurs only when $x(t)$ intersects $\mathcal{Z}_x$. Specifically, let

$R_{ci} \in \mathbb{S}^{m_{ci}}$, $S_{ci} \in \mathbb{R}^{l_{ci}\times m_{ci}}$, $Q_{ci} \in \mathbb{S}^{l_{ci}}$, $R_{di} \in \mathbb{S}^{m_{di}}$, $S_{di} \in \mathbb{R}^{l_{di}\times m_{di}}$, and $Q_{di} \in \mathbb{S}^{l_{di}}$ be given and assume $S_c(u_c, y_c)$ is such that $s_{ci}(u_{ci}, y_{ci}) = y_{ci}^T Q_{ci} y_{ci} + 2y_{ci}^T S_{ci} u_{ci} + u_{ci}^T R_{ci} u_{ci}$ and $S_d(u_d, y_d)$ is such that $s_{di}(u_{di}, y_{di}) = y_{di}^T Q_{di} y_{di} + 2y_{di}^T S_{di} u_{di} + u_{di}^T R_{di} u_{di}$, $i = 1, ..., q$. Furthermore, for the remainder of this section we assume that there exists a continuously differentiable vector storage function $V_s(x)$, $x \in \mathbb{R}^n$, for the large-scale impulsive dynamical system $\mathcal{G}$.

For the statement of the next result recall that $x = [x_1^T, ..., x_q^T]^T$, $u_c = [u_{c1}^T, ..., u_{cq}^T]^T$, $y_c = [y_{c1}^T, ..., y_{cq}^T]^T$, $u_d = [u_{d1}^T, ..., u_{dq}^T]^T$, $y_d = [y_{d1}^T, ..., y_{dq}^T]^T$, $x_i \in \mathbb{R}^{n_i}$, $u_{ci} \in \mathbb{R}^{m_{ci}}$, $y_{ci} \in \mathbb{R}^{l_{ci}}$, $u_{di} \in \mathbb{R}^{m_{di}}$, $y_{di} \in \mathbb{R}^{l_{di}}$, $i = 1, ..., q$, $\sum_{i=1}^q n_i = n$, $\sum_{i=1}^q m_{ci} = m_c$, $\sum_{i=1}^q m_{di} = m_d$, $\sum_{i=1}^q l_{ci} = l_c$, and $\sum_{i=1}^q l_{di} = l_d$. Furthermore, for (4.15)–(4.18) define $\mathcal{F}_c : \mathbb{R}^n \to \mathbb{R}^n$, $G_c : \mathbb{R}^n \to \mathbb{R}^{n\times m_c}$, $h_c : \mathbb{R}^n \to \mathbb{R}^{l_c}$, $J_c : \mathbb{R}^n \to \mathbb{R}^{l_c\times m_c}$, $\mathcal{F}_d : \mathbb{R}^n \to \mathbb{R}^n$, $G_d : \mathbb{R}^n \to \mathbb{R}^{n\times m_d}$, $h_d : \mathbb{R}^n \to \mathbb{R}^{l_d}$, and $J_d : \mathbb{R}^n \to \mathbb{R}^{l_d\times m_d}$ by $\mathcal{F}_c(x) \triangleq [\mathcal{F}_{c1}^T(x), ..., \mathcal{F}_{cq}^T(x)]^T$, $\mathcal{F}_d(x) \triangleq [\mathcal{F}_{d1}^T(x), ..., \mathcal{F}_{dq}^T(x)]^T$, where $\mathcal{F}_{ci}(x) \triangleq f_{ci}(x_i) + \mathcal{I}_{ci}(x)$, $\mathcal{F}_{di}(x) \triangleq f_{di}(x_i) + \mathcal{I}_{di}(x)$, $i = 1, ..., q$, $G_c(x) \triangleq \mathrm{diag}[G_{c1}(x_1), ..., G_{cq}(x_q)]$, $G_d(x) \triangleq \mathrm{diag}[G_{d1}(x_1), ..., G_{dq}(x_q)]$, $h_c(x) \triangleq [h_{c1}^T(x_1), ..., h_{cq}^T(x_q)]^T$, $h_d(x) \triangleq [h_{d1}^T(x_1), ..., h_{dq}^T(x_q)]^T$, $J_c(x) \triangleq \mathrm{diag}[J_{c1}(x_1), ..., J_{cq}(x_q)]$, and $J_d(x) \triangleq \mathrm{diag}[J_{d1}(x_1), ..., J_{dq}(x_q)]$. Moreover, for all $i = 1, ..., q$, define $\hat{R}_{ci} \in \mathbb{S}^{m_c}$, $\hat{S}_{ci} \in \mathbb{R}^{l_c\times m_c}$, $\hat{Q}_{ci} \in \mathbb{S}^{l_c}$, $\hat{R}_{di} \in \mathbb{S}^{m_d}$, $\hat{S}_{di} \in \mathbb{R}^{l_d\times m_d}$, and $\hat{Q}_{di} \in \mathbb{S}^{l_d}$ such that each of these block matrices consists of zero blocks except, respectively, for the matrix blocks $R_{ci} \in \mathbb{S}^{m_{ci}}$, $S_{ci} \in \mathbb{R}^{l_{ci}\times m_{ci}}$, $Q_{ci} \in \mathbb{S}^{l_{ci}}$, $R_{di} \in \mathbb{S}^{m_{di}}$, $S_{di} \in \mathbb{R}^{l_{di}\times m_{di}}$, and $Q_{di} \in \mathbb{S}^{l_{di}}$ on (i, i) position. Finally, we introduce a more general definition of vector dissipativity involving an underlying nonlinear comparison system.

Definition 4.9. The large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) is *vector dissipative* (respectively, *exponentially vector dissipative*) *with respect to the vector hybrid supply rate* $(S_c(u_c, y_c), S_d(u_d, y_d))$ if there exist a continuous, nonnegative definite vector function $V_s = [v_{s1}, ..., v_{sq}]^T : \mathcal{D} \to \overline{\mathbb{R}}_+^q$, called a *vector storage function*, and a class $\mathcal{W}$ function $w_c : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ such that $V_s(0) = 0$, $w_c(0) = 0$, the zero solution $r(t) \equiv 0$ to the comparison system

$$\dot{r}(t) = w_{\mathrm{c}}(r(t)), \quad r(t_0) = r_0, \quad t \geq t_0, \tag{4.71}$$

is Lyapunov (respectively, asymptotically) stable, and the *vector hybrid dissipation inequality*

$$\begin{aligned} V_{\mathrm{s}}(x(T)) \leq\leq\ & V_{\mathrm{s}}(x(t_0)) + \int_{t_0}^{T} w_{\mathrm{c}}(V_{\mathrm{s}}(x(t)))\mathrm{d}t + \int_{t_0}^{T} S_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))\mathrm{d}t \\ & + \sum_{k\in\mathbb{Z}_{[t_0,T)}} S_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)), \quad T \geq t_0, \end{aligned} \tag{4.72}$$

is satisfied, where $x(t)$, $t \geq t_0$, is the solution to (4.15)–(4.18) with $u_{\mathrm{c}} \in \mathcal{U}_{\mathrm{c}}$ and $u_{\mathrm{d}} \in \mathcal{U}_{\mathrm{d}}$. The large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) is *vector lossless with respect to the vector hybrid supply rate* $(S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$ if the vector hybrid dissipation inequality is satisfied as an equality with the zero solution $r(t) \equiv 0$ to (4.71) being Lyapunov stable.

Remark 4.4. If $\mathcal{G}$ is completely reachable and $V_{\mathrm{s}}(\cdot)$ is continuously differentiable, then (4.72) can be equivalently written as

$$\dot{V}_{\mathrm{s}}(x(t)) \leq\leq w_{\mathrm{c}}(V_{\mathrm{s}}(x(t))) + S_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t)), \quad t_k < t \leq t_{k+1}, \tag{4.73}$$

$$V_{\mathrm{s}}(x(t_k) + F_{\mathrm{d}}(x(t_k), u_{\mathrm{d}}(t_k))) \leq\leq V_{\mathrm{s}}(x(t_k)) + S_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)), \quad k \in \overline{\mathbb{Z}}_+, \tag{4.74}$$

with $u_{\mathrm{c}} \in \mathcal{U}_{\mathrm{c}}$ and $u_{\mathrm{d}} \in \mathcal{U}_{\mathrm{d}}$.

Remark 4.5. If in Definition 4.9 the function $w_{\mathrm{c}} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ is such that $w_{\mathrm{c}}(r) = Wr$, where $W \in \mathbb{R}^{q\times q}$, then W is essentially nonnegative and Definition 4.9 collapses to Definition 4.7.

Theorem 4.9. Consider the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18). Let $R_{\mathrm{c}i} \in \mathbb{S}^{m_{\mathrm{c}i}}$, $S_{\mathrm{c}i} \in \mathbb{R}^{l_{\mathrm{c}i}\times m_{\mathrm{c}i}}$, $Q_{\mathrm{c}i} \in \mathbb{S}^{l_{\mathrm{c}i}}$, $R_{\mathrm{d}i} \in \mathbb{S}^{m_{\mathrm{d}i}}$, $S_{\mathrm{d}i} \in \mathbb{R}^{l_{\mathrm{d}i}\times m_{\mathrm{d}i}}$, and $Q_{\mathrm{d}i} \in \mathbb{S}^{l_{\mathrm{d}i}}, i = 1, ..., q$. Then $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the quadratic hybrid supply rate $(S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$, where $s_{\mathrm{c}i}(u_{\mathrm{c}i}, y_{\mathrm{c}i}) =$

$y_{\mathrm{c}i}^{\mathrm{T}}Q_{\mathrm{c}i}y_{\mathrm{c}i}+2y_{\mathrm{c}i}^{\mathrm{T}}S_{\mathrm{c}i}u_{\mathrm{c}i}+u_{\mathrm{c}i}^{\mathrm{T}}R_{\mathrm{c}i}u_{\mathrm{c}i}$ and $s_{\mathrm{d}i}(u_{\mathrm{d}i}, y_{\mathrm{d}i}) = y_{\mathrm{d}i}^{\mathrm{T}}Q_{\mathrm{d}i}y_{\mathrm{d}i}+2y_{\mathrm{d}i}^{\mathrm{T}}S_{\mathrm{d}i}u_{\mathrm{d}i}+u_{\mathrm{d}i}^{\mathrm{T}}R_{\mathrm{d}i}u_{\mathrm{d}i}$, $i = 1, ..., q$, if there exist functions $V_{\mathrm{s}} = [v_{\mathrm{s}1}, ..., v_{\mathrm{s}q}]^{\mathrm{T}} : \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$, $w_{\mathrm{c}} = [w_{\mathrm{c}1}, ..., w_{\mathrm{c}q}]^{\mathrm{T}} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$, $\ell_{\mathrm{c}i} : \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{c}i}}$, $\mathcal{Z}_{\mathrm{c}i} : \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{c}i} \times m_{\mathrm{c}}}$, $\ell_{\mathrm{d}i} : \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{d}i}}$, $\mathcal{Z}_{\mathrm{d}i} : \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{d}i} \times m_{\mathrm{d}}}$, $P_{1i} : \mathbb{R}^n \to \mathbb{R}^{1 \times m_{\mathrm{d}}}$, and $P_{2i} : \mathbb{R}^n \to \mathbb{N}^{m_{\mathrm{d}}}$ such that $v_{\mathrm{s}i}(\cdot)$ is continuously differentiable, $v_{\mathrm{s}i}(0) = 0$, $i = 1, ..., q$, $w_{\mathrm{c}} \in \mathcal{W}$, $w_{\mathrm{c}}(0) = 0$, the zero solution $r(t) \equiv 0$ to (4.71) is Lyapunov (respectively, asymptotically) stable,

$$v_{\mathrm{s}i}(x + \mathcal{F}_{\mathrm{d}}(x) + G_{\mathrm{d}}(x)u_{\mathrm{d}}) = v_{\mathrm{s}i}(x + \mathcal{F}_{\mathrm{d}}(x)) + P_{1i}(x)u_{\mathrm{d}} + u_{\mathrm{d}}^{\mathrm{T}}P_{2i}(x)u_{\mathrm{d}}, \quad x \in \mathcal{Z}_x, \quad u_{\mathrm{d}} \in \mathbb{R}^{m_{\mathrm{d}}}, \tag{4.75}$$

and, for all $i = 1, ..., q$,

$$0 = v_{\mathrm{s}i}'(x)\mathcal{F}_{\mathrm{c}}(x) - h_{\mathrm{c}}^{\mathrm{T}}(x)\hat{Q}_{\mathrm{c}i}h_{\mathrm{c}}(x) - w_{\mathrm{c}i}(V_{\mathrm{s}}(x)) + \ell_{\mathrm{c}i}^{\mathrm{T}}(x)\ell_{\mathrm{c}i}(x), \quad x \notin \mathcal{Z}_x, \tag{4.76}$$

$$0 = \tfrac{1}{2}v_{\mathrm{s}i}'(x)G_{\mathrm{c}}(x) - h_{\mathrm{c}}^{\mathrm{T}}(x)(\hat{S}_{\mathrm{c}i} + \hat{Q}_{\mathrm{c}i}J_{\mathrm{c}}(x)) + \ell_{\mathrm{c}i}^{\mathrm{T}}(x)\mathcal{Z}_{\mathrm{c}i}(x), \quad x \notin \mathcal{Z}_x, \tag{4.77}$$

$$0 = \hat{R}_{\mathrm{c}i} + J_{\mathrm{c}}^{\mathrm{T}}(x)\hat{S}_{\mathrm{c}i} + \hat{S}_{\mathrm{c}i}^{\mathrm{T}}J_{\mathrm{c}}(x) + J_{\mathrm{c}}^{\mathrm{T}}(x)\hat{Q}_{\mathrm{c}i}J_{\mathrm{c}}(x) - \mathcal{Z}_{\mathrm{c}i}^{\mathrm{T}}(x)\mathcal{Z}_{\mathrm{c}i}(x), \quad x \notin \mathcal{Z}_x, \tag{4.78}$$

$$0 = v_{\mathrm{s}i}(x + \mathcal{F}_{\mathrm{d}}(x)) - h_{\mathrm{d}}^{\mathrm{T}}(x)\hat{Q}_{\mathrm{d}i}h_{\mathrm{d}}(x) - v_{\mathrm{s}i}(x) + \ell_{\mathrm{d}i}^{\mathrm{T}}(x)\ell_{\mathrm{d}i}(x), \quad x \in \mathcal{Z}_x, \tag{4.79}$$

$$0 = \tfrac{1}{2}P_{1i}(x) - h_{\mathrm{d}}^{\mathrm{T}}(x)(\hat{S}_{\mathrm{d}i} + \hat{Q}_{\mathrm{d}i}J_{\mathrm{d}}(x)) + \ell_{\mathrm{d}i}^{\mathrm{T}}(x)\mathcal{Z}_{\mathrm{d}i}(x), \quad x \in \mathcal{Z}_x, \tag{4.80}$$

$$0 = \hat{R}_{\mathrm{d}i} + J_{\mathrm{d}}^{\mathrm{T}}(x)\hat{S}_{\mathrm{d}i} + \hat{S}_{\mathrm{d}i}^{\mathrm{T}}J_{\mathrm{d}}(x) + J_{\mathrm{d}}^{\mathrm{T}}(x)\hat{Q}_{\mathrm{d}i}J_{\mathrm{d}}(x) - P_{2i}(x) - \mathcal{Z}_{\mathrm{d}i}^{\mathrm{T}}(x)\mathcal{Z}_{\mathrm{d}i}(x), \quad x \in \mathcal{Z}_x. \tag{4.81}$$

Proof. Suppose that there exist functions $v_{\mathrm{s}i} : \mathbb{R}^n \to \overline{\mathbb{R}}_+$, $\ell_{\mathrm{c}i} : \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{c}i}}$, $\mathcal{Z}_{\mathrm{c}i} : \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{c}i} \times m_{\mathrm{c}}}$, $\ell_{\mathrm{d}i} : \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{d}i}}$, $\mathcal{Z}_{\mathrm{d}i} : \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{d}i} \times m_{\mathrm{d}}}$, $w_{\mathrm{c}} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$, $P_{1i} : \mathbb{R}^n \to \mathbb{R}^{1 \times m_{\mathrm{d}}}$, and $P_{2i} : \mathbb{R}^n \to \mathbb{N}^{m_{\mathrm{d}}}$ such that $v_{\mathrm{s}i}(\cdot)$ is continuously differentiable and nonnegative-definite, $v_{\mathrm{s}i}(0) = 0$, $i = 1, ..., q$, $w_{\mathrm{c}}(0) = 0$, $w_{\mathrm{c}} \in \mathcal{W}$, the zero solution $r(t) \equiv 0$ to (4.71) is Lyapunov (respectively, asymptotically) stable, and (4.75)–(4.81) are satisfied. Then for any $u_{\mathrm{c}} \in \mathcal{U}_{\mathrm{c}}$, $t, \hat{t} \in \mathbb{R}$, $t_k < t \leq \hat{t} \leq t_{k+1}$, $k \in \overline{\mathbb{Z}}_+$, and $i = 1, ..., q$, it follows from (4.76)–(4.78) that

$$\int_t^{\hat{t}} s_{\mathrm{c}i}(u_{\mathrm{c}i}(\sigma), y_{\mathrm{c}i}(\sigma))\mathrm{d}\sigma = \int_t^{\hat{t}} [u_{\mathrm{c}}^{\mathrm{T}}(\sigma)\hat{R}_{\mathrm{c}i}u_{\mathrm{c}}(\sigma) + 2y_{\mathrm{c}}^{\mathrm{T}}(\sigma)\hat{S}_{\mathrm{c}i}u_{\mathrm{c}}(\sigma) + y_{\mathrm{c}}^{\mathrm{T}}(\sigma)\hat{Q}_{\mathrm{c}i}y_{\mathrm{c}}(\sigma)]\mathrm{d}\sigma$$

$$= \int_t^{\hat{t}} [h_{\mathrm{c}}^{\mathrm{T}}(x(\sigma))\hat{Q}_{\mathrm{c}i}h_{\mathrm{c}}(x(\sigma)) + 2h_{\mathrm{c}}^{\mathrm{T}}(x(\sigma))(\hat{S}_{\mathrm{c}i} + \hat{Q}_{\mathrm{c}i}J_{\mathrm{c}}(x(\sigma)))u_{\mathrm{c}}(\sigma)$$

$$
\begin{aligned}
&+u_{\mathrm{c}}^{\mathrm{T}}(\sigma)(J_{\mathrm{c}}^{\mathrm{T}}(x(\sigma))\hat{Q}_{\mathrm{c}i}J_{\mathrm{c}}(x(\sigma))+J_{\mathrm{c}}^{\mathrm{T}}(x(\sigma))\hat{S}_{\mathrm{c}i}+\hat{S}_{\mathrm{c}i}^{\mathrm{T}}J_{\mathrm{c}}(x(\sigma))\\
&+\hat{R}_{\mathrm{c}i})u_{\mathrm{c}}(\sigma)]\mathrm{d}\sigma\\
=&\int_t^{\hat{t}}[v_{\mathrm{s}i}'(x(\sigma))(\mathcal{F}_{\mathrm{c}}(x(\sigma))+G_{\mathrm{c}}(x(\sigma))u_{\mathrm{c}}(\sigma))+\ell_{\mathrm{c}i}^{\mathrm{T}}(x(\sigma))\ell_{\mathrm{c}i}(x(\sigma))\\
&+2\ell_{\mathrm{c}i}^{\mathrm{T}}(x(\sigma))\mathcal{Z}_{\mathrm{c}i}(x(\sigma))u_{\mathrm{c}}(\sigma)+u_{\mathrm{c}}^{\mathrm{T}}(\sigma)\mathcal{Z}_{\mathrm{c}i}^{\mathrm{T}}(x(\sigma))\mathcal{Z}_{\mathrm{c}i}(x(\sigma))u_{\mathrm{c}}(\sigma)\\
&-w_{\mathrm{c}i}(V_{\mathrm{s}}(x(\sigma)))]\mathrm{d}\sigma\\
=&\int_t^{\hat{t}}[\dot{v}_{\mathrm{s}i}(x(\sigma))+[\ell_{\mathrm{c}i}(x(\sigma))+\mathcal{Z}_{\mathrm{c}i}(x(\sigma))u_{\mathrm{c}}(\sigma)]^{\mathrm{T}}[\ell_{\mathrm{c}i}(x(\sigma))\\
&+\mathcal{Z}_{\mathrm{c}i}(x(\sigma))u_{\mathrm{c}}(\sigma)]-w_{\mathrm{c}i}(V_{\mathrm{s}}(x(\sigma)))]\mathrm{d}\sigma\\
\geq&\ v_{\mathrm{s}i}(x(\hat{t}))-v_{\mathrm{s}i}(x(t))-\int_t^{\hat{t}}w_{\mathrm{c}i}(V_{\mathrm{s}}(x(\sigma)))\mathrm{d}\sigma, \qquad (4.82)
\end{aligned}
$$

where $x(\sigma)$, $\sigma\in(t_k,t_{k+1}]$, satisfies (4.15).

Next, for any $u_{\mathrm{d}}\in\mathbb{R}^{m_{\mathrm{d}}}$, $t_k\in\mathbb{R}$, and $k\in\overline{\mathbb{Z}}_+$, it follows from (4.75), (4.79)–(4.81) that

$$
\begin{aligned}
v_{\mathrm{s}i}(x+\mathcal{F}_{\mathrm{d}}(x)+G_{\mathrm{d}}(x)u_{\mathrm{d}})-v_{\mathrm{s}i}(x)=&\ v_{\mathrm{s}i}(x+\mathcal{F}_{\mathrm{d}}(x))-v_{\mathrm{s}i}(x)+P_{1i}(x)u_{\mathrm{d}}+u_{\mathrm{d}}^{\mathrm{T}}P_{2i}(x)u_{\mathrm{d}}\\
=&\ h_{\mathrm{d}}^{\mathrm{T}}(x)\hat{Q}_{\mathrm{d}i}h_{\mathrm{d}}(x)-\ell_{\mathrm{d}i}^{\mathrm{T}}(x)\ell_{\mathrm{d}i}(x)+2[h_{\mathrm{d}}^{\mathrm{T}}(x)(\hat{Q}_{\mathrm{d}i}J_{\mathrm{d}}(x)\\
&+\hat{S}_{\mathrm{d}i})-\ell_{\mathrm{d}i}^{\mathrm{T}}(x)\mathcal{Z}_{\mathrm{d}i}(x)]u_{\mathrm{d}}+u_{\mathrm{d}}^{\mathrm{T}}[\hat{R}_{\mathrm{d}i}+\hat{S}_{\mathrm{d}i}^{\mathrm{T}}J_{\mathrm{d}}(x)\\
&+J_{\mathrm{d}}^{\mathrm{T}}(x)\hat{S}_{\mathrm{d}i}+J_{\mathrm{d}}^{\mathrm{T}}(x)\hat{Q}_{\mathrm{d}i}J_{\mathrm{d}}(x)-\mathcal{Z}_{\mathrm{d}i}^{\mathrm{T}}(x)\mathcal{Z}_{\mathrm{d}i}(x)]u_{\mathrm{d}}\\
=&\ s_{\mathrm{d}i}(u_{\mathrm{d}i},y_{\mathrm{d}i})-[\ell_{\mathrm{d}i}(x)+\mathcal{Z}_{\mathrm{d}i}(x)u_{\mathrm{d}}]^{\mathrm{T}}[\ell_{\mathrm{d}i}(x)+\mathcal{Z}_{\mathrm{d}i}(x)u_{\mathrm{d}}]\\
\leq&\ s_{\mathrm{d}i}(u_{\mathrm{d}i},y_{\mathrm{d}i}). \qquad (4.83)
\end{aligned}
$$

Now, using (4.82) and (4.83) the result is immediate from Remark 4.4 with vector storage function $V_{\mathrm{s}}(x)=[v_{\mathrm{s}1}(x),...,v_{\mathrm{s}q}(x)]^{\mathrm{T}}$, $x\in\mathbb{R}^n$. □

Using (4.76)–(4.81) it follows that for $T\geq t_0\geq 0$, $k\in\mathbb{Z}_{[t_0,T)}$, and $i=1,...,q$,

$$
\begin{aligned}
&\int_{t_0}^T s_{\mathrm{c}i}(u_{\mathrm{c}i}(t),y_{\mathrm{c}i}(t))\mathrm{d}t+\int_{t_0}^T w_{\mathrm{c}i}(V_{\mathrm{s}}(x(t)))\mathrm{d}t+\sum_{k\in\mathbb{Z}_{[t_0,T)}}s_{\mathrm{d}i}(u_{\mathrm{d}}(t_k),y_{\mathrm{d}}(t_k))\\
=&\ v_{\mathrm{s}i}(x(T))-v_{\mathrm{s}i}(x(t_0))+\int_{t_0}^T[\ell_{\mathrm{c}i}(x(t))+\mathcal{Z}_{\mathrm{c}i}(x(t))u_{\mathrm{c}}(t)]^{\mathrm{T}}[\ell_{\mathrm{c}i}(x(t))+\mathcal{Z}_{\mathrm{c}i}(x(t))u_{\mathrm{c}}(t)]\mathrm{d}t\\
&+\sum_{k\in\mathbb{Z}_{[t_0,T)}}[\ell_{\mathrm{d}i}(x(t_k))+\mathcal{Z}_{\mathrm{d}i}(x(t_k))u_{\mathrm{d}}(t_k)]^{\mathrm{T}}[\ell_{\mathrm{d}i}(x(t_k))+\mathcal{Z}_{\mathrm{d}i}(x(t_k))u_{\mathrm{d}}(t_k)], \qquad (4.84)
\end{aligned}
$$

where $V_{\rm s}(x) = [v_{\rm s1}(x), ..., v_{{\rm s}q}(x)]^{\rm T}$, $x \in \mathbb{R}^n$, which can be interpreted as a *generalized energy* balance equation for the ith impulsive subsystem of $\mathcal{G}$ where $v_{{\rm s}i}(x(T)) - v_{{\rm s}i}(x(t_0))$ is the stored or accumulated generalized energy of the ith impulsive subsystem; the two path dependent terms on the left are, respectively, the external supplied energy to the ith subsystem over the continuous-time dynamics and the energy gained over the continuous-time dynamics by the ith subsystem from the net energy flow between all subsystems due to subsystem coupling; the last discrete term on the left corresponds to the external supplied energy to the ith subsystem at the resetting instants; the second path-dependent term on the right corresponds to the dissipated energy from the ith impulsive subsystem over the continuous-time dynamics; and the last discrete term on the right corresponds to the dissipated energy from the ith impulsive subsystem at the resetting instants.

Equivalently, (4.84) can be rewritten as

$$\begin{aligned} \dot{v}_{{\rm s}i}(x(t)) = s_{{\rm c}i}(u_{{\rm c}i}(t), y_{{\rm c}i}(t)) + w_{{\rm c}i}(V_{\rm s}(x(t))) - [\ell_{{\rm c}i}(x(t)) + \mathcal{Z}_{{\rm c}i}(x(t))u_{\rm c}(t)]^{\rm T}[\ell_{{\rm c}i}(x(t)) \\ +\mathcal{Z}_{{\rm c}i}(x(t))u_{\rm c}(t)], \quad t_k < t \le t_{k+1}, \quad i = 1, ..., q, \qquad (4.85) \end{aligned}$$

$$\begin{aligned} v_{{\rm s}i}(x(t_k) + \mathcal{F}_{\rm d}(x(t_k)) + G_{\rm d}(x(t_k))u_{\rm d}(t_k)) - v_{{\rm s}i}(x(t_k)) = s_{{\rm d}i}(u_{\rm d}(t_k), y_{\rm d}(t_k)) - [\ell_{{\rm d}i}(x(t_k)) \\ +\mathcal{Z}_{{\rm d}i}(x(t_k))u_{\rm d}(t_k)]^{\rm T}[\ell_{{\rm d}i}(x(t_k)) + \mathcal{Z}_{{\rm d}i}(x(t_k))u_{\rm d}(t_k)], \quad k \in \overline{\mathbb{Z}}_+, \qquad (4.86) \end{aligned}$$

which yields a set of q generalized energy conservation equations for the large-scale impulsive dynamical system $\mathcal{G}$. Specifically, (4.85) shows that the rate of change in generalized energy, or generalized power, over the time interval $t \in (t_k, t_{k+1}]$ for the ith subsystem of $\mathcal{G}$ is equal to the generalized system power input to the ith subsystem plus the instantaneous rate of energy supplied to the ith subsystem from the net energy flow between all subsystems minus the internal generalized system power dissipated from the ith subsystem; while (4.86) shows that the change of energy at the resetting times t_k, $k \in \overline{\mathbb{Z}}_+$, is equal to the external generalized system supplied energy at the resetting times minus the generalized dissipated energy at the resetting times.

Remark 4.6. Note that if $\mathcal{G}$ with $(u_{\rm c}(t), u_{\rm d}(t_k)) \equiv (0, 0)$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the quadratic hybrid supply rate where $Q_{ci} \leq 0$, $Q_{di} \leq 0$, $i = 1, \cdots, q$, then it follows from the vector hybrid dissipation inequality that for all $k \in \overline{\mathbb{Z}}_+$,

$$\dot{V}_s(x(t)) \leq\leq w_c(V_s(x(t))) + S_c(0, y_c(t)) \leq\leq w_c(V_s(x(t))), \quad t_k < t \leq t_{k+1}, \tag{4.87}$$

$$V_s(x(t_k) + \mathcal{F}_d(x(t_k))) - V_s(x(t_k)) \leq\leq S_d(0, y_d(t_k)) \leq\leq 0, \tag{4.88}$$

where $S_c(0, y_c) = [s_{c1}(0, y_{c1}), \ldots, s_{cq}(0, y_{cq})]^T$, $S_d(0, y_d) = [s_{d1}(0, y_{d1}), \ldots, s_{dq}(0, y_{dq})]^T$, $s_{ci}(0, y_{ci}(t)) = y_{ci}^T(t) Q_{ci} y_{ci}(t) \leq 0$, $s_{di}(0, y_{di}(t_k)) = y_{di}^T(t_k) Q_{di} y_{di}(t_k) \leq 0$, $t_k < t \leq t_{k+1}$, $k \in \overline{\mathbb{Z}}_+$, $i = 1, ..., q$, and $x(t)$, $t \geq t_0$, is the solution to (4.15)–(4.18) with $(u_c(t), u_d(t_k)) \equiv (0, 0)$. If, in addition, there exists $p \in \mathbb{R}_+^q$ such that $p^T V_s(x)$, $x \in \mathbb{R}^n$, is positive definite, then it follows from Theorem 4.1 that the undisturbed $((u_c(t), u_d(t_k)) \equiv (0, 0))$ large-scale impulsive dynamical system (4.15)–(4.18) is Lyapunov (respectively, asymptotically) stable.

Next, we consider a specialization of Theorem 4.9 wherein $\mathcal{G}$ is a linear impulsive dynamical system. Specifically, we assume that $w_c \in \mathcal{W}$ is linear so that $w_c(r) = Wr$, where $W \in \mathbb{R}^{q \times q}$ is essentially nonnegative, and consider the large-scale linear impulsive dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = A_c x(t) + B_c u_c(t), \quad x(t) \notin \mathcal{Z}_x, \tag{4.89}$$

$$\Delta x(t) = (A_d - I_n) x(t) + B_d u_d(t), \quad x(t) \in \mathcal{Z}_x, \tag{4.90}$$

$$y_c(t) = C_c x(t) + D_c u_c(t), \quad x(t) \notin \mathcal{Z}_x, \tag{4.91}$$

$$y_d(t) = C_d x(t) + D_d u_d(t), \quad x(t) \in \mathcal{Z}_x, \tag{4.92}$$

where $A_c \in \mathbb{R}^{n \times n}$ and A_c is partitioned as $A_c \triangleq [A_{cij}], i, j = 1, ..., q$, $A_{cij} \in \mathbb{R}^{n_i \times n_j}$, $\sum_{i=1}^q n_i = n$ $B_c = \text{block–diag}[B_{c1}, \ldots, B_{cq}]$, $C_c = \text{block–diag}[C_{c1}, \ldots, C_{cq}]$, $D_c = \text{block–diag}[D_{c1}, \ldots, D_{cq}]$, $B_{ci} \in \mathbb{R}^{n_i \times m_{ci}}$, $C_{ci} \in \mathbb{R}^{l_{ci} \times n_i}$, $D_{ci} \in \mathbb{R}^{l_{ci} \times m_{ci}}$, $A_d \in \mathbb{R}^{n \times n}$ and A_d is partitioned as $A_d \triangleq [A_{dij}]$, $i, j = 1, ..., q$, $A_{dij} \in \mathbb{R}^{n_i \times n_j}$, $B_d = \text{block–diag}[B_{d1}, \ldots, B_{dq}]$, $C_d = \text{block–diag}[C_{d1}, ..., C_{dq}]$, $D_d = \text{block–diag}[D_{d1}, ..., D_{dq}]$, $B_{di} \in \mathbb{R}^{n_i \times m_{di}}$, $C_{di} \in \mathbb{R}^{l_{di} \times n_i}$, $D_{di} \in \mathbb{R}^{l_{di} \times m_{di}}$, and $i = 1, ..., q$.

Corollary 4.3. Consider the large-scale linear impulsive dynamical system $\mathcal{G}$ given by (4.89)–(4.92). Let $R_{\mathrm{c}i} \in \mathbb{S}^{m_{\mathrm{c}i}}$, $S_{\mathrm{c}i} \in \mathbb{R}^{l_{\mathrm{c}i}\times m_{\mathrm{c}i}}$, $Q_{\mathrm{c}i} \in \mathbb{S}^{l_{\mathrm{c}i}}$, $R_{\mathrm{d}i} \in \mathbb{S}^{m_{\mathrm{d}i}}$, $S_{\mathrm{d}i} \in \mathbb{R}^{l_{\mathrm{d}i}\times m_{\mathrm{d}i}}$, and $Q_{\mathrm{d}i} \in \mathbb{S}^{l_{\mathrm{d}i}}$, $i = 1, ..., q$. Then $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector hybrid supply rate $(S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$, where $s_{\mathrm{c}i}(u_{\mathrm{c}i}, y_{\mathrm{c}i}) = u_{\mathrm{c}i}^{\mathrm{T}}R_{\mathrm{c}i}u_{\mathrm{c}i} + 2y_{\mathrm{c}i}^{\mathrm{T}}S_{\mathrm{c}i}u_{\mathrm{c}i} + y_{\mathrm{c}i}^{\mathrm{T}}Q_{\mathrm{c}i}y_{\mathrm{c}i}$ and $s_{\mathrm{d}i}(u_{\mathrm{d}i}, y_{\mathrm{d}i}) = u_{\mathrm{d}i}^{\mathrm{T}}R_{\mathrm{d}i}u_{\mathrm{d}i} + 2y_{\mathrm{d}i}^{\mathrm{T}}S_{\mathrm{d}i}u_{\mathrm{d}i} + y_{\mathrm{d}i}^{\mathrm{T}}Q_{\mathrm{d}i}y_{\mathrm{d}i}$, $i = 1, ..., q$, if there exist $W \in \mathbb{R}^{q\times q}$, $P_i \in \mathbb{N}^n$, $L_{\mathrm{c}i} \in \mathbb{R}^{s_{\mathrm{c}i}\times n}$, $Z_{\mathrm{c}i} \in \mathbb{R}^{s_{\mathrm{c}i}\times m_{\mathrm{c}}}$, $L_{\mathrm{d}i} \in \mathbb{R}^{s_{\mathrm{d}i}\times n}$, and $Z_{\mathrm{d}i} \in \mathbb{R}^{s_{\mathrm{d}i}\times m_{\mathrm{d}}}$, $i = 1, ..., q$, such that W is essentially nonnegative and semistable (respectively, asymptotically stable), and, for all $i = 1, ..., q$,

$$0 = x^{\mathrm{T}}(A_{\mathrm{c}}^{\mathrm{T}}P_i + P_iA_{\mathrm{c}} - C_{\mathrm{c}}^{\mathrm{T}}\hat{Q}_{\mathrm{c}i}C_{\mathrm{c}} - \sum_{j=1}^{q} W_{(i,j)}P_j + L_{\mathrm{c}i}^{\mathrm{T}}L_{\mathrm{c}i})x, \quad x \notin \mathcal{Z}_x, \tag{4.93}$$

$$0 = x^{\mathrm{T}}(P_iB_{\mathrm{c}} - C_{\mathrm{c}}^{\mathrm{T}}(\hat{S}_{\mathrm{c}i} + \hat{Q}_{\mathrm{c}i}D_{\mathrm{c}}) + L_{\mathrm{c}i}^{\mathrm{T}}Z_{\mathrm{c}i}), \quad x \notin \mathcal{Z}_x, \tag{4.94}$$

$$0 = \hat{R}_{\mathrm{c}i} + D_{\mathrm{c}}^{\mathrm{T}}\hat{S}_{\mathrm{c}i} + \hat{S}_{\mathrm{c}i}^{\mathrm{T}}D_{\mathrm{c}} + D_{\mathrm{c}}^{\mathrm{T}}\hat{Q}_{\mathrm{c}i}D_{\mathrm{c}} - Z_{\mathrm{c}i}^{\mathrm{T}}Z_{\mathrm{c}i}, \tag{4.95}$$

$$0 = x^{\mathrm{T}}(A_{\mathrm{d}}^{\mathrm{T}}P_iA_{\mathrm{d}} - C_{\mathrm{d}}^{\mathrm{T}}\hat{Q}_{\mathrm{d}i}C_{\mathrm{d}} - P_i + L_{\mathrm{d}i}^{\mathrm{T}}L_{\mathrm{d}i})x, \quad x \in \mathcal{Z}_x, \tag{4.96}$$

$$0 = x^{\mathrm{T}}(A_{\mathrm{d}}^{\mathrm{T}}P_iB_{\mathrm{d}} - C_{\mathrm{d}}^{\mathrm{T}}(\hat{S}_{\mathrm{d}i} + \hat{Q}_{\mathrm{d}i}D_{\mathrm{d}}) + L_{\mathrm{d}i}^{\mathrm{T}}Z_{\mathrm{d}i}), \quad x \in \mathcal{Z}_x, \tag{4.97}$$

$$0 = \hat{R}_{\mathrm{d}i} + D_{\mathrm{d}}^{\mathrm{T}}\hat{S}_{\mathrm{d}i} + \hat{S}_{\mathrm{d}i}^{\mathrm{T}}D_{\mathrm{d}} + D_{\mathrm{d}}^{\mathrm{T}}\hat{Q}_{\mathrm{d}i}D_{\mathrm{d}} - B_{\mathrm{d}}^{\mathrm{T}}P_iB_{\mathrm{d}} - Z_{\mathrm{d}i}^{\mathrm{T}}Z_{\mathrm{d}i}. \tag{4.98}$$

Proof. The proof follows from Theorem 4.9 with $\mathcal{F}_{\mathrm{c}}(x) = A_{\mathrm{c}}x$, $G_{\mathrm{c}}(x) = B_{\mathrm{c}}$, $h_{\mathrm{c}}(x) = C_{\mathrm{c}}x$, $J_{\mathrm{c}}(x) = D_{\mathrm{c}}$, $w_{\mathrm{c}}(r) = Wr$, $\ell_{\mathrm{c}i}(x) = L_{\mathrm{c}i}x$, $\mathcal{Z}_{\mathrm{c}i}(x) = Z_{\mathrm{c}i}$, $\mathcal{F}_{\mathrm{d}}(x) = (A_{\mathrm{d}} - I)x$, $G_{\mathrm{d}}(x) = B_{\mathrm{d}}$, $h_{\mathrm{d}}(x) = C_{\mathrm{d}}x$, $J_{\mathrm{d}}(x) = D_{\mathrm{d}}$, $\ell_{\mathrm{d}i}(x) = L_{\mathrm{d}i}x$, $\mathcal{Z}_{\mathrm{d}i}(x) = Z_{\mathrm{d}i}$, $P_{1i}(x) = 2x^{\mathrm{T}}A_{\mathrm{d}}^{\mathrm{T}}P_iB_{\mathrm{d}}$, $P_{2i}(x) = B_{\mathrm{d}}^{\mathrm{T}}P_iB_{\mathrm{d}}$, and $v_{\mathrm{s}i}(x) = x^{\mathrm{T}}P_ix$, $i = 1, ..., q$. $\square$

Remark 4.7. Note that (4.93)–(4.98) are implied by

$$\begin{bmatrix} \mathcal{A}_{\mathrm{c}i} & \mathcal{B}_{\mathrm{c}i} \\ \mathcal{B}_{\mathrm{c}i}^{\mathrm{T}} & \mathcal{C}_{\mathrm{c}i} \end{bmatrix} = -\begin{bmatrix} L_{\mathrm{c}i}^{\mathrm{T}} \\ Z_{\mathrm{c}i}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} L_{\mathrm{c}i} & Z_{\mathrm{c}i} \end{bmatrix} \leq 0, \tag{4.99}$$

$$\begin{bmatrix} \mathcal{A}_{\mathrm{d}i} & \mathcal{B}_{\mathrm{d}i} \\ \mathcal{B}_{\mathrm{d}i}^{\mathrm{T}} & \mathcal{C}_{\mathrm{d}i} \end{bmatrix} = -\begin{bmatrix} L_{\mathrm{d}i}^{\mathrm{T}} \\ Z_{\mathrm{d}i}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} L_{\mathrm{d}i} & Z_{\mathrm{d}i} \end{bmatrix} \leq 0, \quad i = 1, ..., q, \tag{4.100}$$

where, for all $i = 1, ..., q$,

$$\mathcal{A}_{\mathrm{c}i} = A_{\mathrm{c}}^{\mathrm{T}}P_i + P_iA_{\mathrm{c}} - C_{\mathrm{c}}^{\mathrm{T}}\hat{Q}_{\mathrm{c}i}C_{\mathrm{c}} - \sum_{j=1}^{q} W_{(i,j)}P_j, \tag{4.101}$$

$$\mathcal{B}_{ci} = P_i B_c - C_c^T(\hat{S}_{ci} + \hat{Q}_{ci} D_c), \tag{4.102}$$

$$\mathcal{C}_{ci} = -(\hat{R}_{ci} + D_c^T \hat{S}_{ci} + \hat{S}_{ci}^T D_c + D_c^T \hat{Q}_{ci} D_c), \tag{4.103}$$

$$\mathcal{A}_{di} = A_d^T P_i A_d - C_d^T \hat{Q}_{di} C_d - P_i, \tag{4.104}$$

$$\mathcal{B}_{di} = A_d^T P_i B_d - C_d^T(\hat{S}_{di} + \hat{Q}_{di} D_d), \tag{4.105}$$

$$\mathcal{C}_{di} = -(\hat{R}_{di} + D_d^T \hat{S}_{di} + \hat{S}_{di}^T D_d + D_d^T \hat{Q}_{di} D_d - B_d^T P_i B_d). \tag{4.106}$$

Hence, vector dissipativity of large-scale linear impulsive dynamical systems with respect to quadratic hybrid supply rates can be characterized via (cascade) linear matrix inequalities (LMIs) [36].

Next, we extend the notions of passivity and nonexpansivity to vector passivity and vector nonexpansivity.

Definition 4.10. The large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) with $m_{ci} = l_{ci}$, $m_{di} = l_{di}$, $i = 1, ..., q$, is *vector passive* (respectively, *vector exponentially passive*) if it is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector hybrid supply rate $(S_c(u_c, y_c), S_d(u_d, y_d))$, where $s_{ci}(u_{ci}, y_{ci}) = 2y_{ci}^T u_{ci}$ and $s_{di}(u_{di}, y_{di}) = 2y_{di}^T u_{di}$, $i = 1, ..., q$.

Definition 4.11. The large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) is *vector nonexpansive* (respectively, *vector exponentially nonexpansive*) if it is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector hybrid supply rate $(S_c(u_c, y_c), S_d(u_d, y_d))$, where $s_{ci}(u_{ci}, y_{ci}) = \gamma_{ci}^2 u_{ci}^T u_{ci} - y_{ci}^T y_{ci}$ and $s_{di}(u_{di}, y_{di}) = \gamma_{di}^2 u_{di}^T u_{di} - y_{di}^T y_{di}$, $i = 1, ..., q$, and $\gamma_{ci} > 0$, $\gamma_{di} > 0$, $i = 1, ..., q$, are given.

Remark 4.8. Note that a mixed vector passive-nonexpansive formulation of $\mathcal{G}$ can also be considered. Specifically, one can consider large-scale impulsive dynamical systems $\mathcal{G}$ which are vector dissipative with respect to hybrid vector supply rates $(S_c(u_c, y_c), S_d(u_d, y_d))$, where $s_{ci}(u_{ci}, y_{ci}) = 2y_{ci}^T u_{ci}$, $s_{di}(u_{di}, y_{di}) = 2y_{di}^T u_{di}$, $i \in \mathbb{Z}_p$, $s_{cj}(u_{cj}, y_{cj}) = \gamma_{cj}^2 u_{cj}^T u_{cj} - y_{cj}^T y_{cj}$,

$\gamma_{cj} > 0$, $s_{dj}(u_{dj}, y_{dj}) = \gamma_{dj}^2 u_{dj}^{\mathrm{T}} u_{dj} - y_{dj}^{\mathrm{T}} y_{dj}$, $\gamma_{dj} > 0$, $j \in \mathbb{Z}_{\mathrm{ne}}$, $\mathbb{Z}_{\mathrm{p}} \cap \mathbb{Z}_{\mathrm{ne}} = \emptyset$, and $\mathbb{Z}_{\mathrm{p}} \cup \mathbb{Z}_{\mathrm{ne}} = \{1, ..., q\}$. Furthermore, hybrid supply rates for vector input strict passivity, vector output strict passivity, and vector input-output strict passivity generalizing the passivity notions given in [118] can also be considered. However, for simplicity of exposition we do not do so here.

The next result presents constructive sufficient conditions guaranteeing vector dissipativity of $\mathcal{G}$ with respect to a quadratic hybrid supply rate for the case where the vector storage function $V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$, is component decoupled, that is, $V_{\mathrm{s}}(x) = [v_{\mathrm{s}1}(x_1), ..., v_{\mathrm{s}q}(x_q)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$.

Theorem 4.10. Consider the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18). Assume that there exist functions $V_{\mathrm{s}} = [v_{\mathrm{s}1}, ..., v_{\mathrm{s}q}]^{\mathrm{T}} : \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$, $w_{\mathrm{c}} = [w_{\mathrm{c}1}, ..., w_{\mathrm{c}q}]^{\mathrm{T}} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$, $\ell_{\mathrm{c}i} : \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{c}i}}$, $\mathcal{Z}_{\mathrm{c}i} : \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{c}i} \times m_{\mathrm{c}i}}$, $\ell_{\mathrm{d}i} : \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{d}i}}$, $\mathcal{Z}_{\mathrm{d}i} : \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{d}i} \times m_{\mathrm{d}i}}$, $P_{1i} : \mathbb{R}^n \to \mathbb{R}^{1 \times m_{\mathrm{d}i}}$, $P_{2i} : \mathbb{R}^n \to \mathbb{N}^{m_{\mathrm{d}i}}$ such that $v_{\mathrm{s}i}(\cdot)$ is continuously differentiable, $v_{\mathrm{s}i}(0) = 0$, $i = 1, ..., q$, $w_{\mathrm{c}} \in \mathcal{W}$, $w_{\mathrm{c}}(0) = 0$, the zero solution $r(t) \equiv 0$ to (4.71) is Lyapunov (respectively, asymptotically) stable, and, for all $x \in \mathbb{R}^n$ and $i = 1, ..., q$,

$$0 \leq v_{\mathrm{s}i}(x_i + \mathcal{F}_{\mathrm{d}i}(x)) - v_{\mathrm{s}i}(x_i + \mathcal{F}_{\mathrm{d}i}(x) + G_{\mathrm{d}i}(x_i) u_{\mathrm{d}i}) + P_{1i}(x) u_{\mathrm{d}i} + u_{\mathrm{d}i}^{\mathrm{T}} P_{2i}(x) u_{\mathrm{d}i}, \quad x \in \mathcal{Z}_x, \quad u_{\mathrm{d}i} \in \mathbb{R}^{m_{\mathrm{d}i}}, \tag{4.107}$$

$$0 \geq v_{\mathrm{s}i}'(x_i) \mathcal{F}_{\mathrm{c}i}(x) - h_{\mathrm{c}i}^{\mathrm{T}}(x_i) Q_{\mathrm{c}i} h_{\mathrm{c}i}(x_i) - w_{\mathrm{c}i}(V_{\mathrm{s}}(x)) + \ell_{\mathrm{c}i}^{\mathrm{T}}(x_i) \ell_{\mathrm{c}i}(x_i), \quad x \notin \mathcal{Z}_x, \tag{4.108}$$

$$0 = \tfrac{1}{2} v_{\mathrm{s}i}'(x_i) G_{\mathrm{c}i}(x_i) - h_{\mathrm{c}i}^{\mathrm{T}}(x_i)(S_{\mathrm{c}i} + Q_{\mathrm{c}i} J_{\mathrm{c}i}(x_i)) + \ell_{\mathrm{c}i}^{\mathrm{T}}(x_i) \mathcal{Z}_{\mathrm{c}i}(x_i), \quad x \notin \mathcal{Z}_x, \tag{4.109}$$

$$0 \leq R_{\mathrm{c}i} + J_{\mathrm{c}i}^{\mathrm{T}}(x_i) S_{\mathrm{c}i} + S_{\mathrm{c}i}^{\mathrm{T}} J_{\mathrm{c}i}(x_i) + J_{\mathrm{c}i}^{\mathrm{T}}(x_i) Q_{\mathrm{c}i} J_{\mathrm{c}i}(x_i) - \mathcal{Z}_{\mathrm{c}i}^{\mathrm{T}}(x_i) \mathcal{Z}_{\mathrm{c}i}(x_i), \quad x \notin \mathcal{Z}_x, \tag{4.110}$$

$$0 \geq v_{\mathrm{s}i}(x_i + \mathcal{F}_{\mathrm{d}i}(x)) - h_{\mathrm{d}i}^{\mathrm{T}}(x_i) Q_{\mathrm{d}i} h_{\mathrm{d}i}(x_i) - v_{\mathrm{s}i}(x_i) + \ell_{\mathrm{d}i}^{\mathrm{T}}(x_i) \ell_{\mathrm{d}i}(x_i), \quad x \in \mathcal{Z}_x, \tag{4.111}$$

$$0 = \tfrac{1}{2} P_{1i}(x) - h_{\mathrm{d}i}^{\mathrm{T}}(x_i)(S_{\mathrm{d}i} + Q_{\mathrm{d}i} J_{\mathrm{d}i}(x_i)) + \ell_{\mathrm{d}i}^{\mathrm{T}}(x_i) \mathcal{Z}_{\mathrm{d}i}(x_i), \quad x \in \mathcal{Z}_x, \tag{4.112}$$

$$0 \leq R_{\mathrm{d}i} + J_{\mathrm{d}i}^{\mathrm{T}}(x_i) S_{\mathrm{d}i} + S_{\mathrm{d}i}^{\mathrm{T}} J_{\mathrm{d}i}(x_i) + J_{\mathrm{d}i}^{\mathrm{T}}(x_i) Q_{\mathrm{d}i} J_{\mathrm{d}i}(x_i) - P_{2i}(x) - \mathcal{Z}_{\mathrm{d}i}^{\mathrm{T}}(x_i) \mathcal{Z}_{\mathrm{d}i}(x_i), \quad x \in \mathcal{Z}_x. \tag{4.113}$$

Then $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector hybrid supply rate $(S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$, where $s_{\mathrm{c}i}(u_{\mathrm{c}i}, y_{\mathrm{c}i}) = u_{\mathrm{c}i}^{\mathrm{T}} R_{\mathrm{c}i} u_{\mathrm{c}i} + 2y_{\mathrm{c}i}^{\mathrm{T}} S_{\mathrm{c}i} u_{\mathrm{c}i} + y_{\mathrm{c}i}^{\mathrm{T}} Q_{\mathrm{c}i} y_{\mathrm{c}i}$ and $s_{\mathrm{d}i}(u_{\mathrm{d}i}, y_{\mathrm{d}i}) = u_{\mathrm{d}i}^{\mathrm{T}} R_{\mathrm{d}i} u_{\mathrm{d}i} + 2y_{\mathrm{d}i}^{\mathrm{T}} S_{\mathrm{d}i} u_{\mathrm{d}i} + y_{\mathrm{d}i}^{\mathrm{T}} Q_{\mathrm{d}i} y_{\mathrm{d}i}$, $i = 1, ..., q$.

Proof. The proof is similar to the proof of Theorem 4.9 and, hence, is omitted. □

Finally, we provide necessary and sufficient conditions for the case where the large-scale impulsive dynamical system $\mathcal{G}$ is vector lossless with respect to a quadratic hybrid supply rate.

Theorem 4.11. Consider the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18). Let $R_{\mathrm{c}i} \in \mathbb{S}^{m_{\mathrm{c}i}}$, $S_{\mathrm{c}i} \in \mathbb{R}^{l_{\mathrm{c}i} \times m_{\mathrm{c}i}}$, $Q_{\mathrm{c}i} \in \mathbb{S}^{l_{\mathrm{c}i}}$, $R_{\mathrm{d}i} \in \mathbb{S}^{m_{\mathrm{d}i}}$, $S_{\mathrm{d}i} \in \mathbb{R}^{l_{\mathrm{d}i} \times m_{\mathrm{d}i}}$, and $Q_{\mathrm{d}i} \in \mathbb{S}^{l_{\mathrm{d}i}}$, $i = 1, ..., q$. Then $\mathcal{G}$ is vector lossless with respect to the quadratic hybrid supply rate $(S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$, where $s_{\mathrm{c}i}(u_{\mathrm{c}i}, y_{\mathrm{c}i}) = u_{\mathrm{c}i}^{\mathrm{T}} R_{\mathrm{c}i} u_{\mathrm{c}i} + 2y_{\mathrm{c}i}^{\mathrm{T}} S_{\mathrm{c}i} u_{\mathrm{c}i} + y_{\mathrm{c}i}^{\mathrm{T}} Q_{\mathrm{c}i} y_{\mathrm{c}i}$ and $s_{\mathrm{d}i}(u_{\mathrm{d}i}, y_{\mathrm{d}i}) = u_{\mathrm{d}i}^{\mathrm{T}} R_{\mathrm{d}i} u_{\mathrm{d}i} + 2y_{\mathrm{d}i}^{\mathrm{T}} S_{\mathrm{d}i} u_{\mathrm{d}i} + y_{\mathrm{d}i}^{\mathrm{T}} Q_{\mathrm{d}i} y_{\mathrm{d}i}$, $i = 1, ..., q$, if and only if there exist functions $V_{\mathrm{s}} = [v_{\mathrm{s}1}, ..., v_{\mathrm{s}q}]^{\mathrm{T}} : \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$, $P_{1i} : \mathbb{R}^n \to \mathbb{R}^{1 \times m_{\mathrm{d}}}$, $P_{2i} : \mathbb{R}^n \to \mathbb{N}^{m_{\mathrm{d}}}$, and $w_{\mathrm{c}} = [w_{\mathrm{c}1}, ..., w_{\mathrm{c}q}]^{\mathrm{T}} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ such that $v_{\mathrm{s}i}(\cdot)$ is continuously differentiable, $v_{\mathrm{s}i}(0) = 0$, $i = 1, ..., q$, $w_{\mathrm{c}} \in \mathcal{W}$, $w_{\mathrm{c}}(0) = 0$, the zero solution $r(t) \equiv 0$ to (4.71) is Lyapunov stable, and, for all $x \in \mathbb{R}^n$, $i = 1, ..., q$, (4.75) holds and

$$0 = v_{\mathrm{s}i}'(x)\mathcal{F}_{\mathrm{c}}(x) - h_{\mathrm{c}}^{\mathrm{T}}(x)\hat{Q}_{\mathrm{c}i}h_{\mathrm{c}}(x) - w_{\mathrm{c}i}(V_{\mathrm{s}}(x)), \quad x \notin \mathcal{Z}_x, \tag{4.114}$$

$$0 = \tfrac{1}{2}v_{\mathrm{s}i}'(x)G_{\mathrm{c}}(x) - h_{\mathrm{c}}^{\mathrm{T}}(x)(\hat{S}_{\mathrm{c}i} + \hat{Q}_{\mathrm{c}i}J_{\mathrm{c}}(x)), \quad x \notin \mathcal{Z}_x, \tag{4.115}$$

$$0 = \hat{R}_{\mathrm{c}i} + J_{\mathrm{c}}^{\mathrm{T}}(x)\hat{S}_{\mathrm{c}i} + \hat{S}_{\mathrm{c}i}^{\mathrm{T}}J_{\mathrm{c}}(x) + J_{\mathrm{c}}^{\mathrm{T}}(x)\hat{Q}_{\mathrm{c}i}J_{\mathrm{c}}(x), \quad x \notin \mathcal{Z}_x, \tag{4.116}$$

$$0 = v_{\mathrm{s}i}(x + \mathcal{F}_{\mathrm{d}}(x)) - h_{\mathrm{d}}^{\mathrm{T}}(x)\hat{Q}_{\mathrm{d}i}h_{\mathrm{d}}(x) - v_{\mathrm{s}i}(x), \quad x \in \mathcal{Z}_x, \tag{4.117}$$

$$0 = \tfrac{1}{2}P_{1i}(x) - h_{\mathrm{d}}^{\mathrm{T}}(x)(\hat{S}_{\mathrm{d}i} + \hat{Q}_{\mathrm{d}i}J_{\mathrm{d}}(x)), \quad x \in \mathcal{Z}_x, \tag{4.118}$$

$$0 = \hat{R}_{\mathrm{d}i} + J_{\mathrm{d}}^{\mathrm{T}}(x)\hat{S}_{\mathrm{d}i} + \hat{S}_{\mathrm{d}i}^{\mathrm{T}}J_{\mathrm{d}}(x) + J_{\mathrm{d}}^{\mathrm{T}}(x)\hat{Q}_{\mathrm{d}i}J_{\mathrm{d}}(x) - P_{2i}(x), \quad x \in \mathcal{Z}_x. \tag{4.119}$$

Proof. Sufficiency follows as in the proof of Theorem 4.9. To show necessity, suppose that $\mathcal{G}$ is lossless with respect to the quadratic hybrid supply rate $(S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$. Then, there exist continuous functions $V_{\rm s} = [v_{\rm s1}, ..., v_{{\rm s}q}]^{\rm T} : \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$ and $w_{\rm c} = [w_{\rm c1}, ..., w_{{\rm c}q}]^{\rm T} : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ such that $V_{\rm s}(0) = 0$, the zero solution $r(t) \equiv 0$ to (4.71) is Lyapunov stable and for all $k \in \overline{\mathbb{Z}}_+$, $i = 1, ..., q$,

$$v_{{\rm s}i}(x(\hat{t})) - v_{{\rm s}i}(x(t)) = \int_t^{\hat{t}} s_{{\rm c}i}(u_{{\rm c}i}(\sigma), y_{{\rm c}i}(\sigma)){\rm d}\sigma + \int_t^{\hat{t}} w_{{\rm c}i}(V_{\rm s}(x(\sigma))){\rm d}\sigma, \quad t_k < t \le \hat{t} \le t_{k+1} \tag{4.120}$$

and

$$v_{{\rm s}i}(x(t_k) + \mathcal{F}_{\rm d}(x(t_k)) + G_{\rm d}(x(t_k))u_{\rm d}(t_k)) = v_{{\rm s}i}(x(t_k)) + s_{{\rm d}i}(u_{{\rm d}i}(t_k), y_{{\rm d}i}(t_k)). \tag{4.121}$$

Now, dividing (4.120) by $\hat{t} - t^+$ and letting $\hat{t} \to t^+$, (4.120) is equivalent to

$$\begin{aligned} \dot{v}_{{\rm s}i}(x(t)) &= v'_{{\rm s}i}(x(t))[\mathcal{F}_{\rm c}(x(t)) + G_{\rm c}(x(t))u_{\rm c}(t)] \\ &= s_{{\rm c}i}(u_{{\rm c}i}(t), y_{{\rm c}i}(t)) + w_{{\rm c}i}(V_{\rm s}(x(t))), \quad t_k < t \le t_{k+1}. \end{aligned} \tag{4.122}$$

Next, with $t = t_0$, it follows from (4.122) that

$$\begin{aligned} v'_{{\rm s}i}(x_0)[\mathcal{F}_{\rm c}(x_0) + G_{\rm c}(x_0)u_{\rm c}(t_0)] &= s_{{\rm c}i}(u_{{\rm c}i}(t_0), y_{{\rm c}i}(t_0)) + w_{{\rm c}i}(V_{\rm s}(x_0)), \\ & \qquad x_0 \notin \mathcal{Z}_x, \quad u_{\rm c}(t_0) \in \mathbb{R}^{m_{\rm c}}. \end{aligned} \tag{4.123}$$

Since $x_0 \notin \mathcal{Z}_x$ is arbitrary, it follows that

$$\begin{aligned} v'_{{\rm s}i}(x)[\mathcal{F}_{\rm c}(x) + G_{\rm c}(x)u_{\rm c}] &= w_{{\rm c}i}(V_{\rm s}(x)) + u_{\rm c}^{\rm T}\hat{R}_{{\rm c}i}u_{\rm c} + 2y_{\rm c}^{\rm T}\hat{S}_{{\rm c}i}u_{\rm c} + y_{\rm c}^{\rm T}\hat{Q}_{{\rm c}i}y_{\rm c} \\ &= w_{{\rm c}i}(V_{\rm s}(x)) + h_{\rm c}^{\rm T}(x)\hat{Q}_{{\rm c}i}h_{\rm c}(x) + 2h_{\rm c}^{\rm T}(x)(\hat{Q}_{{\rm c}i}J_{\rm c}(x) + \hat{S}_{{\rm c}i})u_{\rm c} \\ &\quad + u_{\rm c}^{\rm T}(\hat{R}_{{\rm c}i} + \hat{S}_{{\rm c}i}^{\rm T}J_{\rm c}(x) + J_{\rm c}^{\rm T}(x)\hat{S}_{{\rm c}i} + J_{\rm c}^{\rm T}(x)\hat{Q}_{{\rm c}i}J_{\rm c}(x))u_{\rm c}, \\ & \qquad x \in \mathbb{R}^n, \quad u_{\rm c} \in \mathbb{R}^{m_{\rm c}}. \end{aligned} \tag{4.124}$$

Now, equating coefficients of equal powers yields (4.114)–(4.116).

Next, it follows from (4.121) with $k = 1$ that

$$v_{{\rm s}i}(x(t_1) + \mathcal{F}_{\rm d}(x(t_1)) + G_{\rm d}(x(t_1))u_{\rm d}(t_1)) = v_{{\rm s}i}(x(t_1)) + s_{{\rm d}i}(u_{{\rm d}i}(t_1), y_{{\rm d}i}(t_1)). \tag{4.125}$$

Now, since the continuous-time dynamics (4.15) are Lipschitz, it follows that for arbitrary $x \in \mathcal{Z}_x$ there exists $x_0 \notin \mathcal{Z}_x$ such that $x(t_1) = x$. Hence, it follows from (4.125) that

$$\begin{aligned} v_{\mathrm{s}i}(x + \mathcal{F}_{\mathrm{d}}(x) + G_{\mathrm{d}}(x)u_{\mathrm{d}}) = &\ v_{\mathrm{s}i}(x) + u_{\mathrm{d}}^{\mathrm{T}}\hat{R}_{\mathrm{d}i}u_{\mathrm{d}} + 2y_{\mathrm{d}}^{\mathrm{T}}\hat{S}_{\mathrm{d}i}u_{\mathrm{d}} + y_{\mathrm{d}}^{\mathrm{T}}\hat{Q}_{\mathrm{d}i}y_{\mathrm{d}} \\ = &\ v_{\mathrm{s}i}(x) + h_{\mathrm{d}}^{\mathrm{T}}(x)\hat{Q}_{\mathrm{d}i}h_{\mathrm{d}}(x) + 2h_{\mathrm{d}}^{\mathrm{T}}(x)(\hat{Q}_{\mathrm{d}i}J_{\mathrm{d}}(x) + \hat{S}_{\mathrm{d}i})u_{\mathrm{d}} \\ &+ u_{\mathrm{d}}^{\mathrm{T}}(\hat{R}_{\mathrm{d}i} + \hat{S}_{\mathrm{d}i}^{\mathrm{T}}J_{\mathrm{d}}(x) + J_{\mathrm{d}}^{\mathrm{T}}(x)\hat{S}_{\mathrm{d}i} + J_{\mathrm{d}}^{\mathrm{T}}(x)\hat{Q}_{\mathrm{d}i}J_{\mathrm{d}}(x))u_{\mathrm{d}}, \\ & x \in \mathbb{R}^n, \quad u_{\mathrm{d}} \in \mathbb{R}^{m_{\mathrm{d}}}. \qquad (4.126) \end{aligned}$$

Since the right-hand-side of (4.126) is quadratic in u_{d} it follows that $v_{\mathrm{s}i}(x+\mathcal{F}_{\mathrm{d}}(x)+G_{\mathrm{d}}(x)u_{\mathrm{d}})$ is quadratic in u_{d}, and hence, there exist $P_{1i} : \mathbb{R}^n \to \mathbb{R}^{1\times m_{\mathrm{d}}}$ and $P_{2i} : \mathbb{R}^n \to \mathbb{N}^{m_{\mathrm{d}}}$ such that

$$\begin{aligned} v_{\mathrm{s}i}(x + \mathcal{F}_{\mathrm{d}}(x) + G_{\mathrm{d}}(x)u_{\mathrm{d}}) = v_{\mathrm{s}i}(x + \mathcal{F}_{\mathrm{d}}(x)) + P_{1i}(x)u_{\mathrm{d}} + u_{\mathrm{d}}^{\mathrm{T}}P_{2i}(x)u_{\mathrm{d}}, \\ x \in \mathbb{R}^n, \quad u_{\mathrm{d}} \in \mathbb{R}^{m_{\mathrm{d}}}. \qquad (4.127) \end{aligned}$$

Now, using (4.127) and equating coefficients of equal powers in (4.126) yields (4.117)–(4.119).

□

4.5. Stability of Feedback Interconnections of Large-Scale Impulsive Dynamical Systems

In this section, we consider stability of feedback interconnections of large-scale impulsive dynamical systems. Specifically, for the large-scale impulsive dynamical system $\mathcal{G}$ given by (4.15)–(4.18) we consider either a dynamic or static large-scale feedback system $\mathcal{G}_{\mathrm{c}}$. Then by appropriately combining vector storage functions for each system we show stability of the feedback interconnection. We begin by considering the large-scale impulsive dynamical system (4.15)–(4.18) with the large-scale feedback system $\mathcal{G}_{\mathrm{c}}$ given by

$$\dot{x}_{\mathrm{c}}(t) = F_{\mathrm{cc}}(x_{\mathrm{c}}(t), u_{\mathrm{cc}}(t)), \quad x_{\mathrm{c}}(t_0) = x_{\mathrm{c}0}, \quad (x_{\mathrm{c}}(t), u_{\mathrm{cc}}(t)) \notin \mathcal{Z}_{\mathrm{c}}, \qquad (4.128)$$

$$\Delta x_{\mathrm{c}}(t) = F_{\mathrm{dc}}(x_{\mathrm{c}}(t), u_{\mathrm{dc}}(t)), \quad (x_{\mathrm{c}}(t), u_{\mathrm{cc}}(t)) \in \mathcal{Z}_{\mathrm{c}}, \qquad (4.129)$$

$$y_{\mathrm{cc}}(t) = H_{\mathrm{cc}}(x_{\mathrm{c}}(t), u_{\mathrm{cc}}(t)), \quad (x_{\mathrm{c}}(t), u_{\mathrm{cc}}(t)) \notin \mathcal{Z}_{\mathrm{c}}, \qquad (4.130)$$

$$y_{\mathrm{dc}}(t) = H_{\mathrm{dc}}(x_{\mathrm{c}}(t), u_{\mathrm{dc}}(t)), \quad (x_{\mathrm{c}}(t), u_{\mathrm{cc}}(t)) \in \mathcal{Z}_{\mathrm{c}}, \tag{4.131}$$

where $F_{\mathrm{cc}} : \mathbb{R}^{n_{\mathrm{c}}} \times \mathcal{U}_{\mathrm{cc}} \to \mathbb{R}^{n_{\mathrm{c}}}$, $F_{\mathrm{dc}} : \mathbb{R}^{n_{\mathrm{c}}} \times \mathcal{U}_{\mathrm{dc}} \to \mathbb{R}^{n_{\mathrm{c}}}$, $H_{\mathrm{cc}} : \mathbb{R}^{n_{\mathrm{c}}} \times \mathcal{U}_{\mathrm{cc}} \to \mathcal{Y}_{\mathrm{cc}}$, $H_{\mathrm{dc}} : \mathbb{R}^{n_{\mathrm{c}}} \times \mathcal{U}_{\mathrm{dc}} \to \mathcal{Y}_{\mathrm{dc}}$, $F_{\mathrm{cc}} \triangleq [F_{\mathrm{cc}1}^{\mathrm{T}}, ..., F_{\mathrm{cc}q}^{\mathrm{T}}]^{\mathrm{T}}$, $F_{\mathrm{dc}} \triangleq [F_{\mathrm{dc}1}^{\mathrm{T}}, ..., F_{\mathrm{dc}q}^{\mathrm{T}}]^{\mathrm{T}}$, $H_{\mathrm{cc}} \triangleq [H_{\mathrm{cc}1}^{\mathrm{T}}, ..., H_{\mathrm{cc}q}^{\mathrm{T}}]^{\mathrm{T}}$, $H_{\mathrm{dc}} \triangleq [H_{\mathrm{dc}1}^{\mathrm{T}}, ..., H_{\mathrm{dc}q}^{\mathrm{T}}]^{\mathrm{T}}$, $\mathcal{U}_{\mathrm{cc}} \subseteq \mathbb{R}^{l_{\mathrm{c}}}$, $\mathcal{U}_{\mathrm{dc}} \subseteq \mathbb{R}^{l_{\mathrm{d}}}$, $\mathcal{Y}_{\mathrm{cc}} \subseteq \mathbb{R}^{m_{\mathrm{c}}}$, $\mathcal{Y}_{\mathrm{dc}} \subseteq \mathbb{R}^{m_{\mathrm{d}}}$.

Moreover, for all $i = 1, ..., q$, we assume that

$$F_{\mathrm{cc}i}(x_{\mathrm{c}}, u_{\mathrm{cc}i}) = f_{\mathrm{cc}i}(x_{\mathrm{c}i}) + \mathcal{I}_{\mathrm{cc}i}(x_{\mathrm{c}}) + G_{\mathrm{cc}i}(x_{\mathrm{c}i})u_{\mathrm{cc}i}, \tag{4.132}$$

$$F_{\mathrm{dc}i}(x_{\mathrm{c}}, u_{\mathrm{dc}i}) = f_{\mathrm{dc}i}(x_{\mathrm{c}i}) + \mathcal{I}_{\mathrm{dc}i}(x_{\mathrm{c}}) + G_{\mathrm{dc}i}(x_{\mathrm{c}i})u_{\mathrm{dc}i}, \tag{4.133}$$

$$H_{\mathrm{cc}i}(x_{\mathrm{c}i}, u_{\mathrm{cc}i}) = h_{\mathrm{cc}i}(x_{\mathrm{c}i}) + J_{\mathrm{cc}i}(x_{\mathrm{c}i})u_{\mathrm{cc}i}, \tag{4.134}$$

$$H_{\mathrm{dc}i}(x_{\mathrm{c}i}, u_{\mathrm{dc}i}) = h_{\mathrm{dc}i}(x_{\mathrm{c}i}) + J_{\mathrm{dc}i}(x_{\mathrm{c}i})u_{\mathrm{dc}i}, \tag{4.135}$$

where $u_{\mathrm{cc}i} \in \mathcal{U}_{\mathrm{cc}i} \subseteq \mathbb{R}^{l_{\mathrm{c}i}}$, $u_{\mathrm{dc}i} \in \mathcal{U}_{\mathrm{dc}i} \subseteq \mathbb{R}^{l_{\mathrm{d}i}}$, $y_{\mathrm{cc}i} \triangleq H_{\mathrm{cc}i}(x_{\mathrm{c}i}, u_{\mathrm{cc}i}) \in \mathcal{Y}_{\mathrm{cc}i} \subseteq \mathbb{R}^{m_{\mathrm{c}i}}$, $y_{\mathrm{dc}i} \triangleq H_{\mathrm{dc}i}(x_{\mathrm{c}i}, u_{\mathrm{dc}i}) \in \mathcal{Y}_{\mathrm{dc}i} \subseteq \mathbb{R}^{m_{\mathrm{d}i}}$, $f_{\mathrm{cc}i} : \mathbb{R}^{n_{\mathrm{c}i}} \to \mathbb{R}^{n_{\mathrm{c}i}}$ and $\mathcal{I}_{\mathrm{cc}i} : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}i}}$ satisfy $f_{\mathrm{cc}i}(0) = 0$ and $\mathcal{I}_{\mathrm{cc}i}(0) = 0$, $f_{\mathrm{dc}i} : \mathbb{R}^{n_{\mathrm{c}i}} \to \mathbb{R}^{n_{\mathrm{c}i}}$, $\mathcal{I}_{\mathrm{dc}i} : \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}i}}$, $G_{\mathrm{cc}i} : \mathbb{R}^{n_{\mathrm{c}i}} \to \mathbb{R}^{n_{\mathrm{c}i} \times l_{\mathrm{c}i}}$, $G_{\mathrm{dc}i} : \mathbb{R}^{n_{\mathrm{c}i}} \to \mathbb{R}^{n_{\mathrm{c}i} \times l_{\mathrm{d}i}}$, $h_{\mathrm{cc}i} : \mathbb{R}^{n_{\mathrm{c}i}} \to \mathbb{R}^{m_{\mathrm{c}i}}$ and satisfies $h_{\mathrm{cc}i}(0) = 0$, $h_{\mathrm{dc}i} : \mathbb{R}^{n_{\mathrm{c}i}} \to \mathbb{R}^{m_{\mathrm{d}i}}$, $J_{\mathrm{cc}i} : \mathbb{R}^{n_{\mathrm{c}i}} \to \mathbb{R}^{m_{\mathrm{c}i} \times l_{\mathrm{c}i}}$, $J_{\mathrm{dc}i} : \mathbb{R}^{n_{\mathrm{c}i}} \to \mathbb{R}^{m_{\mathrm{d}i} \times l_{\mathrm{d}i}}$, and $\sum_{i=1}^{q} n_{\mathrm{c}i} = n_{\mathrm{c}}$. Furthermore, we define the composite input and composite output for the system $\mathcal{G}_{\mathrm{c}}$ as $u_{\mathrm{cc}} \triangleq [u_{\mathrm{cc}1}^{\mathrm{T}}, ..., u_{\mathrm{cc}q}^{\mathrm{T}}]^{\mathrm{T}}$, $u_{\mathrm{dc}} \triangleq [u_{\mathrm{dc}1}^{\mathrm{T}}, ..., u_{\mathrm{dc}q}^{\mathrm{T}}]^{\mathrm{T}}$, $y_{\mathrm{cc}} \triangleq [y_{\mathrm{cc}1}^{\mathrm{T}}, ..., y_{\mathrm{cc}q}^{\mathrm{T}}]^{\mathrm{T}}$, and $y_{\mathrm{dc}} \triangleq [y_{\mathrm{dc}1}^{\mathrm{T}}, ..., y_{\mathrm{dc}q}^{\mathrm{T}}]^{\mathrm{T}}$, respectively. In this case, $\mathcal{U}_{\mathrm{cc}} = \mathcal{U}_{\mathrm{cc}1} \times \cdots \times \mathcal{U}_{\mathrm{cc}q}$, $\mathcal{U}_{\mathrm{dc}} = \mathcal{U}_{\mathrm{dc}1} \times \cdots \times \mathcal{U}_{\mathrm{dc}q}$, $\mathcal{Y}_{\mathrm{cc}} = \mathcal{Y}_{\mathrm{cc}1} \times \cdots \times \mathcal{Y}_{\mathrm{cc}q}$, and $\mathcal{Y}_{\mathrm{dc}} = \mathcal{Y}_{\mathrm{dc}1} \times \cdots \times \mathcal{Y}_{\mathrm{dc}q}$.

Note that with the feedback interconnection given by Figure 4.1, $(u_{\mathrm{cc}}, u_{\mathrm{dc}}) = (y_{\mathrm{c}}, y_{\mathrm{d}})$ and $(y_{\mathrm{cc}}, y_{\mathrm{dc}}) = (-u_{\mathrm{c}}, -u_{\mathrm{d}})$. We assume that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is well posed, that is, $\det(I_{m_{\mathrm{c}i}} + J_{\mathrm{cc}i}(x_{\mathrm{c}i})J_{\mathrm{c}i}(x_i)) \neq 0$, $\det(I_{m_{\mathrm{d}i}} + J_{\mathrm{dc}i}(x_{\mathrm{c}i})J_{\mathrm{d}i}(x_i)) \neq 0$ for all $x_i \in \mathbb{R}^{n_i}$, $x_{\mathrm{c}i} \in \mathbb{R}^{n_{\mathrm{c}i}}$, and $i = 1, ..., q$. Next, we assume that the set $\mathcal{Z}_{\mathrm{c}} \triangleq \mathcal{Z}_{\mathrm{c}x_{\mathrm{c}}} \times \mathcal{Z}_{\mathrm{c}u_{\mathrm{cc}}} = \{(x_{\mathrm{c}}, u_{\mathrm{cc}}) : \mathcal{X}_{\mathrm{c}}(x_{\mathrm{c}}, u_{\mathrm{cc}}) = 0\}$, where $\mathcal{X}_{\mathrm{c}} : \mathbb{R}^{n_{\mathrm{c}}} \times \mathcal{U}_{\mathrm{cc}} \to \mathbb{R}$, and define the closed-loop resetting set

$$\tilde{\mathcal{Z}}_{\tilde{x}} \triangleq \mathcal{Z}_x \times \mathcal{Z}_{\mathrm{c}x_{\mathrm{c}}} \cup \{(x, x_{\mathrm{c}}) : (\mathcal{L}_{\mathrm{cc}}(x, x_{\mathrm{c}}), \mathcal{L}_{\mathrm{c}}(x, x_{\mathrm{c}})) \in \mathcal{Z}_{\mathrm{c}u_{\mathrm{cc}}} \times \mathcal{Z}_{u_{\mathrm{c}}}\}, \tag{4.136}$$

where $\mathcal{L}_{\mathrm{cc}}(\cdot, \cdot)$ and $\mathcal{L}_{\mathrm{c}}(\cdot, \cdot)$ are functions of x and x_{c} arising from the algebraic loops due to

u_{cc} and u_{c}, respectively. Note that since the feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is well posed, it follows that $\tilde{\mathcal{Z}}_{\tilde{x}}$ is well defined and depends on the closed-loop states $\tilde{x} \triangleq [x^{\mathrm{T}}\, x_{\mathrm{c}}^{\mathrm{T}}]^{\mathrm{T}}$. Furthermore, we assume that for the large-scale systems $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$, the conditions of Theorem 4.6 are satisfied; that is, if $V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$, and $V_{\mathrm{cs}}(x_{\mathrm{c}})$, $x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}$, are vector storage functions for $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$, respectively, then there exist $p \in \mathbb{R}_+^q$ and $p_{\mathrm{c}} \in \mathbb{R}_+^q$ such that the functions $v_{\mathrm{s}}(x) = p^{\mathrm{T}} V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$, and $v_{\mathrm{cs}}(x_{\mathrm{c}}) = p_{\mathrm{c}}^{\mathrm{T}} V_{\mathrm{cs}}(x_{\mathrm{c}})$, $x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}$, are positive definite.

The following result gives sufficient conditions for Lyapunov and asymptotic stability of the feedback interconnection given by Figure 4.1. For the statement of this result let $\mathcal{T}^{\mathrm{c}}_{x0,u_{\mathrm{c}}}$ denote the set of resetting times of $\mathcal{G}$, let $\mathcal{T}_{x0,u_{\mathrm{c}}}$ denote the complement of $\mathcal{T}^{\mathrm{c}}_{x0,u_{\mathrm{c}}}$, that is, $[0,\infty)\backslash\mathcal{T}^{\mathrm{c}}_{x0,u_{\mathrm{c}}}$, let $\mathcal{T}^{\mathrm{c}}_{x_{\mathrm{c}}0,u_{\mathrm{cc}}}$ denote the set of resetting times of $\mathcal{G}_{\mathrm{c}}$ and let $\mathcal{T}_{x_{\mathrm{c}}0,u_{\mathrm{cc}}}$ denote the complement of $\mathcal{T}^{\mathrm{c}}_{x_{\mathrm{c}}0,u_{\mathrm{cc}}}$, that is, $[0,\infty)\backslash\mathcal{T}^{\mathrm{c}}_{x_{\mathrm{c}}0,u_{\mathrm{cc}}}$.

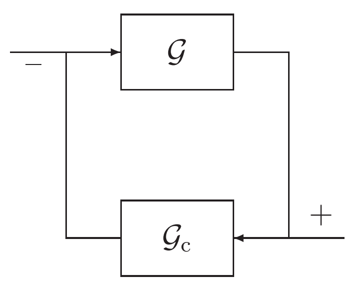

Figure 4.1: Feedback interconnection of large-scale systems $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$

Theorem 4.12. Consider the large-scale impulsive dynamical systems $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ given by (4.15)–(4.18) and (4.128)–(4.131), respectively. Assume that $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are vector dissipative with respect to the vector hybrid supply rates $(S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$ and $(S_{\mathrm{cc}}(u_{\mathrm{cc}}, y_{\mathrm{cc}}), S_{\mathrm{dc}}(u_{\mathrm{dc}}, y_{\mathrm{dc}}))$, and with continuously differentiable vector storage functions $V_{\mathrm{s}}(\cdot)$ and $V_{\mathrm{cs}}(\cdot)$ and dissipation matrices $W \in \mathbb{R}^{q\times q}$ and $W_{\mathrm{c}} \in \mathbb{R}^{q\times q}$, respectively.

i) If there exists $\Sigma \triangleq \mathrm{diag}[\sigma_1, ..., \sigma_q] > 0$ such that $S_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}) + \Sigma S_{\mathrm{cc}}(u_{\mathrm{cc}}, y_{\mathrm{cc}}) \leq\leq 0$, $S_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}) + \Sigma S_{\mathrm{dc}}(u_{\mathrm{dc}}, y_{\mathrm{dc}}) \leq\leq 0$, and $\tilde{W} \in \mathbb{R}^{q\times q}$ is semistable (respectively, asymptotically stable), where $\tilde{W}_{(i,j)} \triangleq \max\{W_{(i,j)}, (\Sigma W_{\mathrm{c}}\Sigma^{-1})_{(i,j)}\} = \max\{W_{(i,j)}, \frac{\sigma_i}{\sigma_j} W_{\mathrm{c}(i,j)}\}$,

$i,j = 1,...,q$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is Lyapunov (respectively, asymptotically) stable.

ii) Let $Q_{{\rm c}i} \in \mathbb{S}^{l_{{\rm c}i}}$, $S_{{\rm c}i} \in \mathbb{R}^{l_{{\rm c}i}\times m_{{\rm c}i}}$, $R_{{\rm c}i} \in \mathbb{S}^{m_{{\rm c}i}}$, $Q_{{\rm d}i} \in \mathbb{S}^{l_{{\rm d}i}}$, $S_{{\rm d}i} \in \mathbb{R}^{l_{{\rm d}i}\times m_{{\rm d}i}}$, $R_{{\rm d}i} \in \mathbb{S}^{m_{{\rm d}i}}$, $Q_{{\rm cc}i} \in \mathbb{S}^{m_{{\rm c}i}}$, $S_{{\rm cc}i} \in \mathbb{R}^{m_{{\rm c}i}\times l_{{\rm c}i}}$, $R_{{\rm cc}i} \in \mathbb{S}^{l_{{\rm c}i}}$, $Q_{{\rm dc}i} \in \mathbb{S}^{m_{{\rm d}i}}$, $S_{{\rm dc}i} \in \mathbb{R}^{m_{{\rm d}i}\times l_{{\rm d}i}}$, and $R_{{\rm dc}i} \in \mathbb{S}^{l_{{\rm d}i}}$, and suppose $S_{\rm c}(u_{\rm c}, y_{\rm c}) = [s_{\rm c1}(u_{\rm c1}, y_{\rm c1}), ..., s_{{\rm c}q}(u_{{\rm c}q}, y_{{\rm c}q})]^{\rm T}$, $S_{\rm d}(u_{\rm d}, y_{\rm d}) = [s_{\rm d1}(u_{\rm d1}, y_{\rm d1}), ..., s_{{\rm d}q}(u_{{\rm d}q}, y_{{\rm d}q})]^{\rm T}$, $S_{\rm cc}(u_{\rm cc}, y_{\rm cc}) = [s_{\rm cc1}(u_{\rm cc1}, y_{\rm cc1}), ..., s_{{\rm cc}q}(u_{{\rm cc}q}, y_{{\rm cc}q})]^{\rm T}$, and $S_{\rm dc}(u_{\rm dc}, y_{\rm dc}) = [s_{\rm dc1}(u_{\rm dc1}, y_{\rm dc1}), ..., s_{{\rm dc}q}(u_{{\rm dc}q}, y_{{\rm dc}q})]^{\rm T}$, where $s_{{\rm c}i}(u_{{\rm c}i}, y_{{\rm c}i}) = u_{{\rm c}i}^{\rm T} R_{{\rm c}i} u_{{\rm c}i} + 2y_{{\rm c}i}^{\rm T} S_{{\rm c}i} u_{{\rm c}i} + y_{{\rm c}i}^{\rm T} Q_{{\rm c}i} y_{{\rm c}i}$, $s_{{\rm d}i}(u_{{\rm d}i}, y_{{\rm d}i}) = u_{{\rm d}i}^{\rm T} R_{{\rm d}i} u_{{\rm d}i} + 2y_{{\rm d}i}^{\rm T} S_{{\rm d}i} u_{{\rm d}i} + y_{{\rm d}i}^{\rm T} Q_{{\rm d}i} y_{{\rm d}i}$, $s_{{\rm cc}i}(u_{{\rm cc}i}, y_{{\rm cc}i}) = u_{{\rm cc}i}^{\rm T} R_{{\rm cc}i} u_{{\rm cc}i} + 2y_{{\rm cc}i}^{\rm T} S_{{\rm cc}i} u_{{\rm cc}i} + y_{{\rm cc}i}^{\rm T} Q_{{\rm cc}i} y_{{\rm cc}i}$, and $s_{{\rm dc}i}(u_{{\rm dc}i}, y_{{\rm dc}i}) = u_{{\rm dc}i}^{\rm T} R_{{\rm dc}i} u_{{\rm dc}i} + 2y_{{\rm dc}i}^{\rm T} S_{{\rm dc}i} u_{{\rm dc}i} + y_{{\rm dc}i}^{\rm T} Q_{{\rm dc}i} y_{{\rm dc}i}$, $i = 1, ..., q$. If there exists $\Sigma \triangleq {\rm diag}[\sigma_1, ..., \sigma_q] > 0$ such that for all $i = 1, ..., q$,

$$\tilde{Q}_{{\rm c}i} \triangleq \begin{bmatrix} Q_{{\rm c}i} + \sigma_i R_{{\rm cc}i} & -S_{{\rm c}i} + \sigma_i S_{{\rm cc}i}^{\rm T} \\ -S_{{\rm c}i}^{\rm T} + \sigma_i S_{{\rm cc}i} & R_{{\rm c}i} + \sigma_i Q_{{\rm cc}i} \end{bmatrix} \leq 0, \tag{4.137}$$

$$\tilde{Q}_{{\rm d}i} \triangleq \begin{bmatrix} Q_{{\rm d}i} + \sigma_i R_{{\rm dc}i} & -S_{{\rm d}i} + \sigma_i S_{{\rm dc}i}^{\rm T} \\ -S_{{\rm d}i}^{\rm T} + \sigma_i S_{{\rm dc}i} & R_{{\rm d}i} + \sigma_i Q_{{\rm dc}i} \end{bmatrix} \leq 0, \tag{4.138}$$

and $\tilde{W} \in \mathbb{R}^{q\times q}$ is semistable (respectively, asymptotically stable), where $\tilde{W}_{(i,j)} \triangleq \max\{W_{(i,j)}, (\Sigma W_{\rm c} \Sigma^{-1})_{(i,j)}\} = \max\{W_{(i,j)}, \frac{\sigma_i}{\sigma_j} W_{{\rm c}(i,j)}\}$, $i,j = 1,...,q$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is Lyapunov (respectively, asymptotically) stable.

Proof. Let $\tilde{\mathcal{T}}^{\rm c} \triangleq \mathcal{T}^{\rm c}_{x_0,u_{\rm c}} \cup \mathcal{T}^{\rm c}_{x_{\rm c0},u_{\rm cc}}$ and $t_k \in \tilde{\mathcal{T}}^{\rm c}$, $k \in \overline{\mathbb{Z}}_+$. First, note that it follows from Assumptions A1 and A2 that the resetting times $t_k (= \tau_k(\tilde{x}_0))$ for the feedback system are well defined and distinct for every closed-loop trajectory. *i)* Consider the vector Lyapunov function candidate $V(x, x_{\rm c}) = V_{\rm s}(x) + \Sigma V_{\rm cs}(x_{\rm c})$, $(x, x_{\rm c}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\rm c}}$, and note that the corresponding vector Lyapunov derivative of $V(x, x_{\rm c})$ along the state trajectories $(x(t), x_{\rm c}(t))$, $t \in (t_k, t_{k+1})$, is given by

$$\begin{aligned} \dot{V}(x(t), x_{\rm c}(t)) &= \dot{V}_{\rm s}(x(t)) + \Sigma \dot{V}_{\rm cs}(x_{\rm c}(t)) \\ &\leq\leq S_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)) + \Sigma S_{\rm cc}(u_{\rm cc}(t), y_{\rm cc}(t)) + W V_{\rm s}(x(t)) + \Sigma W_{\rm c} V_{\rm cs}(x_{\rm c}(t)) \\ &\leq\leq W V_{\rm s}(x(t)) + \Sigma W_{\rm c} \Sigma^{-1} \Sigma V_{\rm cs}(x_{\rm c}(t)) \end{aligned}$$

$$
\begin{aligned}
&\leq\leq \tilde{W}(V_{\mathrm{s}}(x(t))+\Sigma V_{\mathrm{cs}}(x_{\mathrm{c}}(t)))\\
&= \tilde{W}V(x(t),x_{\mathrm{c}}(t)), \quad (x(t),x_{\mathrm{c}}(t))\notin \tilde{\mathcal{Z}}_{\tilde{x}},
\end{aligned} \tag{4.139}
$$

and the Lyapunov difference of $V(x,x_{\mathrm{c}})$ at the resetting times t_k, $k\in\overline{\mathbb{Z}}_+$, is given by

$$
\begin{aligned}
\Delta V(x(t_k),x_{\mathrm{c}}(t_k)) &= \Delta V_{\mathrm{s}}(x(t_k))+\Sigma\Delta V_{\mathrm{cs}}(x_{\mathrm{c}}(t_k))\\
&\leq\leq S_{\mathrm{d}}(u_{\mathrm{d}}(t_k),y_{\mathrm{d}}(t_k))+\Sigma S_{\mathrm{dc}}(u_{\mathrm{dc}}(t_k),y_{\mathrm{dc}}(t_k))\\
&\leq\leq 0, \quad (x(t),x_{\mathrm{c}}(t))\in\tilde{\mathcal{Z}}_{\tilde{x}}.
\end{aligned} \tag{4.140}
$$

Next, since for $V_{\mathrm{s}}(x)$, $x\in\mathbb{R}^n$, and $V_{\mathrm{cs}}(x_{\mathrm{c}})$, $x_{\mathrm{c}}\in\mathbb{R}^{n_{\mathrm{c}}}$, there exist, by assumption, $p\in\mathbb{R}^q_+$ and $p_{\mathrm{c}}\in\mathbb{R}^q_+$ such that the functions $v_{\mathrm{s}}(x)=p^{\mathrm{T}}V_{\mathrm{s}}(x)$, $x\in\mathbb{R}^n$, and $v_{\mathrm{cs}}(x_{\mathrm{c}})=p_{\mathrm{c}}^{\mathrm{T}}V_{\mathrm{cs}}(x_{\mathrm{c}})$, $x_{\mathrm{c}}\in\mathbb{R}^{n_{\mathrm{c}}}$, are positive definite and noting that $v_{\mathrm{cs}}(x_{\mathrm{c}})\leq\max_{i=1,\ldots,q}\{p_{\mathrm{c}i}\}\mathbf{e}^{\mathrm{T}}V_{\mathrm{cs}}(x_{\mathrm{c}})$, where $p_{\mathrm{c}i}$ is the ith element of p_{c} and $\mathbf{e}\triangleq[1,...,1]^{\mathrm{T}}$, it follows that $\mathbf{e}^{\mathrm{T}}V_{\mathrm{cs}}(x_{\mathrm{c}})$, $x_{\mathrm{c}}\in\mathbb{R}^{n_{\mathrm{c}}}$, is positive definite. Now, since $\min_{i=1,\ldots,q}\{p_i\sigma_i\}\mathbf{e}^{\mathrm{T}}V_{\mathrm{cs}}(x_{\mathrm{c}})\leq p^{\mathrm{T}}\Sigma V_{\mathrm{cs}}(x_{\mathrm{c}})$, it follows that $p^{\mathrm{T}}\Sigma V_{\mathrm{cs}}(x_{\mathrm{c}})$, $x_{\mathrm{c}}\in\mathbb{R}^{n_{\mathrm{c}}}$, is positive definite. Hence, the function $v(x,x_{\mathrm{c}})=p^{\mathrm{T}}V(x,x_{\mathrm{c}})$, $(x,x_{\mathrm{c}})\in\mathbb{R}^n\times\mathbb{R}^{n_{\mathrm{c}}}$, is positive definite. Now, the result is a direct consequence of Theorem 4.1.

ii) The proof follows from *i*) by noting that, for all $i=1,..,q$,

$$
s_{\mathrm{c}i}(u_{\mathrm{c}i},y_{\mathrm{c}i})+\sigma_i s_{\mathrm{cc}i}(u_{\mathrm{cc}i},y_{\mathrm{cc}i}) = \begin{bmatrix} y_{\mathrm{c}}\\ y_{\mathrm{cc}}\end{bmatrix}^{\mathrm{T}}\tilde{Q}_{\mathrm{c}i}\begin{bmatrix} y_{\mathrm{c}}\\ y_{\mathrm{cc}}\end{bmatrix}, \tag{4.141}
$$

$$
s_{\mathrm{d}i}(u_{\mathrm{d}i},y_{\mathrm{d}i})+\sigma_i s_{\mathrm{dc}i}(u_{\mathrm{dc}i},y_{\mathrm{dc}i}) = \begin{bmatrix} y_{\mathrm{d}}\\ y_{\mathrm{dc}}\end{bmatrix}^{\mathrm{T}}\tilde{Q}_{\mathrm{d}i}\begin{bmatrix} y_{\mathrm{d}}\\ y_{\mathrm{dc}}\end{bmatrix}, \tag{4.142}
$$

and hence, $S_{\mathrm{c}}(u_{\mathrm{c}},y_{\mathrm{c}})+\Sigma S_{\mathrm{cc}}(u_{\mathrm{cc}},y_{\mathrm{cc}})\leq\leq 0$ and $S_{\mathrm{d}}(u_{\mathrm{d}},y_{\mathrm{d}})+\Sigma S_{\mathrm{dc}}(u_{\mathrm{dc}},y_{\mathrm{dc}})\leq\leq 0$. □

For the next result note that if the large-scale impulsive dynamical system $\mathcal{G}$ is vector dissipative with respect to the vector hybrid supply rate $(S_{\mathrm{c}}(u_{\mathrm{c}},y_{\mathrm{c}}),S_{\mathrm{d}}(u_{\mathrm{d}},y_{\mathrm{d}}))$, where $s_{\mathrm{c}i}(u_{\mathrm{c}i},y_{\mathrm{c}i})=2y_{\mathrm{c}i}^{\mathrm{T}}u_{\mathrm{c}i}$ and $s_{\mathrm{d}i}(u_{\mathrm{d}i},y_{\mathrm{d}i})=2y_{\mathrm{d}i}^{\mathrm{T}}u_{\mathrm{d}i}$, $i=1,...,q$, then with $\kappa_{\mathrm{c}i}(y_{\mathrm{c}i})=-\kappa_{\mathrm{c}i}y_{\mathrm{c}i}$ and $\kappa_{\mathrm{d}i}(y_{\mathrm{d}i})=-\kappa_{\mathrm{d}i}y_{\mathrm{d}i}$, where $\kappa_{\mathrm{c}i}>0$, $\kappa_{\mathrm{d}i}>0$, $i=1,...,q$, it follows that $s_{\mathrm{c}i}(\kappa_{\mathrm{c}i}(y_{\mathrm{c}i}),y_{\mathrm{c}i})=-\kappa_{\mathrm{c}i}y_{\mathrm{c}i}^{\mathrm{T}}y_{\mathrm{c}i}<0$ and $s_{\mathrm{d}i}(\kappa_{\mathrm{d}i}(y_{\mathrm{d}i}),y_{\mathrm{d}i})=-\kappa_{\mathrm{d}i}y_{\mathrm{d}i}^{\mathrm{T}}y_{\mathrm{d}i}<0$, $y_{\mathrm{c}i}\neq 0$, $y_{\mathrm{d}i}\neq 0$, $i=1,...,q$. Alternatively, if $\mathcal{G}$ is vector dissipative with respect to the vector hybrid supply rate $(S_{\mathrm{c}}(u_{\mathrm{c}},y_{\mathrm{c}}),S_{\mathrm{d}}(u_{\mathrm{d}},$

$y_{\rm d})$), where $s_{{\rm c}i}(u_{{\rm c}i}, y_{{\rm c}i}) = \gamma_{{\rm c}i}^2 u_{{\rm c}i}^{\rm T} u_{{\rm c}i} - y_{{\rm c}i}^{\rm T} y_{{\rm c}i}$ and $s_{{\rm d}i}(u_{{\rm d}i}, y_{{\rm d}i}) = \gamma_{{\rm d}i}^2 u_{{\rm d}i}^{\rm T} u_{{\rm d}i} - y_{{\rm d}i}^{\rm T} y_{{\rm d}i}$, where $\gamma_{{\rm c}i} > 0$, $\gamma_{{\rm d}i} > 0$, $i = 1, ..., q$, then with $\kappa_{{\rm c}i}(y_{{\rm c}i}) = 0$ and $\kappa_{{\rm d}i}(y_{{\rm d}i}) = 0$, it follows that $s_{{\rm c}i}(\kappa_{{\rm c}i}(y_{{\rm c}i}), y_{{\rm c}i}) = -y_{{\rm c}i}^{\rm T} y_{{\rm c}i} < 0$ and $s_{{\rm d}i}(\kappa_{{\rm d}i}(y_{{\rm d}i}), y_{{\rm d}i}) = -y_{{\rm d}i}^{\rm T} y_{{\rm d}i} < 0$, $y_{{\rm c}i} \neq 0$, $y_{{\rm d}i} \neq 0$, $i = 1, ..., q$. Hence, if $\mathcal{G}$ is zero-state observable and the dissipation matrix W is such that there exist $\alpha \geq 0$ and $p \in \mathbb{R}_+^q$ such that (4.2) holds, then it follows from Theorem 4.6 that (scalar) storage functions of the form $v_{\rm s}(x) = p^{\rm T} V_{\rm s}(x)$, $x \in \mathbb{R}^n$, where $V_{\rm s}(\cdot)$ is a vector storage function for $\mathcal{G}$, are positive definite. If $\mathcal{G}$ is exponentially vector dissipative, then p is positive.

Corollary 4.4. Consider the large-scale impulsive dynamical systems $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ given by (4.15)–(4.18) and (4.128)–(4.131), respectively. Assume that $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are zero-state observable and the dissipation matrices $W \in \mathbb{R}^{q\times q}$ and $W_{\rm c} \in \mathbb{R}^{q\times q}$ are such that there exist, respectively, $\alpha \geq 0$, $p \in \mathbb{R}_+^q$, $\alpha_{\rm c} \geq 0$, and $p_{\rm c} \in \mathbb{R}_+^q$ such that (4.2) is satisfied. Then the following statements hold:

i) If $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are vector passive and $\tilde{W} \in \mathbb{R}^{q\times q}$ is asymptotically stable, where $\tilde{W}_{(i,j)} \triangleq \max\{W_{(i,j)}, W_{{\rm c}(i,j)}\}$, $i, j = 1, ..., q$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is asymptotically stable.

ii) If $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are vector nonexpansive and $\tilde{W} \in \mathbb{R}^{q\times q}$ is asymptotically stable, where $\tilde{W}_{(i,j)} \triangleq \max\{W_{(i,j)}, W_{{\rm c}(i,j)}\}$, $i, j = 1, ..., q$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is asymptotically stable.

Proof. The proof is a direct consequence of Theorem 4.12. Specifically, *i)* follows from Theorem 4.12 with $R_{{\rm c}i} = 0$, $S_{{\rm c}i} = I_{m_{{\rm c}i}}$, $Q_{{\rm c}i} = 0$, $R_{{\rm d}i} = 0$, $S_{{\rm d}i} = I_{m_{{\rm d}i}}$, $Q_{{\rm d}i} = 0$, $R_{{\rm cc}i} = 0$, $S_{{\rm cc}i} = I_{m_{{\rm c}i}}$, $Q_{{\rm cc}i} = 0$, $R_{{\rm dc}i} = 0$, $S_{{\rm dc}i} = I_{m_{{\rm d}i}}$, $Q_{{\rm dc}i} = 0$, $i = 1, ..., q$, and $\Sigma = I_q$; while *ii)* follows from Theorem 4.12 with $R_{{\rm c}i} = \gamma_{{\rm c}i}^2 I_{m_{{\rm c}i}}$, $S_{{\rm c}i} = 0$, $Q_{{\rm c}i} = -I_{l_{{\rm c}i}}$, $R_{{\rm d}i} = \gamma_{{\rm d}i}^2 I_{m_{{\rm d}i}}$, $S_{{\rm d}i} = 0$, $Q_{{\rm d}i} = -I_{l_{{\rm d}i}}$, $R_{{\rm cc}i} = \gamma_{{\rm cc}i}^2 I_{l_{{\rm c}i}}$, $S_{{\rm cc}i} = 0$, $Q_{{\rm cc}i} = -I_{m_{{\rm c}i}}$, $R_{{\rm dc}i} = \gamma_{{\rm dc}i}^2 I_{l_{{\rm d}i}}$, $S_{{\rm dc}i} = 0$, $Q_{{\rm dc}i} = -I_{m_{{\rm d}i}}$, $i = 1, ..., q$, and $\Sigma = I_q$. $\square$

Chapter 5

Energy- and Entropy-Based Stabilization for Nonlinear Systems via Hybrid Controllers

5.1. Introduction

Energy is a concept that underlies our understanding of all physical phenomena and is a measure of the ability of a dynamical system to produce changes (motion) in its own system state as well as changes in the system states of its surroundings. In control engineering, dissipativity theory [236], which encompasses passivity theory, provides a fundamental framework for the analysis and control design of dynamical systems using an input, state, and output system description based on system energy related considerations [161, 189, 218]. The notion of energy here refers to abstract energy notions for which a physical system energy interpretation is not necessary. The dissipation hypothesis on dynamical systems results in a fundamental constraint on their dynamic behavior, wherein a dissipative dynamical system can only deliver a fraction of its energy to its surroundings and can only store a fraction of the work done to it. Thus, dissipativity theory provides a powerful framework for the analysis and control design of dynamical systems based on generalized energy considerations by exploiting the notion that numerous physical systems have certain input, state, and output properties related to conservation, dissipation, and transport of energy. Such conservation laws are prevalent in dynamical systems such as mechanical, fluid, electromechanical, electrical, combustion, structural vibration, biological, physiological, power, telecommunications, and economic systems, to cite but a few examples.

Energy-based control for Euler-Lagrange dynamical systems and Hamiltonian dynamical systems based on passivity notions has received considerable attention in the literature [181, 188–190, 217, 221]. This controller design technique achieves system stabilization by shaping the energy of the closed-loop system which involves the physical system energy and the controller emulated energy. Specifically, *energy shaping* is achieved by modifying the system potential energy in such a way so that the shaped potential energy function for the closed-loop system possesses a unique global minimum at a desired equilibrium point. Next, damping is *injected* via feedback control modifying the system dissipation to guarantee asymptotic stability of the closed-loop system. A central feature of this energy-based stabilization approach is that the Lagrangian system form is preserved at the closed-loop system level. Furthermore, the control action has a clear physical energy interpretation, wherein the total energy of the closed-loop Euler-Lagrange system corresponds to the difference between the physical system energy and the emulated energy supplied by the controller.

More recently, a passivity-based control framework for port-controlled Hamiltonian systems is established in [191, 218]. Specifically, the authors in [191] develop a controller design methodology that achieves stabilization via system passivation. In particular, the interconnection and damping matrix functions of the port-controlled Hamiltonian system are shaped so that the physical (Hamiltonian) system structure is preserved at the closed-loop level, and the closed-loop energy function is equal to the difference between the physical energy of the system and the energy supplied by the controller. Since the Hamiltonian structure is preserved at the closed-loop level, the passivity-based controller is *robust* with respect to unmodeled passive dynamics. Furthermore, passivity-based control architectures are extremely appealing since the control action has a clear *physical* energy interpretation which can considerably simplify controller implementation.

In this chapter, we develop a novel energy dissipating hybrid control framework for lossless dynamical systems. These dynamical systems cover a very broad spectrum of applications including mechanical, electrical, electromechanical, structural, biological, and power systems. The dynamic, energy-based hybrid controller is a hybrid controller that emulates an *approximately lossless* hybrid dynamical system and exploits the feature that the states of the dynamic controller may be reset to enhance the overall energy dissipation in the closed-loop system. An important feature of the hybrid controller is that its structure can be associated with an energy function. In a mechanical Euler-Lagrange system, positions typically correspond to elastic deformations, which contribute to the potential energy of the system, whereas velocities typically correspond to momenta, which contribute to the kinetic energy of the system. On the other hand, while our energy-based hybrid controller has dynamical states that emulate the motion of a physical lossless system, these states only "exist" as numerical representations inside the processor. Consequently, while one can associate an *emulated energy* with these states, this energy is merely a mathematical construct and does not correspond to any physical form of energy.

The concept of an energy-based hybrid controller can be viewed as a feedback control technique that exploits the coupling between a physical dynamical system and an energy-based controller to efficiently remove energy from the physical system. Specifically, if a dissipative or lossless plant is at high energy level, and a lossless feedback controller at a low energy level is attached to it, then energy will generally tend to flow from the plant into the controller, decreasing the plant energy and increasing the controller energy [142]. Of course, emulated energy, and not physical energy, is accumulated by the controller. Conversely, if the attached controller is at a high energy level and a plant is at a low energy level, then energy can flow from the controller to the plant, since a controller can generate real, physical energy to effect the required energy flow. Hence, if and when the controller states coincide with a high emulated energy level, then we can *reset* these states to remove the emulated energy so that the emulated energy is not returned to the plant. In this case, the overall closed-loop system consisting of the plant and the controller possesses discontinuous flows since it combines logical switchings with continuous dynamics, leading to impulsive differential equations [14–16, 52, 98, 105, 147, 215]. Within the context of vibration control using resetting virtual absorbers, these ideas were first explored in [44].

5.2. Hybrid Control and Impulsive Dynamical Systems

In this section, we establish definitions, notation, and review some basic results on impulsive dynamical systems [98]. Let $\overline{\mathbb{R}}_+$ denote the set of nonnegative real numbers, let $\overline{\mathbb{Z}}_+$ denote the set of nonnegative integers, and let $\partial\mathcal{S}$, $\overset{\circ}{\mathcal{S}}$, and $\overline{\mathcal{S}}$ denote the boundary, the interior, and the closure of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. We write $x(t) \to \mathcal{M}$ as $t \to \infty$ to denote that $x(t)$ approaches the set $\mathcal{M}$, that is, for each $\varepsilon > 0$ there exists $T > 0$ such that $\operatorname{dist}(x(t), \mathcal{M}) < \varepsilon$ for all $t > T$, where $\operatorname{dist}(p, \mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \|p - x\|$.

In the first part of this chapter, we consider continuous-time nonlinear dynamical systems of the form

$$\dot{x}_{\mathrm{p}}(t) = f_{\mathrm{p}}(x_{\mathrm{p}}(t), u(t)), \quad x_{\mathrm{p}}(0) = x_{\mathrm{p}0}, \quad t \geq 0, \tag{5.1}$$

$$y(t) = h_{\mathrm{p}}(x_{\mathrm{p}}(t)), \tag{5.2}$$

where $t \geq 0$, $x_{\mathrm{p}}(t) \in \mathcal{D}_{\mathrm{p}} \subseteq \mathbb{R}^{n_{\mathrm{p}}}$, $\mathcal{D}_{\mathrm{p}}$ is an open set with $0 \in \mathcal{D}_{\mathrm{p}}$, $u(t) \in \mathbb{R}^m$, $f_{\mathrm{p}} : \mathcal{D}_{\mathrm{p}} \times \mathbb{R}^m \to \mathbb{R}^{n_{\mathrm{p}}}$ is smooth (i.e., infinitely differentiable) on $\mathcal{D}_{\mathrm{p}} \times \mathbb{R}^m$ and satisfies $f_{\mathrm{p}}(0,0) = 0$, and $h_{\mathrm{p}} : \mathcal{D}_{\mathrm{p}} \to \mathbb{R}^l$ is smooth and satisfies $h_{\mathrm{p}}(0) = 0$. Furthermore, we consider hybrid (resetting) dynamic controllers of the form

$$\dot{x}_{\mathrm{c}}(t) = f_{\mathrm{cc}}(x_{\mathrm{c}}(t), y(t)), \quad x_{\mathrm{c}}(0) = x_{\mathrm{c}0}, \quad (x_{\mathrm{c}}(t), y(t)) \notin \mathcal{Z}_{\mathrm{c}}, \tag{5.3}$$

$$\Delta x_{\mathrm{c}}(t) = f_{\mathrm{dc}}(x_{\mathrm{c}}(t), y(t)), \quad (x_{\mathrm{c}}(t), y(t)) \in \mathcal{Z}_{\mathrm{c}}, \tag{5.4}$$

$$u(t) = h_{\mathrm{cc}}(x_{\mathrm{c}}(t), y(t)), \tag{5.5}$$

where $t \geq 0$, $x_{\mathrm{c}}(t) \in \mathcal{D}_{\mathrm{c}} \subseteq \mathbb{R}^{n_{\mathrm{c}}}$, $\mathcal{D}_{\mathrm{c}}$ is an open set with $0 \in \mathcal{D}_{\mathrm{c}}$, $\Delta x_{\mathrm{c}}(t) \triangleq x_{\mathrm{c}}(t^+) - x_{\mathrm{c}}(t)$, where $x_{\mathrm{c}}(t^+) \triangleq x_{\mathrm{c}}(t) + f_{\mathrm{dc}}(x_{\mathrm{c}}(t), y(t)) = \lim_{\varepsilon\to 0^+} x_{\mathrm{c}}(t+\varepsilon)$, $(x_{\mathrm{c}}(t), y(t)) \in \mathcal{Z}_{\mathrm{c}}$, $f_{\mathrm{cc}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l \to \mathbb{R}^{n_{\mathrm{c}}}$ is smooth on $\mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l$ and satisfies $f_{\mathrm{cc}}(0,0) = 0$, $h_{\mathrm{cc}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l \to \mathbb{R}^m$ is smooth and satisfies $h_{\mathrm{cc}}(0,0) = 0$, $f_{\mathrm{dc}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l \to \mathbb{R}^{n_{\mathrm{c}}}$ is continuous, and $\mathcal{Z}_{\mathrm{c}} \subset \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l$ is the resetting set. Note that, for generality, we allow the hybrid dynamic controller to be of fixed dimension n_{c} which may be less than the plant order n_{p}.

The equations of motion for the closed-loop dynamical system (5.1)–(5.5) have the form

$$\dot{x}(t) = f_{\mathrm{c}}(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}, \tag{5.6}$$

$$\Delta x(t) = f_{\mathrm{d}}(x(t)), \quad x(t) \in \mathcal{Z}, \tag{5.7}$$

where

$$x \triangleq \begin{bmatrix} x_{\mathrm{p}} \\ x_{\mathrm{c}} \end{bmatrix} \in \mathbb{R}^n, \quad f_{\mathrm{c}}(x) \triangleq \begin{bmatrix} f_{\mathrm{p}}(x_{\mathrm{p}}, h_{\mathrm{cc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}}))) \\ f_{\mathrm{cc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) \end{bmatrix}, \quad f_{\mathrm{d}}(x) \triangleq \begin{bmatrix} 0 \\ f_{\mathrm{dc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) \end{bmatrix}, \tag{5.8}$$

and $\mathcal{Z} \triangleq \{x \in \mathcal{D} : (x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) \in \mathcal{Z}_{\mathrm{c}}\}$, with $n \triangleq n_{\mathrm{p}} + n_{\mathrm{c}}$ and $\mathcal{D} \triangleq \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}}$. We refer to the differential equation (5.6) as the *continuous-time dynamics*, and we refer to the difference equation (5.7) as the *resetting law*. Note that although the closed-loop state vector consists of plant states and controller states, it is clear from (5.8) that only those states associated with the controller are reset. To ensure well-posedness of the solutions to (5.6) and (5.7), we make the following additional assumptions [98]:

Assumption 1. If $x \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$, then there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$, $\psi(\delta, x) \notin \mathcal{Z}$, where $\psi(\cdot, \cdot)$ denotes the solution to the continuous-time dynamics (5.6).

Assumption 2. If $x \in \mathcal{Z}$, then $x + f_{\mathrm{d}}(x) \notin \mathcal{Z}$.

Assumption 1 ensures that if a trajectory reaches the closure of $\mathcal{Z}$ at a point that does not belong to $\mathcal{Z}$, then the trajectory must be directed away from $\mathcal{Z}$, that is, a trajectory cannot enter $\mathcal{Z}$ through a point that belongs to the closure of $\mathcal{Z}$ but not to $\mathcal{Z}$. Furthermore, Assumption 2 ensures that when a trajectory intersects the resetting set $\mathcal{Z}$, it instantaneously exits $\mathcal{Z}$. Finally, we note that if $x_0 \in \mathcal{Z}$, then the system initially resets to $x_0^+ = x_0 + f_{\mathrm{d}}(x_0) \notin \mathcal{Z}$, which serves as the initial condition for the continuous-time dynamics (5.6).

A function $x : \mathcal{I}_{x_0} \to \mathcal{D}$ is a *solution* to the impulsive dynamical system (5.6) and (5.7) on the interval $\mathcal{I}_{x_0} \subseteq \mathbb{R}$ with initial condition $x(0) = x_0$, where $\mathcal{I}_{x_0}$ denotes the maximal interval of existence of a solution to (5.6) and (5.7), if $x(\cdot)$ is left-continuous and $x(t)$ satisfies

(5.6) and (5.7) for all $t \in \mathcal{I}_{x_0}$. For further discussion on solutions to impulsive differential equations, see [14, 15, 41, 52, 98, 147, 175, 215, 241]. For convenience, we use the notation $s(t, x_0)$ to denote the solution $x(t)$ of (5.6) and (5.7) at time $t \geq 0$ with initial condition $x(0) = x_0$.

For a particular closed-loop trajectory $x(t)$, we let $t_k \triangleq \tau_k(x_0)$ denote the kth instant of time at which $x(t)$ intersects $\mathcal{Z}$, and we call the times t_k the *resetting times*. Thus, the trajectory of the closed-loop system (5.6) and (5.7) from the initial condition $x(0) = x_0$ is given by $\psi(t, x_0)$ for $0 < t \leq t_1$. If and when the trajectory reaches a state $x_1 \triangleq x(t_1)$ satisfying $x_1 \in \mathcal{Z}$, then the state is instantaneously transferred to $x_1^+ \triangleq x_1 + f_{\rm d}(x_1)$ according to the resetting law (5.7). The trajectory $x(t)$, $t_1 < t \leq t_2$, is then given by $\psi(t - t_1, x_1^+)$, and so on. Our convention here is that the solution $x(t)$ of (5.6) and (5.7) is left continuous, that is, it is continuous everywhere except at the resetting times t_k, and $x_k \triangleq x(t_k) = \lim_{\varepsilon \to 0^+} x(t_k - \varepsilon)$ and $x_k^+ \triangleq x(t_k) + f_{\rm d}(x(t_k)) = \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon)$ for $k = 1, 2, \ldots$.

It follows from Assumptions 1 and 2 that for a particular initial condition, the resetting times $t_k = \tau_k(x_0)$ are distinct and well defined [98]. Since the resetting set $\mathcal{Z}$ is a subset of the state space and is independent of time, impulsive dynamical systems of the form (5.6) and (5.7) are time-invariant systems. These systems are called *state-dependent impulsive dynamical systems* [98]. Since the resetting times are well defined and distinct, and since the solution to (5.6) exists and is unique, it follows that the solution of the impulsive dynamical system (5.6) and (5.7) also exists and is unique over a forward time interval. For details on the existence and uniqueness of solutions of impulsive dynamical systems in forward time see [14, 15, 147, 215].

Remark 5.1. Let $x^* \in \mathcal{D}$ satisfy $f_{\rm d}(x^*) = 0$. Then $x^* \notin \mathcal{Z}$. To see this, suppose $x^* \in \mathcal{Z}$. Then $x^* + f_{\rm d}(x^*) = x^* \in \mathcal{Z}$, which contradicts the assumption that if $x \in \mathcal{Z}$, then $x + f_{\rm d}(x) \notin \mathcal{Z}$. Furthermore, if $x = 0$ is an equilibrium point of (5.6) and (5.7), then $0 \notin \mathcal{Z}$.

For the statement of the next result the following key assumption is needed.

Assumption 3. Consider the impulsive dynamical system (5.6) and (5.7), and let $s(t, x_0)$, $t \geq 0$, denote the solution to (5.6) and (5.7) with initial condition x_0. Then for every $x_0 \notin \mathcal{Z}$ and every $\varepsilon > 0$ and $t \neq t_k$, there exists $\delta(\varepsilon, x_0, t) > 0$ such that if $\|x_0 - z\| < \delta(\varepsilon, x_0, t)$, $z \in \mathcal{D}$, then $\|s(t, x_0) - s(t, z)\| < \varepsilon$.

Assumption 3 is a weakened version of the quasi-continuous dependence assumption given in [52, 98], and is a generalization of the standard continuous dependence property for dynamical systems with continuous flows to dynamical systems with left-continuous flows. Specifically, by letting $t \in [0, \infty)$, Assumption 3 specializes to the classical continuous dependence of solutions of a given dynamical system with respect to the system's initial conditions $x_0 \in \mathcal{D}$ for every time instant. It should be noted that the standard continuous dependence property for dynamical systems with continuous flows is defined uniformly in time on compact intervals. Since solutions of impulsive dynamical systems are not continuous in time and solutions are not continuous functions of the system initial conditions, Assumption 3 involving point-wise continuous dependence is needed to apply the hybrid invariance principle developed in [52, 98] to hybrid closed-loop systems. Sufficient conditions that guarantee that the impulsive dynamical system (5.6) and (5.7) satisfies a stronger version of Assumption 3 are given in [52] (see also [84]). The following proposition provides a generalization of Proposition 4.1 in [52] for establishing sufficient conditions for guaranteeing that the impulsive dynamical system (5.6) and (5.7) satisfies Assumption 3.

Proposition 5.1. Consider the impulsive dynamical system $\mathcal{G}$ given by (5.6) and (5.7). Assume that Assumptions 1 and 2 hold, $\tau_1(\cdot)$ is continuous at every $x \notin \overline{\mathcal{Z}}$ such that $0 < \tau_1(x) < \infty$, and if $x \in \mathcal{Z}$, then $x + f_{\mathrm{d}}(x) \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$. Furthermore, for every $x \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$ such that $0 < \tau_1(x) < \infty$, assume that the following statements hold:

i) If a sequence $\{x_i\}_{i=1}^{\infty} \in \mathcal{D}$ is such that $\lim_{i \to \infty} x_i = x$ and $\lim_{i \to \infty} \tau_1(x_i)$ exists, then either $f_{\mathrm{d}}(x) = 0$ and $\lim_{i \to \infty} \tau_1(x_i) = 0$, or $\lim_{i \to \infty} \tau_1(x_i) = \tau_1(x)$.

ii) If a sequence $\{x_i\}_{i=1}^{\infty} \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$ is such that $\lim_{i \to \infty} x_i = x$ and $\lim_{i \to \infty} \tau_1(x_i)$ exists, then

$\lim_{i\to\infty}\tau_1(x_i)=\tau_1(x)$.

Then $\mathcal{G}$ satisfies Assumption 3.

Proof. Let $x_0\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$ and let $\{x_i\}_{i=1}^{\infty}\in\mathcal{D}$ be such that $\lim_{i\to\infty}x_i=x_0$, $f_{\rm d}(x_0)=0$, and $\lim_{i\to\infty}\tau_1(x_i)=0$ hold. Define $z_i\triangleq s(\tau_1(x_i),x_i)+f_{\rm d}(s(\tau_1(x_i),x_i))=\psi(\tau_1(x_i),x_i)+f_{\rm d}(\psi(\tau_1(x_i),x_i))$, $i=1,2,\ldots$, where $\psi(t,x_0)$ denotes the solution to the continuous-time dynamics (5.6), and note that, since $f_{\rm d}(x_0)=0$ and $\lim_{i\to\infty}\tau_1(x_i)=0$, it follows that $\lim_{i\to\infty}z_i=x_0$. Hence, since by assumption $z_i\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$, $i=1,2,\ldots$, it follows from *ii*) that $\lim_{i\to\infty}\tau_1(z_i)=\tau_1(x_0)$ or, equivalently, $\lim_{i\to\infty}\tau_2(x_i)=\tau_1(x_0)$. Similarly, it can be shown that $\lim_{i\to\infty}\tau_{k+1}(x_i)=\tau_k(x_0)$, $k=2,3,\ldots$. Next, note that

$$\begin{aligned}\lim_{i\to\infty}s(\tau_2(x_i),x_i)&=\lim_{i\to\infty}\psi(\tau_2(x_i)-\tau_1(x_i),s(\tau_1(x_i),x_i)+f_{\rm d}(s(\tau_1(x_i),x_i)))\\&=\psi(\tau_1(x_0),x_0)=s(\tau_1(x_0),x_0).\end{aligned}$$

Now, using mathematical induction it can be shown that $\lim_{i\to\infty}s(\tau_{k+1}(x_i),x_i)=s(\tau_k(x_0),x_0)$, $k=2,3,\ldots$.

Next, let $k\in\{1,2,\ldots\}$ and let $t\in(\tau_k(x_0),\tau_{k+1}(x_0))$. Since $\lim_{i\to\infty}\tau_{k+1}(x_i)=\tau_k(x_0)$, it follows that there exists $I\in\{1,2,\ldots\}$ such that $\tau_{k+1}(x_i)<t$ and $\tau_{k+2}(x_i)>t$ for all $i>I$. Hence, it follows that for every $t\in(\tau_k(x_0),\tau_{k+1}(x_0))$,

$$\begin{aligned}\lim_{i\to\infty}s(t,x_i)&=\lim_{i\to\infty}\psi(t-\tau_{k+1}(x_i),s(\tau_{k+1}(x_i),x_i)+f_{\rm d}(s(\tau_{k+1}(x_i),x_i)))\\&=\psi(t-\tau_k(x_0),s(\tau_k(x_0),x_0)+f_{\rm d}(s(\tau_k(x_0),x_0)))=s(t,x_0).\end{aligned}$$

Alternatively, if $x_0\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$ is such that $\lim_{i\to\infty}\tau_1(x_i)=\tau_1(x_0)$ for $\{x_i\}_{i=1}^{\infty}\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$, then using identical arguments as above, it can be shown that $\lim_{i\to\infty}s(t,x_i)=s(t,x_0)$ for every $t\in(\tau_k(x_0),\tau_{k+1}(x_0))$, $k=1,2,\ldots$.

Finally, let $x_0\notin\overline{\mathcal{Z}}$, $0<\tau_1(x_0)<\infty$, and assume $\tau_1(\cdot)$ is continuous. In this case, it follows from the definition of $\tau_1(x_0)$ that for every $x_0\notin\overline{\mathcal{Z}}$ and $t\in(\tau_1(x_0),\tau_2(x_0)]$,

$$s(t,x_0)=\psi(t-\tau_1(x_0),s(\tau_1(x_0),x_0)+f_{\rm d}(s(\tau_1(x_0),x_0))).\tag{5.9}$$

Since $\psi(\cdot,\cdot)$ is continuous in both its arguments, $\tau_1(\cdot)$ is continuous at x_0, and $f_{\rm d}(\cdot)$ is continuous, it follows that $s(t,\cdot)$ is continuous at x_0 for every $t \in (\tau_1(x_0), \tau_2(x_0))$. Next, for every sequence $\{x_i\}_{i=1}^{\infty} \in \mathcal{D}$ such that $\lim_{i\to\infty} x_i = x_0$, it follows that $\lim_{i\to\infty} s(\tau_1(x_i), x_i) = \lim_{i\to\infty} \psi(\tau_1(x_i), x_i) = \psi(\tau_1(x_0), x_0) = s(\tau_1(x_0), x_0)$. Furthermore, note that by assumption $z_i \triangleq s(\tau_1(x_i), x_i) + f_{\rm d}(s(\tau_1(x_i), x_i)) \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$, $i = 0, 1, \ldots$. Hence, it follows that for all $t \in (\tau_k(z_0), \tau_{k+1}(z_0))$, $k = 1, 2, \ldots$, $\lim_{i\to\infty} s(t, z_i) = s(t, z_0)$ or, equivalently, for all $t \in (\tau_k(x_0), \tau_{k+1}(x_0))$, $k = 2, 3, \ldots$, $\lim_{i\to\infty} s(t, x_i) = s(t, x_0)$, which proves the result. □

The following result provides sufficient conditions for establishing continuity of $\tau_1(\cdot)$ at $x_0 \notin \overline{\mathcal{Z}}$ and *sequential continuity* of $\tau_1(\cdot)$ at $x_0 \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$, that is, $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$ for $\{x_i\}_{i=1}^{\infty} \notin \mathcal{Z}$ and $\lim_{i\to\infty} x_i = x_0$. For this result, the following definition is needed. First, however, recall that the *Lie derivative* of a smooth function $\mathcal{X} : \mathcal{D} \to \mathbb{R}$ along the vector field of the continuous-time dynamics $f_{\rm c}(x)$ is given by $L_{f_{\rm c}}\mathcal{X}(x) \triangleq \frac{\rm d}{{\rm d}t}\mathcal{X}(\psi(t,x))|_{t=0} = \frac{\partial \mathcal{X}(x)}{\partial x} f_{\rm c}(x)$, and the *zeroth* and *higher-order Lie derivatives* are, respectively, defined by $L^0_{f_{\rm c}}\mathcal{X}(x) \triangleq \mathcal{X}(x)$ and $L^k_{f_{\rm c}}\mathcal{X}(x) \triangleq L_{f_{\rm c}}(L^{k-1}_{f_{\rm c}}\mathcal{X}(x))$, where $k \geq 1$.

Definition 5.1. Let $\mathcal{Q} \triangleq \{x \in \mathcal{D} : \ \mathcal{X}(x) = 0\}$, where $\mathcal{X} : \mathcal{D} \to \mathbb{R}$ is an infinitely differentiable function. A point $x \in \mathcal{Q}$ such that $f_{\rm c}(x) \neq 0$ is *k-transversal* to (5.6) if there exists $k \in \{1, 2, \ldots\}$ such that

$$L^r_{f_{\rm c}}\mathcal{X}(x) = 0, \quad r = 0, \ldots, 2k-2, \quad L^{2k-1}_{f_{\rm c}}\mathcal{X}(x) \neq 0. \tag{5.10}$$

Proposition 5.2. Consider the impulsive dynamical system (5.6) and (5.7). Let $\mathcal{X} : \mathcal{D} \to \mathbb{R}$ be an infinitely differentiable function such that $\overline{\mathcal{Z}} = \{x \in \mathcal{D} : \mathcal{X}(x) = 0\}$, and assume that every $x \in \overline{\mathcal{Z}}$ is k-transversal to (5.6). Then at every $x_0 \notin \overline{\mathcal{Z}}$ such that $0 < \tau_1(x_0) < \infty$, $\tau_1(\cdot)$ is continuous. Furthermore, if $x_0 \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$ is such that $\tau_1(x_0) \in (0, \infty)$ and *i)* $\{x_i\}_{i=1}^{\infty} \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$ or *ii)* $\lim_{i\to\infty} \tau_1(x_i) > 0$, where $\{x_i\}_{i=1}^{\infty} \notin \overline{\mathcal{Z}}$ is such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} \tau_1(x_i)$ exists, then $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$.

Proof. Let $x_0 \notin \overline{\mathcal{Z}}$ be such that $0 < \tau_1(x_0) < \infty$. It follows from the definition of $\tau_1(\cdot)$

that $s(t,x_0)=\psi(t,x_0)$, $t\in[0,\tau_1(x_0)]$, $\mathcal{X}(s(t,x_0))\neq 0$, $t\in(0,\tau_1(x_0))$, and $\mathcal{X}(s(\tau_1(x_0),x_0))=0$. Without loss of generality, let $\mathcal{X}(s(t,x_0))>0$, $t\in(0,\tau_1(x_0))$. Since $\hat{x}\triangleq\psi(\tau_1(x_0),x_0)\in\overline{\mathcal{Z}}$ is k-transversal to (5.6), it follows that there exists $\theta>0$ such that $\mathcal{X}(\psi(t,\hat{x}))>0$, $t\in[-\theta,0)$, and $\mathcal{X}(\psi(t,\hat{x}))<0$, $t\in(0,\theta]$. (This fact can be easily shown by expanding $\mathcal{X}(\psi(t,x))$ via a Taylor series expansion about $\hat{x}$ and using the fact that $\hat{x}$ is k-transversal to (5.6).) Hence, $\mathcal{X}(\psi(t,x_0))>0$, $t\in[\hat{t}_1,\tau_1(x_0))$, and $\mathcal{X}(\psi(t,x_0))<0$, $t\in(\tau_1(x_0),\hat{t}_2]$, where $\hat{t}_1\triangleq\tau_1(x_0)-\theta$ and $\hat{t}_2\triangleq\tau_1(x_0)+\theta$.

Next, let $\varepsilon\triangleq\min\{|\mathcal{X}(\psi(\hat{t}_1,x_0))|,|\mathcal{X}(\psi(\hat{t}_2,x_0))|\}$. Now, it follows from the continuity of $\mathcal{X}(\cdot)$ and the continuous dependence of $\psi(\cdot,\cdot)$ on the system initial conditions that there exists $\delta>0$ such that

$$\sup_{0\le t\le\hat{t}_2}|\mathcal{X}(\psi(t,x))-\mathcal{X}(\psi(t,x_0))|<\varepsilon,\quad x\in\mathcal{B}_\delta(x_0),\tag{5.11}$$

which implies that $\mathcal{X}(\psi(\hat{t}_1,x))>0$ and $\mathcal{X}(\psi(\hat{t}_2,x))<0$, $x\in\mathcal{B}_\delta(x_0)$. Hence, it follows that $\hat{t}_1<\tau_1(x)<\hat{t}_2$, $x\in\mathcal{B}_\delta(x_0)$. The continuity of $\tau_1(\cdot)$ at x_0 now follows immediately by noting that θ can be chosen arbitrarily small.

Finally, let $x_0\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$ be such that $\lim_{i\to\infty}x_i=x_0$ for some sequence $\{x_i\}_{i=1}^\infty\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$. Then using similar arguments as above it can be shown that $\lim_{i\to\infty}\tau_1(x_i)=\tau_1(x_0)$. Alternatively, if $x_0\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$ is such that $\lim_{i\to\infty}x_i=x_0$ and $\lim_{i\to\infty}\tau_1(x_i)>0$ for some sequence $\{x_i\}_{i=1}^\infty\notin\mathcal{Z}$, then it follows that there exists sufficiently small $\hat{t}>0$ and $I\in\mathbb{Z}_+$ such that $s(\hat{t},x_i)=\psi(\hat{t},x_i)$, $i=I,I+1,\ldots$, which implies that $\lim_{i\to\infty}s(\hat{t},x_i)=s(\hat{t},x_0)$. Next, define $z_i\triangleq\psi(\hat{t},x_i)$, $i=0,1,\ldots$, so that $\lim_{i\to\infty}z_i=z_0$, and note that it follows from the k-transversality assumption that $z_0\notin\overline{\mathcal{Z}}$, which implies that $\tau_1(\cdot)$ is continuous at z_0. Hence, $\lim_{i\to\infty}\tau_1(z_i)=\tau_1(z_0)$. The result now follows by noting that $\tau_1(x_i)=\hat{t}+\tau_1(z_i)$, $i=1,2,\ldots$.

□

Remark 5.2. Let $x_0\notin\mathcal{Z}$ be such that $\lim_{i\to\infty}\tau_1(x_i)\neq\tau_1(x_0)$ for some sequence

$\{x_i\}_{i=1}^{\infty} \notin \mathcal{Z}$ with $\lim_{i\to\infty} x_i = x_0$. Then it follows from Proposition 5.2 that $\lim_{i\to\infty} \tau_1(x_i) = 0$.

Remark 5.3. The notion of k-transversality introduced here differs from the well-known notion of transversality [68, 88] involving an orthogonality condition between a vector field and a differentiable submanifold. In the case where $k = 1$, Definition 5.1 coincides with the standard notion of transversality and guarantees that the solution of the closed-loop system (5.6) and (5.7) is not tangent to the closure of the resetting set $\mathcal{Z}$ at the intersection with $\overline{\mathcal{Z}}$ [105]. In general, however, k-transversality guarantees that the sign of $\mathcal{X}(x(t))$ changes as the closed-loop system trajectory $x(t)$ transverses the closure of the resetting set $\mathcal{Z}$ at the intersection with $\overline{\mathcal{Z}}$.

Remark 5.4. Proposition 5.2 is a nontrivial generalization of Proposition 4.2 of [52] and Lemma 3 of [84]. Specifically, Proposition 5.2 establishes the continuity of $\tau(\cdot)$ in the case where the resetting set $\mathcal{Z}$ is not a closed set. In addition, the k-transversality condition given in Definition 5.1 is also a generalization of the transversality conditions given in [52], [105], and [84] by considering higher-order derivatives of the function $\mathcal{X}(\cdot)$ rather than simply considering the first-order derivative as in [52, 84].

The next result characterizes impulsive dynamical system limit sets in terms of continuously differentiable functions. In particular, we show that the system trajectories of a state-dependent impulsive dynamical system converge to an invariant set contained in a union of level surfaces characterized by the continuous-time system dynamics and the resetting system dynamics. Note that for addressing the stability of the zero solution of an impulsive dynamical system the usual stability definitions are valid [14, 15, 52, 98, 147, 215]. Specifically, the zero solution $x(t) \equiv 0$ to (5.6) and (5.7) is *Lyapunov stable* if and only if, for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|x(0)\| < \delta$, then $\|x(t)\| < \varepsilon$, $t \geq 0$. The zero solution to (5.6) and (5.7) is *asymptotically stable* if and only if it is Lyapunov stable and there exists $\delta > 0$ such that if $\|x(0)\| < \delta$, then $\lim_{t\to\infty} x(t) = 0$. Asymptotic stability is *global* if the previous statement holds for all $x(0) \in \mathbb{R}^n$.

It is important to note here that since state-dependent impulsive dynamical systems are time-invariant [14], the notions of asymptotic stability and uniform asymptotic stability with respect to initial times are equivalent. However, unlike continuous-time and discrete-time dynamical systems wherein asymptotic stability of autonomous systems is equivalent to the existence of class $\mathcal{K}$ and $\mathcal{L}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, respectively, such that if $\|x_0\| < \delta$, $\delta > 0$, then $\|x(t)\| \leq \alpha(\|x_0\|)\beta(t)$, $t \geq 0$, this is not generally true for state-dependent impulsive dynamical systems. That is, asymptotic stability might not be uniform with respect to compact sets of initial conditions. If, however, for every compact set the first time-to-impact function $\tau_1(x_0)$ is uniformly bounded with respect to the system initial conditions, then it can be shown that asymptotic stability is uniform with respect to compact sets of initial conditions. In the case where $\mathcal{G}_\mathrm{p}$ is *dissipative* with respect to the supply rate $s_\mathrm{p}(u, y)$ global asymptotic stability can be shown to be uniform with respect to compact sets of initial conditions. For further details on this subtle point see [83].

Theorem 5.1. Consider the impulsive dynamical system (5.6) and (5.7), and assume Assumptions 1–3 hold. Assume $\mathcal{D}_\mathrm{ci} \subset \mathcal{D}$ is a compact positively invariant set with respect to (5.6) and (5.7), assume that if $x_0 \in \mathcal{Z}$ then $x_0 + f_\mathrm{d}(x_0) \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$, and assume that there exists a continuously differentiable function $V : \mathcal{D}_\mathrm{ci} \to \mathbb{R}$ such that

$$V'(x)f_\mathrm{c}(x) \leq 0, \quad x \in \mathcal{D}_\mathrm{ci}, \quad x \notin \mathcal{Z}, \tag{5.12}$$

$$V(x + f_\mathrm{d}(x)) \leq V(x), \quad x \in \mathcal{D}_\mathrm{ci}, \quad x \in \mathcal{Z}. \tag{5.13}$$

Let $\mathcal{R} \triangleq \{x \in \mathcal{D}_\mathrm{ci} : x \notin \mathcal{Z}, V'(x)f_\mathrm{c}(x) = 0\} \cup \{x \in \mathcal{D}_\mathrm{ci} : x \in \mathcal{Z}, V(x + f_\mathrm{d}(x)) = V(x)\}$ and let $\mathcal{M}$ denote the largest invariant set contained in $\mathcal{R}$. If $x_0 \in \mathcal{D}_\mathrm{ci}$, then $x(t) \to \mathcal{M}$ as $t \to \infty$. Furthermore, if $0 \in \overset{\circ}{\mathcal{D}}_\mathrm{ci}$, $V(0) = 0$, $V(x) > 0$, $x \neq 0$, and the set $\mathcal{R}$ contains no invariant set other than the set $\{0\}$, then the zero solution $x(t) \equiv 0$ to (5.6) and (5.7) is asymptotically stable and $\mathcal{D}_\mathrm{ci}$ is a subset of the domain of attraction of (5.6) and (5.7).

Proof. The proof is similar to the proof of Corollary 5.1 given in [52] and, hence, is omitted. $\square$

Remark 5.5. Setting $\mathcal{D} = \mathbb{R}^n$ and requiring $V(x) \to \infty$ as $\|x\| \to \infty$ in Theorem 5.1, it follows that the zero solution $x(t) \equiv 0$ to (5.6) and (5.7) is globally asymptotically stable. A similar remark holds for Theorem 5.2 below.

Theorem 5.2. Consider the impulsive dynamical system (5.6) and (5.7), and assume Assumptions 1–3 hold. Assume $\mathcal{D}_{\mathrm{ci}} \subset \mathcal{D}$ is a compact positively invariant set with respect to (5.6) and (5.7) such that $0 \in \overset{\circ}{\mathcal{D}}_{\mathrm{ci}}$, assume that if $x_0 \in \mathcal{Z}$ then $x_0 + f_{\mathrm{d}}(x_0) \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$, and assume that for every $x_0 \in \mathcal{D}_{\mathrm{ci}}$, $x_0 \neq 0$, there exists $\tau \geq 0$ such that $x(\tau) \in \mathcal{Z}$, where $x(t)$, $t \geq 0$, denotes the solution to (5.6) and (5.7) with the initial condition x_0. Furthermore, assume there exists a continuously differentiable function $V : \mathcal{D}_{\mathrm{ci}} \to \mathbb{R}$ such that $V(0) = 0$, $V(x) > 0$, $x \neq 0$,

$$V(x + f_{\mathrm{d}}(x)) < V(x), \quad x \in \mathcal{D}_{\mathrm{ci}}, \quad x \in \mathcal{Z}, \tag{5.14}$$

and (5.12) is satisfied. Then the zero solution $x(t) \equiv 0$ to (5.6) and (5.7) is asymptotically stable and $\mathcal{D}_{\mathrm{ci}}$ is a subset of the domain of attraction of (5.6) and (5.7).

Proof. It follows from (5.14) that $\mathcal{R} = \{x \in \mathcal{D}_{\mathrm{ci}} : x \notin \mathcal{Z}, V'(x) f_{\mathrm{c}}(x) = 0\}$. Since for every $x_0 \in \mathcal{D}_{\mathrm{ci}}$, $x_0 \neq 0$, there exists $\tau \geq 0$ such that $x(\tau) \in \mathcal{Z}$, it follows that the largest invariant set contained in $\mathcal{R}$ is $\{0\}$. Now, the result is a direct consequence of Theorem 5.1. $\square$

5.3. Hybrid Control Design for Lossless Dynamical Systems

In this section, we present a hybrid controller design framework for lossless dynamical systems [236]. Specifically, we consider nonlinear dynamical systems $\mathcal{G}_{\mathrm{p}}$ of the form given by (5.1) and (5.2). Furthermore, we consider hybrid resetting dynamic controllers $\mathcal{G}_{\rm c}$ of the form

$$\dot{x}_{\rm c}(t) = f_{\rm cc}(x_{\rm c}(t), y(t)), \quad x_{\rm c}(0) = x_{\rm c0}, \quad (x_{\rm c}(t), y(t)) \notin \mathcal{Z}_{\rm c}, \tag{5.15}$$

$$\Delta x_{\rm c}(t) = \eta(y(t)) - x_{\rm c}(t), \quad (x_{\rm c}(t), y(t)) \in \mathcal{Z}_{\rm c}, \tag{5.16}$$

$$y_{\rm c}(t) = h_{\rm cc}(x_{\rm c}(t), y(t)), \tag{5.17}$$

where $x_{\rm c}(t) \in \mathcal{D}_{\rm c} \subseteq \mathbb{R}^{n_{\rm c}}$, $\mathcal{D}_{\rm c}$ is an open set with $0 \in \mathcal{D}_{\rm c}$, $y(t) \in \mathbb{R}^l$, $y_{\rm c}(t) \in \mathbb{R}^m$, $f_{\rm cc} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \mathbb{R}^{n_{\rm c}}$ is smooth on $\mathcal{D}_{\rm c} \times \mathbb{R}^l$ and satisfies $f_{\rm cc}(0,0) = 0$, $\eta : \mathbb{R}^l \to \mathcal{D}_{\rm c}$ is continuous and satisfies $\eta(0) = 0$, and $h_{\rm cc} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \mathbb{R}^m$ is smooth and satisfies $h_{\rm cc}(0,0) = 0$.

Recall that for the dynamical system $\mathcal{G}_{\rm p}$ given by (5.1) and (5.2), a function $s_{\rm p}(u,y)$, where $s_{\rm p} : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ is such that $s_{\rm p}(0,0) = 0$, is called a *supply rate* [236] if it is locally integrable for all input-output pairs satisfying (5.1) and (5.2), that is, for all input-output pairs $u \in \mathcal{U}$ and $y \in \mathcal{Y}$ satisfying (5.1) and (5.2), $s_{\rm p}(\cdot,\cdot)$ satisfies $\int_t^{\hat{t}} |s_{\rm p}(u(\sigma), y(\sigma))| {\rm d}\sigma < \infty$, $t, \hat{t} \geq 0$. Here, $\mathcal{U}$ and $\mathcal{Y}$ are input and output spaces, respectively, that are assumed to be closed under the shift operator. Furthermore, we assume that $\mathcal{G}_{\rm p}$ is *lossless with respect to the supply rate* $s_{\rm p}(u,y)$ with a continuously differentiable nonnegative-definite *storage function* $V_{\rm s} : \mathcal{D}_{\rm p} \to \overline{\mathbb{R}}_+$ such that $V_{\rm s}(0) = 0$ and

$$V_{\rm s}(x_{\rm p}(t)) = V_{\rm s}(x_{\rm p}(t_0)) + \int_{t_0}^{t} s_{\rm p}(u(\sigma), y(\sigma)) {\rm d}\sigma, \quad t \geq t_0, \tag{5.18}$$

for all $t_0, t \geq 0$, where $x_{\rm p}(t)$, $t \geq t_0$, is the solution to (5.1) with $u \in \mathcal{U}$. In addition, we assume that the nonlinear dynamical system $\mathcal{G}_{\rm p}$ is *completely reachable* [236] and *zero-state observable* [236], and there exists a function $\kappa : \mathbb{R}^l \to \mathbb{R}^m$ such that $\kappa(0) = 0$ and $s_{\rm p}(\kappa(y), y) < 0$, $y \neq 0$, so that all storage functions $V_{\rm s}(x_{\rm p})$, $x_{\rm p} \in \mathcal{D}_{\rm p}$, of $\mathcal{G}_{\rm p}$ are positive definite [119].

Consider the negative feedback interconnection of $\mathcal{G}_{\rm p}$ and $\mathcal{G}_{\rm c}$ given by $y = u_{\rm c}$ and $u = -y_{\rm c}$. In this case, the closed-loop system $\mathcal{G}$ is given by

$$\dot{x}(t) = f_{\rm c}(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}, \quad t \geq 0, \tag{5.19}$$

$$\Delta x(t) = f_{\rm d}(x(t)), \quad x(t) \in \mathcal{Z}, \tag{5.20}$$

where $t \geq 0$, $x(t) \triangleq [x_{\rm p}^{\rm T}(t), x_{\rm c}^{\rm T}(t)]^{\rm T}$, $\mathcal{Z} \triangleq \{x \in \mathcal{D} : (x_{\rm c}, h_{\rm p}(x_{\rm p})) \in \mathcal{Z}_{\rm c}\}$,

$$f_{\rm c}(x) = \begin{bmatrix} f_{\rm p}(x_{\rm p}, -h_{\rm cc}(x_{\rm c}, h_{\rm p}(x_{\rm p}))) \\ f_{\rm cc}(x_{\rm c}, h_{\rm p}(x_{\rm p})) \end{bmatrix}, \quad f_{\rm d}(x) = \begin{bmatrix} 0 \\ \eta(h_{\rm p}(x_{\rm p})) - x_{\rm c} \end{bmatrix}. \tag{5.21}$$

Assume that there exists an infinitely differentiable function $V_{\rm c} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \overline{\mathbb{R}}_+$ such that $V_{\rm c}(x_{\rm c}, y) \geq 0$, $x_{\rm c} \in \mathcal{D}_{\rm c}$, $y \in \mathbb{R}^l$, and $V_{\rm c}(x_{\rm c}, y) = 0$ if and only if $x_{\rm c} = \eta(y)$ and

$$\dot{V}_{\rm c}(x_{\rm c}(t), y(t)) = s_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)), \quad (x_{\rm c}(t), y(t)) \notin \mathcal{Z}, \quad t \geq 0, \tag{5.22}$$

where $s_{\rm c} : \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$ is such that $s_{\rm c}(0,0) = 0$.

We associate with the plant a positive-definite, continuously differentiable function $V_{\rm p}(x_{\rm p}) \triangleq V_{\rm s}(x_{\rm p})$, which we will refer to as the *plant energy*. Furthermore, we associate with the controller a nonnegative-definite, infinitely differentiable function $V_{\rm c}(x_{\rm c}, y)$ called the controller *emulated energy*. Finally, we associate with the closed-loop system the function $V(x) \triangleq V_{\rm p}(x_{\rm p}) + V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p}))$, called the *total energy*.

Next, we construct the resetting set for the closed-loop system $\mathcal{G}$ in the following form

$$\mathcal{Z} = \{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c} : L_{f_{\rm c}} V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p})) = 0 \text{ and } V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p})) > 0\}. \tag{5.23}$$

The resetting set $\mathcal{Z}$ is thus defined to be the set of all points in the closed-loop state space that correspond to the instant when the controller is at the verge of decreasing its emulated energy. By resetting the controller states, the plant energy can never increase after the first resetting event. Furthermore, if the closed-loop system total energy is conserved between resetting events, then a decrease in plant energy is accompanied by a corresponding increase in emulated energy. Hence, this approach allows the plant energy to flow to the controller, where it increases the emulated energy but does not allow the emulated energy to flow back to the plant after the first resetting event. This energy dissipating hybrid controller effectively enforces a one-way energy transfer between the plant and the controller after the first resetting event. For practical implementation, knowledge of $x_{\rm c}$ and y is sufficient to determine whether or not the closed-loop state vector is in the set $\mathcal{Z}$. That is, the full state $x_{\rm p}$ need not be known in order to determine whether or not the closed-loop state vector is in the set $\mathcal{Z}$.

The next theorem gives sufficient conditions for asymptotic stability of the closed-loop system $\mathcal{G}$ using state-dependent hybrid controllers.

Theorem 5.3. Consider the closed-loop impulsive dynamical system $\mathcal{G}$ given by (5.19) and (5.20). Assume that $\mathcal{D}_{\rm ci} \subset \mathcal{D}$ is a compact positively invariant set with respect to $\mathcal{G}$ such that $0 \in \overset{\circ}{\mathcal{D}}_{\rm ci}$, assume that $\mathcal{G}_{\rm p}$ is lossless with respect to the supply rate $s_{\rm p}(u, y)$ and with a positive definite, continuously differentiable storage function $V_{\rm p}(x_{\rm p})$, $x_{\rm p} \in \mathcal{D}_{\rm p}$, and assume there exists a smooth (i.e., infinitely differentiable) function $V_{\rm c} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \overline{\mathbb{R}}_+$ such that $V_{\rm c}(x_{\rm c}, y) \geq 0$, $x_{\rm c} \in \mathcal{D}_{\rm c}$, $y \in \mathbb{R}^l$, and $V_{\rm c}(x_{\rm c}, y) = 0$ if and only if $x_{\rm c} = \eta(y)$ and (5.22) holds. Furthermore, assume that every $x_0 \in \overline{\mathcal{Z}}$ is k-transversal to (5.19) and

$$s_{\rm p}(u, y) + s_{\rm c}(u_{\rm c}, y_{\rm c}) = 0, \quad x \notin \mathcal{Z}, \tag{5.24}$$

where $y = u_{\rm c} = h_{\rm p}(x_{\rm p})$, $u = -y_{\rm c} = -h_{\rm cc}(x_{\rm c}, h_{\rm p}(x_{\rm p}))$, and $\mathcal{Z}$ is given by (5.23). Then the zero solution $x(t) \equiv 0$ to the closed-loop system $\mathcal{G}$ is asymptotically stable. Finally, if $\mathcal{D}_{\rm p} = \mathbb{R}^{n_{\rm p}}$, $\mathcal{D}_{\rm c} = \mathbb{R}^{n_{\rm c}}$, and $V(\cdot)$ is radially unbounded, then the zero solution $x(t) \equiv 0$ to $\mathcal{G}$ is globally asymptotically stable.

Proof. First, note that since $V_{\rm c}(x_{\rm c}, y) \geq 0$, $x_{\rm c} \in \mathcal{D}_{\rm c}$, $y \in \mathbb{R}^l$, it follows that

$$\begin{aligned} \overline{\mathcal{Z}} &= \{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c} : L_{f_{\rm c}} V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p})) = 0 \text{ and } V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p})) \geq 0\} \\ &= \{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c} : \mathcal{X}(x) = 0\}, \end{aligned} \tag{5.25}$$

where $\mathcal{X}(x) = L_{f_{\rm c}} V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p}))$. Next, we show that if the k-transversality condition (5.10) holds, then Assumptions 1–3 hold and, for every $x_0 \in \mathcal{D}_{\rm ci}$, there exists $\tau \geq 0$ such that $x(\tau) \in \mathcal{Z}$. Note that if $x_0 \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$, that is, $V_{\rm c}(x_{\rm c}(0), h_{\rm p}(x_{\rm p}(0))) = 0$ and $L_{f_{\rm c}} V_{\rm c}(x_{\rm c}(0), h_{\rm p}(x_{\rm p}(0))) = 0$, it follows from the k-transversality condition that there exists $\delta > 0$ such that for all $t \in$

$(0,\delta]$, $L_{f_{\rm c}}V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))\neq 0$. Hence, since $V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))=V_{\rm c}(x_{\rm c}(0),h_{\rm p}(x_{\rm p}(0)))+tL_{f_{\rm c}}V_{\rm c}(x_{\rm c}(\tau),h_{\rm p}(x_{\rm p}(\tau)))$ for some $\tau\in(0,t]$ and $V_{\rm c}(x_{\rm c},y)\geq 0$, $x_{\rm c}\in\mathcal{D}_{\rm c}$, $y\in\mathbb{R}^l$, it follows that $V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))>0$, $t\in(0,\delta]$, which implies that Assumption 1 is satisfied. Furthermore, if $x\in\mathcal{Z}$ then, since $V_{\rm c}(x_{\rm c},y)=0$ if and only if $x_{\rm c}=\eta(y)$, it follows from (5.20) that $x+f_{\rm d}(x)\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$. Hence, Assumption 2 holds.

Next, consider the set $\mathcal{M}_\gamma\triangleq\{x\in\mathcal{D}_{\rm ci}:V_{\rm c}(x_{\rm c},h_{\rm p}(x_{\rm p}))=\gamma\}$, where $\gamma\geq 0$. It follows from the k-transversality condition that for every $\gamma\geq 0$, $\mathcal{M}_\gamma$ does not contain any nontrivial trajectory of $\mathcal{G}$. To see this, suppose, *ad absurdum*, there exists a nontrivial trajectory $x(t)\in\mathcal{M}_\gamma$, $t\geq 0$, for some $\gamma\geq 0$. In this case, it follows that $\frac{{\rm d}^k}{{\rm d}t^k}V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))=L^k_{f_{\rm c}}V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))\equiv 0$, $k=1,2,\ldots$, which contradicts the k-transversality condition.

Next, we show that for every $x_0\notin\mathcal{Z}$, $x_0\neq 0$, there exists $\tau>0$ such that $x(\tau)\in\mathcal{Z}$. To see this, suppose, *ad absurdum*, $x(t)\notin\mathcal{Z}$, $t\geq 0$, which implies that

$$\frac{\rm d}{{\rm d}t}V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))\neq 0,\quad t\geq 0,\tag{5.26}$$

or

$$V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))=0,\quad t\geq 0.\tag{5.27}$$

If (5.26) holds, then it follows that $V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))$ is a (decreasing or increasing) monotonic function of time. Hence, $V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))\to\gamma$ as $t\to\infty$, where $\gamma\geq 0$ is a constant, which implies that the positive limit set of the closed-loop system is contained in $\mathcal{M}_\gamma$ for some $\gamma\geq 0$, and hence, is a contradiction. Similarly, if (5.27) holds then $\mathcal{M}_0$ contains a nontrivial trajectory of $\mathcal{G}$ also leading to a contradiction. Hence, for every $x_0\notin\mathcal{Z}$, there exists $\tau>0$ such that $x(\tau)\in\mathcal{Z}$. Thus, it follows that for every $x_0\notin\mathcal{Z}$, $0<\tau_1(x_0)<\infty$. Now, it follows from Proposition 5.2 that $\tau_1(\cdot)$ is continuous at $x_0\notin\overline{\mathcal{Z}}$. Furthermore, for all $x_0\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$ and for every sequence $\{x_i\}_{i=1}^\infty\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$ converging to $x_0\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$, it follows from the k-transversality condition and Proposition 5.2 that $\lim_{i\to\infty}\tau_1(x_i)=\tau_1(x_0)$. Next, let $x_0\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$ and let $\{x_i\}_{i=1}^\infty\in\mathcal{D}_{\rm ci}$ be such that $\lim_{i\to\infty}x_i=x_0$ and $\lim_{i\to\infty}\tau_1(x_i)$ exists. In this case, it follows from Proposition 5.2 that either $\lim_{i\to\infty} \tau_1(x_i) = 0$ or $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$. Furthermore, since $x_0 \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$ corresponds to the case where $V_{\rm c}(x_{\rm c0}, h_{\rm p}(x_{\rm p0})) = 0$, it follows that $x_{\rm c0} = \eta(h_{\rm p}(x_{\rm p0}))$, and hence, $f_{\rm d}(x_0) = 0$. Now, it follows from Proposition 5.1 that Assumption 3 holds.

Next, note that if $x_0 \in \mathcal{Z}$ and $x_0 + f_{\rm d}(x_0) \neq 0$, then it follows from the above analysis that there exists $\tau > 0$ such that $x(\tau) \in \mathcal{Z}$. Alternatively, if $x_0 \in \mathcal{Z}$ and $x_0 + f_{\rm d}(x_0) = 0$, then $x(t) = 0$, $t \geq 0$. In this case, the solution of the closed-loop system reaches the origin in finite time which is a stronger condition than reaching the origin as $t \to \infty$.

To show that the zero solution $x(t) \equiv 0$ to $\mathcal{G}$ is asymptotically stable, consider the Lyapunov function candidate $V(x) = V_{\rm p}(x_{\rm p}) + V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p}))$ corresponding to the total energy function. Since $\mathcal{G}_{\rm p}$ is lossless with respect to the supply rate $s_{\rm p}(u, y)$, and (5.22) and (5.24) hold, it follows that

$$\dot{V}(x(t)) = s_{\rm p}(u(t), y(t)) + s_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)) = 0, \quad x(t) \notin \mathcal{Z}. \tag{5.28}$$

Furthermore, it follows from (5.21) and (5.23) that

$$\begin{aligned}\Delta V(x(t_k)) &= V_{\rm c}(x_{\rm c}(t_k^+), h_{\rm p}(x_{\rm p}(t_k^+))) - V_{\rm c}(x_{\rm c}(t_k), h_{\rm p}(x_{\rm p}(t_k)))\\ &= V_{\rm c}(\eta(h_{\rm p}(x_{\rm p}(t_k))), h_{\rm p}(x_{\rm p}(t_k))) - V_{\rm c}(x_{\rm c}(t_k), h_{\rm p}(x_{\rm p}(t_k)))\\ &= -V_{\rm c}(x_{\rm c}(t_k), h_{\rm p}(x_{\rm p}(t_k))) < 0, \quad x(t_k) \in \mathcal{Z}, \quad k \in \overline{\mathbb{Z}}_+.\end{aligned} \tag{5.29}$$

Thus, it follows from Theorem 5.2 that the zero solution $x(t) \equiv 0$ to $\mathcal{G}$ is asymptotically stable. Finally, if $\mathcal{D}_{\rm p} = \mathbb{R}^{n_{\rm p}}$, $\mathcal{D}_{\rm c} = \mathbb{R}^{n_{\rm c}}$, and $V(\cdot)$ is radially unbounded, then global asymptotic stability is immediate. □

Remark 5.6. Theorem 5.3 can be generalized to the case where $\mathcal{G}_{\rm p}$ is *dissipative* with respect to the supply rate $s_{\rm p}(u, y)$ since a dissipation rate function does not add any additional complexity to the hybrid stabilization process. Specifically, in this case (5.28) becomes $\dot{V}(x(t)) = d(x_{\rm p}(t)) \leq 0$, $x(t) \in \mathcal{Z}$, where $d : \mathcal{D}_{\rm p} \to \mathbb{R}$ is a continuous, nonnegative-definite dissipation rate function. Now, Theorem 5.3 holds with the additional assumption that the only invariant set contained in $\mathcal{R} \triangleq \{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm ci} : d(x_{\rm p}) = 0\}$ is $\mathcal{M} = \{(0,0)\}$. Furthermore, in this case, global asymptotic stability can be shown to be uniform with respect to compact sets of initial conditions. Similar remarks hold for Euler-Lagrange systems with Rayleigh dissipation functions considered in the next section.

Finally, we specialize the hybrid controller design framework just presented to *port-controlled Hamiltonian systems* [161]. Specifically, consider the port-controlled Hamiltonian system given by

$$\dot{x}_{\rm p}(t) = \mathcal{J}_{\rm p}(x_{\rm p}(t)) \left(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}(t))\right)^{\rm T} + G_{\rm p}(x_{\rm p}(t))u(t), \quad x_{\rm p}(0) = x_{\rm p0}, \quad t \geq 0, \tag{5.30}$$

$$y(t) = G_{\rm p}^{\rm T}(x_{\rm p}(t)) \left(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}(t))\right)^{\rm T}, \tag{5.31}$$

where $x_{\rm p}(t) \in \mathcal{D}_{\rm p} \subseteq \mathbb{R}^{n_{\rm p}}$, $\mathcal{D}_{\rm p}$ is an open set with $0 \in \mathcal{D}_{\rm p}$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, $\mathcal{H}_{\rm p} : \mathcal{D}_{\rm p} \to \mathbb{R}$ is an infinitely differentiable Hamiltonian function for the system (5.30) and (5.31), $\mathcal{J}_{\rm p} : \mathcal{D}_{\rm p} \to \mathbb{R}^{n_{\rm p} \times n_{\rm p}}$ is such that $\mathcal{J}_{\rm p}(x_{\rm p}) = -\mathcal{J}_{\rm p}^{\rm T}(x_{\rm p})$, $x_{\rm p} \in \mathcal{D}_{\rm p}$, $\mathcal{J}_{\rm p}(x_{\rm p})(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}))^{\rm T}$, $x_{\rm p} \in \mathcal{D}_{\rm p}$, is smooth on $\mathcal{D}_{\rm p}$, and $G_{\rm p} : \mathcal{D}_{\rm p} \to \mathbb{R}^{n_{\rm p} \times m}$. The skew-symmetric matrix function $\mathcal{J}_{\rm p}(x_{\rm p})$, $x_{\rm p} \in \mathcal{D}_{\rm p}$, captures the internal system interconnection structure. Furthermore, we assume that $\mathcal{H}_{\rm p}(0) = 0$ and $\mathcal{H}_{\rm p}(x_{\rm p}) > 0$ for all $x_{\rm p} \neq 0$ and $x_{\rm p} \in \mathcal{D}_{\rm p}$.

Next, consider the dynamic, energy-based hybrid controller

$$\dot{x}_{\rm c}(t) = \mathcal{J}_{\rm cc}(x_{\rm c}(t)) \left(\frac{\partial \mathcal{H}_{\rm c}}{\partial x_{\rm c}}(x_{\rm c}(t))\right)^{\rm T} + G_{\rm cc}(x_{\rm c}(t))y(t),$$
$$x_{\rm c}(0) = x_{\rm c0}, \quad (x_{\rm p}(t), x_{\rm c}(t)) \notin \mathcal{Z}, \tag{5.32}$$

$$\Delta x_{\rm c}(t) = -x_{\rm c}(t), \quad (x_{\rm p}(t), x_{\rm c}(t)) \in \mathcal{Z}, \tag{5.33}$$

$$u(t) = -G_{\rm cc}^{\rm T}(x_{\rm c}(t)) \left(\frac{\partial \mathcal{H}_{\rm c}}{\partial x_{\rm c}}(x_{\rm c}(t))\right)^{\rm T}, \tag{5.34}$$

where $t \geq 0$, $x_{\rm c}(t) \in \mathcal{D}_{\rm c} \subseteq \mathbb{R}^{n_{\rm c}}$, $\mathcal{D}_{\rm c}$ is an open set with $0 \in \mathcal{D}_{\rm c}$, $\Delta x_{\rm c}(t) \triangleq x_{\rm c}(t^+) - x_{\rm c}(t)$, $\mathcal{H}_{\rm c} : \mathcal{D}_{\rm c} \to \mathbb{R}$ is an infinitely differentiable Hamiltonian function for (5.32), $\mathcal{J}_{\rm cc} : \mathcal{D}_{\rm c} \to \mathbb{R}^{n_{\rm c} \times n_{\rm c}}$ is such that $\mathcal{J}_{\rm cc}(x_{\rm c}) = -\mathcal{J}_{\rm cc}^{\rm T}(x_{\rm c})$, $x_{\rm c} \in \mathcal{D}_{\rm c}$, $\mathcal{J}_{\rm cc}(x_{\rm c})(\frac{\partial \mathcal{H}_{\rm c}}{\partial x_{\rm c}}(x_{\rm c}))^{\rm T}$, $x_{\rm c} \in \mathcal{D}_{\rm c}$, is smooth on $\mathcal{D}_{\rm c}$,

$G_{\mathrm{cc}} : \mathcal{D}_{\mathrm{c}} \to \mathbb{R}^{n_{\mathrm{c}}\times m}$, and resetting set $\mathcal{Z} \subset \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}}$ is given by

$$\mathcal{Z} \triangleq \left\{(x_{\mathrm{p}}, x_{\mathrm{c}}) \in \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}} : \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}) = 0 \text{ and } \mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}) > 0\right\}, \tag{5.35}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}(t)) \triangleq \lim_{\tau\to t^-} \frac{1}{t-\tau}[\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}(t)) - \mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}(\tau))]$ whenever the limit on the right-hand side exists. Here, we assume that $\mathcal{H}_{\mathrm{c}}(0) = 0$ and $\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}) > 0$ for all $x_{\mathrm{c}} \neq 0$ and $x_{\mathrm{c}} \in \mathcal{D}_{\mathrm{c}}$.

Note that $\mathcal{H}_{\mathrm{p}}(x_{\mathrm{p}})$, $x_{\mathrm{p}} \in \mathcal{D}_{\mathrm{p}}$, is the plant energy and $\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}})$, $x_{\mathrm{c}} \in \mathcal{D}_{\mathrm{c}}$, is the controller emulated energy. Furthermore, the closed-loop system energy is given by $\mathcal{H}(x_{\mathrm{p}}, x_{\mathrm{c}}) \triangleq \mathcal{H}_{\mathrm{p}}(x_{\mathrm{p}}) + \mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}})$. Next, note that total energy function $\mathcal{H}(x_{\mathrm{p}}, x_{\mathrm{c}})$ along the trajectories of the closed-loop dynamics (5.30)–(5.34) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x_{\mathrm{p}}(t), x_{\mathrm{c}}(t)) = 0, \quad (x_{\mathrm{p}}(t), x_{\mathrm{c}}(t)) \notin \mathcal{Z}, \tag{5.36}$$

$$\Delta\mathcal{H}(x_{\mathrm{p}}(t_k), x_{\mathrm{c}}(t_k)) = -\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}(t_k)), \quad (x_{\mathrm{p}}(t_k), x_{\mathrm{c}}(t_k)) \in \mathcal{Z}, \quad k \in \overline{\mathbb{Z}}_+. \tag{5.37}$$

Here, we assume that every $(x_{\mathrm{p}0}, x_{\mathrm{c}0}) \in \overline{\mathcal{Z}}$ is transversal to the closed-loop dynamical system given by (5.30)–(5.34). Furthermore, we assume $\mathcal{D}_{\mathrm{ci}} \subset \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}}$ is a compact positively invariant set with respect to the closed-loop dynamical system (5.30)–(5.34) such that $0 \in \overset{\circ}{\mathcal{D}}_{\mathrm{ci}}$. In this case, it follows from Theorem 5.3, with $V_{\mathrm{s}}(x_{\mathrm{p}}) = \mathcal{H}_{\mathrm{p}}(x_{\mathrm{p}})$, $V_{\mathrm{c}}(x_{\mathrm{c}}, y) = \mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}})$, $s(u, y) = u^{\mathrm{T}}y$, and $s_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}) = u_{\mathrm{c}}^{\mathrm{T}}y_{\mathrm{c}}$, that the zero solution $(x_{\mathrm{p}}(t), x_{\mathrm{c}}(t)) \equiv (0, 0)$ to the closed-loop system (5.30)–(5.34), with $\mathcal{Z}$ given by (5.35), is asymptotically stable.

5.4. Hybrid Control Design for Euler-Lagrange Systems

Consider the governing equations of motion of an $\hat{n}_{\mathrm{p}}$ degree-of-freedom dynamical system given by the *Euler-Lagrange* equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial\dot{q}}(q(t), \dot{q}(t))\right]^{\mathrm{T}} - \left[\frac{\partial\mathcal{L}}{\partial q}(q(t), \dot{q}(t))\right]^{\mathrm{T}} = u(t), \quad q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0, \tag{5.38}$$

where $t \geq 0$, $q \in \mathbb{R}^{\hat{n}_{\mathrm{p}}}$ represents the generalized system positions, $\dot{q} \in \mathbb{R}^{\hat{n}_{\mathrm{p}}}$ represents the generalized system velocities, $\mathcal{L} : \mathbb{R}^{\hat{n}_{\mathrm{p}}} \times \mathbb{R}^{\hat{n}_{\mathrm{p}}} \to \mathbb{R}$ denotes the system Lagrangian given by $\mathcal{L}(q, \dot{q}) = T(q, \dot{q}) - U(q)$, where $T : \mathbb{R}^{\hat{n}_{\mathrm{p}}} \times \mathbb{R}^{\hat{n}_{\mathrm{p}}} \to \mathbb{R}$ is the system kinetic energy and

$U : \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}$ is the system potential energy, and $u \in \mathbb{R}^{\hat{n}_{\rm p}}$ is the vector of generalized control forces acting on the system. Furthermore, let $\mathcal{H} : \mathbb{R}^{\hat{n}_{\rm p}} \times \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}$ denote the *Legendre transformation* of the Lagrangian function $\mathcal{L}(q, \dot{q})$ with respect to the generalized velocity $\dot{q}$ defined by $\mathcal{H}(q, p) \triangleq \dot{q}^{\rm T} p - \mathcal{L}(q, \dot{q})$, where p denotes the vector of generalized momenta given by $p(q, \dot{q}) = \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q})\right]^{\rm T}$, and where the map from the generalized velocities $\dot{q}$ to the generalized momenta p is assumed to be *bijective* (i.e., one-to-one and onto).

Next, we present a hybrid feedback control framework for Euler-Lagrange dynamical systems. Specifically, consider the Lagrangian system (5.38) with outputs

$$y = \begin{bmatrix} h_1(q) \\ h_2(\dot{q}) \end{bmatrix} = \begin{bmatrix} h_1(q) \\ h_2\left(\frac{\partial \mathcal{H}}{\partial p}(q, p)\right) \end{bmatrix}, \tag{5.39}$$

where $h_1 : \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}^{l_1}$ and $h_2 : \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}^{l - l_1}$ are continuously differentiable, $h_1(0) = 0$, $h_2(0) = 0$, and $h_1(q) \not\equiv 0$. We assume that the system kinetic energy is such that $T(q, \dot{q}) = \frac{1}{2}\dot{q}^{\rm T}[\frac{\partial T}{\partial \dot{q}}(q, \dot{q})]^{\rm T}$, $T(q, 0) = 0$, and $T(q, \dot{q}) > 0$, $\dot{q} \neq 0$, $\dot{q} \in \mathbb{R}^{\hat{n}_{\rm p}}$. We also assume that the system potential energy $U(\cdot)$ is such that $U(0) = 0$ and $U(q) > 0$, $q \neq 0$, $q \in \mathcal{D}_{\rm q} \subseteq \mathbb{R}^{\hat{n}_{\rm p}}$, which implies that $\mathcal{H}(q, p) = T(q, \dot{q}) + U(q) > 0$, $(q, \dot{q}) \neq 0$, $(q, \dot{q}) \in \mathcal{D}_{\rm q} \times \mathbb{R}^{\hat{n}_{\rm p}}$.

Next, consider the energy-based hybrid controller

$$\frac{\rm d}{{\rm d}t}\left[\frac{\partial \mathcal{L}_{\rm c}}{\partial \dot{q}_{\rm c}}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\right]^{\rm T} - \left[\frac{\partial \mathcal{L}_{\rm c}}{\partial q_{\rm c}}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\right]^{\rm T} = 0, \quad q_{\rm c}(0) = q_{\rm c0}, \quad \dot{q}_{\rm c}(0) = \dot{q}_{\rm c0},$$
$$(q_{\rm c}(t), \dot{q}_{\rm c}(t), y(t)) \notin \mathcal{Z}_{\rm c}, \tag{5.40}$$

$$\begin{bmatrix} \Delta q_{\rm c}(t) \\ \Delta \dot{q}_{\rm c}(t) \end{bmatrix} = \begin{bmatrix} \eta(y_q(t)) - q_{\rm c}(t) \\ -\dot{q}_{\rm c}(t) \end{bmatrix}, \quad (q_{\rm c}(t), \dot{q}_{\rm c}(t), y(t)) \in \mathcal{Z}_{\rm c}, \tag{5.41}$$

$$u(t) = \left[\frac{\partial \mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\right]^{\rm T}, \tag{5.42}$$

where $t \geq 0$, $q_{\rm c} \in \mathbb{R}^{\hat{n}_{\rm c}}$ represents virtual controller positions, $\dot{q}_{\rm c} \in \mathbb{R}^{\hat{n}_{\rm c}}$ represents virtual controller velocities, $y_q \triangleq h_1(q)$, $\mathcal{L}_{\rm c} : \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{l_1} \to \mathbb{R}$ denotes the controller Lagrangian given by $\mathcal{L}_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, y_q) \triangleq T_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}) - U_{\rm c}(q_{\rm c}, y_q)$, where $T_{\rm c} : \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{\hat{n}_{\rm c}} \to \mathbb{R}$ is the controller kinetic energy, $U_{\rm c} : \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{l_1} \to \mathbb{R}$ is the controller potential energy, $\eta(\cdot)$ is a continuously differentiable function such that $\eta(0) = 0$, $\mathcal{Z}_{\rm c} \subset \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{l}$ is the resetting set, $\Delta q_{\rm c}(t) \triangleq$

$q_{\rm c}(t^+) - q_{\rm c}(t)$, and $\Delta\dot{q}_{\rm c}(t) \triangleq \dot{q}_{\rm c}(t^+) - \dot{q}_{\rm c}(t)$. We assume that the controller kinetic energy $T_{\rm c}(q_{\rm c}, \dot{q}_{\rm c})$ is such that $T_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}) = \frac{1}{2}\dot{q}_{\rm c}^{\rm T}[\frac{\partial T_{\rm c}}{\partial \dot{q}_{\rm c}}(q_{\rm c}, \dot{q}_{\rm c})]^{\rm T}$, with $T_{\rm c}(q_{\rm c}, 0) = 0$ and $T_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}) > 0$, $\dot{q}_{\rm c} \neq 0$, $\dot{q}_{\rm c} \in \mathbb{R}^{\hat{n}_{\rm c}}$. Furthermore, we assume that $U_{\rm c}(\eta(y_q), y_q) = 0$ and $U_{\rm c}(q_{\rm c}, y_q) > 0$ for $q_{\rm c} \neq \eta(y_q)$, $q_{\rm c} \in \mathcal{D}_{\rm q_c} \subseteq \mathbb{R}^{\hat{n}_{\rm c}}$.

As in Section 5.3, note that $V_{\rm p}(q, \dot{q}) \triangleq T(q, \dot{q}) + U(q)$ is the plant energy, $V_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, y_q) \triangleq T_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}) + U_{\rm c}(q_{\rm c}, y_q)$ is the controller emulated energy, and $V(q, \dot{q}, q_{\rm c}, \dot{q}_{\rm c}) \triangleq V_{\rm p}(q, \dot{q}) + V_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, y_q)$ is the total energy of the closed-loop system. It is important to note that the Lagrangian dynamical system (5.40) is *not* lossless with inputs y_q or y. Next, we study the behavior of the total energy function $V(q, \dot{q}, q_{\rm c}, \dot{q}_{\rm c})$ along the trajectories of the closed-loop system dynamics. For the closed-loop system, we define our resetting set as $\mathcal{Z} \triangleq \{(q, \dot{q}, q_{\rm c}, \dot{q}_{\rm c}) : (q_{\rm c}, \dot{q}_{\rm c}, y) \in \mathcal{Z}_{\rm c}\}$. Note that $\frac{\rm d}{{\rm d}t}V_{\rm p}(q, \dot{q}) = \frac{\rm d}{{\rm d}t}\mathcal{H}(q, p) = u^{\rm T}\dot{q}$, $(q, \dot{q}, q_{\rm c}, \dot{q}_{\rm c}) \notin \mathcal{Z}$.

To obtain an expression for $\frac{\rm d}{{\rm d}t}V_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, y_q)$ when $(q, \dot{q}, q_{\rm c}, \dot{q}_{\rm c}) \notin \mathcal{Z}$, define the controller Hamiltonian by $\mathcal{H}_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, p_{\rm c}, y_q) \triangleq \dot{q}_{\rm c}^{\rm T}p_{\rm c} - \mathcal{L}_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, y_q)$, where the virtual controller momentum $p_{\rm c}$ is given by $p_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, y_q) = \left[\frac{\partial \mathcal{L}_{\rm c}}{\partial \dot{q}_{\rm c}}(q_{\rm c}, \dot{q}_{\rm c}, y_q)\right]^{\rm T}$. Then $\mathcal{H}_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, p_{\rm c}, y_q) = T_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}) + U_{\rm c}(q_{\rm c}, y_q)$. Now, it follows from (5.40) and the structure of $T_{\rm c}(q_{\rm c}, \dot{q}_{\rm c})$ that, for $t \in (t_k, t_{k+1}]$,

$$
\begin{aligned}
0 &= \frac{\rm d}{{\rm d}t}\left[p_{\rm c}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\right]^{\rm T}\dot{q}_{\rm c}(t) - \frac{\partial \mathcal{L}_{\rm c}}{\partial q_{\rm c}}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\dot{q}_{\rm c}(t) \\
&= \frac{\rm d}{{\rm d}t}\left[p_{\rm c}^{\rm T}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\dot{q}_{\rm c}(t)\right] - p_{\rm c}^{\rm T}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\ddot{q}_{\rm c}(t) + \frac{\partial \mathcal{L}_{\rm c}}{\partial \dot{q}_{\rm c}}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\ddot{q}_{\rm c}(t) \\
&\quad + \frac{\partial \mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\dot{q}(t) - \frac{\rm d}{{\rm d}t}\mathcal{L}_{\rm c}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t)) \\
&= \frac{\rm d}{{\rm d}t}[p_{\rm c}^{\rm T}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\dot{q}_{\rm c}(t) - \mathcal{L}_{\rm c}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))] + \frac{\partial \mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\dot{q}(t) \\
&= \frac{\rm d}{{\rm d}t}V_{\rm c}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t)) + \frac{\partial \mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\dot{q}(t), \quad (q(t), \dot{q}(t), q_{\rm c}(t), \dot{q}_{\rm c}(t)) \notin \mathcal{Z}.
\end{aligned}
\tag{5.43}
$$

Hence,

$$
\begin{aligned}
\frac{\rm d}{{\rm d}t}V(q(t), \dot{q}(t), q_{\rm c}(t), \dot{q}_{\rm c}(t)) &= u(t)^{\rm T}\dot{q}(t) - \frac{\partial \mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\dot{q}(t) \\
&= 0, \quad (q(t), \dot{q}(t), q_{\rm c}(t), \dot{q}_{\rm c}(t)) \notin \mathcal{Z}, \quad t_k < t \leq t_{k+1},
\end{aligned}
\tag{5.44}
$$

which implies that the total energy of the closed-loop system between resetting events is conserved.

The total energy difference across resetting events is given by

$$\begin{aligned}\Delta V(q(t_k),\dot{q}(t_k),q_{\rm c}(t_k),\dot{q}_{\rm c}(t_k)) &= T_{\rm c}(q_{\rm c}(t_k^+),\dot{q}_{\rm c}(t_k^+))+U_{\rm c}(q_{\rm c}(t_k^+),y_q(t_k))\\ &\quad -V_{\rm c}(q_{\rm c}(t_k),\dot{q}_{\rm c}(t_k),y_q(t_k))\\ &= -V_{\rm c}(q_{\rm c}(t_k),\dot{q}_{\rm c}(t_k),y_q(t_k)),\\ &\qquad (q(t_k),\dot{q}(t_k),q_{\rm c}(t_k),\dot{q}_{\rm c}(t_k))\in\mathcal{Z},\quad k\in\overline{\mathbb{Z}}_+,\end{aligned}\tag{5.45}$$

which implies that the resetting law (5.41) ensures the total energy decrease across resetting events by an amount equal to the accumulated emulated energy.

Here, we concentrate on an energy dissipating state-dependent resetting controller that affects a one-way energy transfer between the plant and the controller. Specifically, consider the closed-loop system (5.38), (5.39)–(5.42), where $\mathcal{Z}$ is defined by

$$\mathcal{Z}\triangleq\left\{(q,\dot{q},q_{\rm c},\dot{q}_{\rm c}):\frac{{\rm d}}{{\rm d}t}V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q)=0 \text{ and } V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q)>0\right\}.\tag{5.46}$$

Once again, for practical implementation, knowledge of $q_{\rm c}$, $\dot{q}_{\rm c}$, and y_q is sufficient to determine whether or not the closed-loop state vector is in the set $\mathcal{Z}$.

The next theorem gives sufficient conditions for stabilization of Euler-Lagrange dynamical systems using state-dependent hybrid controllers. For this result define the closed-loop system states $x\triangleq[q^{\rm T},\dot{q}^{\rm T},q_{\rm c}^{\rm T},\dot{q}_{\rm c}^{\rm T}]^{\rm T}$.

Theorem 5.4. Consider the closed-loop dynamical system $\mathcal{G}$ given by (5.38), (5.39)–(5.42), with the resetting set $\mathcal{Z}$ given by (5.46). Assume that $\mathcal{D}_{\rm ci}\subset\mathcal{D}_{\rm q}\times\mathbb{R}^{\hat{n}_{\rm p}}\times\mathcal{D}_{\rm q_c}\times\mathbb{R}^{\hat{n}_{\rm c}}$ is a compact positively invariant set with respect to $\mathcal{G}$ such that $0\in\overset{\circ}{\mathcal{D}}_{\rm ci}$. Furthermore, assume that the k-transversality condition (5.10) holds with $\mathcal{X}(x)=\frac{{\rm d}}{{\rm d}t}V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q)$. Then the zero solution $x(t)\equiv 0$ to $\mathcal{G}$ is asymptotically stable. Finally, if $\mathcal{D}_{\rm q}=\mathbb{R}^{\hat{n}_{\rm p}}$, $\mathcal{D}_{\rm q_c}=\mathbb{R}^{\hat{n}_{\rm c}}$, and the total energy function $V(x)$ is radially unbounded, then the zero solution $x(t)\equiv 0$ to $\mathcal{G}$ is globally asymptotically stable.

Proof. The result is a direct consequence of Theorem 5.3 with $V_{\rm p}(x_{\rm p}) = V_{\rm p}(q,\dot{q})$, $V_{\rm c}(x_{\rm c},y) = V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q)$, $y = u_{\rm c} = x_{\rm p}$, $u = -y_{\rm c} = \frac{\partial \mathcal{L}_{\rm c}}{\partial q}$, $s_{\rm p}(u,y) = u^{\rm T}\rho(y)$, $s_{\rm c}(u_{\rm c},y_{\rm c}) = y_{\rm c}^{\rm T}\rho(u_{\rm c})$, where $\rho(y) = \rho\left(\left[\begin{array}{c} q \\ \dot{q} \end{array}\right]\right) = \dot{q}$, and $\eta(y)$ replaced by $\left[\begin{array}{c} \eta(y_q) \\ 0 \end{array}\right]$. □

5.5. Thermodynamic Stabilization

In this section, we present yet another form of the resetting set that provides a hybrid controller architecture that is based on entropy notions and is consistent with thermodynamic stabilization. In particular, we use the recently developed notion of system thermodynamics [104] to develop thermodynamically consistent hybrid controllers for lossless dynamical systems. Specifically, since our energy-based hybrid controller architecture involves the exchange of energy with conservation laws describing transfer, accumulation, and dissipation of energy between the controller and the plant, we construct a modified hybrid controller that guarantees that the closed-loop system is consistent with basic thermodynamic principles after the first resetting event. To develop thermodynamically consistent hybrid controllers consider the closed-loop system $\mathcal{G}$ given by (5.19) and (5.20) with $\mathcal{Z}$ given by

$$\mathcal{Z} \triangleq \{x \in \mathcal{D} : \phi(x)(V_{\rm p}(x) - V_{\rm c}(x)) = 0 \text{ and } V_{\rm c}(x) > 0\}, \tag{5.47}$$

where $\phi(x) \triangleq -\dot{V}_{\rm c}(x)$, $x \notin \mathcal{Z}$. It follows from (5.28) that $\phi(\cdot)$ is the net energy flow from the plant to the controller, and hence, we refer to $\phi(\cdot)$ as the *net energy flow* function.

We assume that the energy flow function $\phi(x)$ is infinitely differentiable and the k-transversality condition (5.10) holds with $\mathcal{X}(x) = \phi(x)(V_{\rm p}(x) - V_{\rm c}(x))$. If $\mathcal{D}_{\rm ci} \subset \mathcal{D}$ is a compact positively invariant set with respect to the closed-loop dynamical system $\mathcal{G}$ given by (5.19) and (5.20) such that $0 \in \overset{\circ}{\mathcal{D}}_{\rm ci}$, and the k-transversality condition (5.10) holds with $\mathcal{X}(x) = \phi(x)(V_{\rm p}(x) - V_{\rm c}(x))$, then using similar arguments as in the proof of Theorem 5.3 it can be shown that the zero solution $x(t) \equiv 0$ of the closed-loop system $\mathcal{G}$, with resetting set $\mathcal{Z}$ given by (5.47), is asymptotically stable. Specifically, note that the resetting set given by (5.23) is a subset of the resetting set given by (5.47) which simply involves additional resettings when $V_{\rm p}(x) = V_{\rm c}(x)$. Hence, identical arguments as in the proof of Theorem 5.3 can be used to show asymptotic stability of the closed-loop system.

To ensure a thermodynamically consistent energy flow between the plant and controller after the first resetting event, the controller resetting logic must be designed in such a way so as to satisfy three key thermodynamic axioms on the closed-loop system level. Namely, between resettings the energy flow function $\phi(\cdot)$ must satisfy the following two assumptions [104]:

Assumption 4. For the *connectivity matrix* $\mathcal{C} \in \mathbb{R}^{2\times 2}$ [104, p. 56] associated with the closed-loop system $\mathcal{G}$ defined by

$$\mathcal{C}_{(i,j)} \triangleq \begin{cases} 0, & \text{if } \phi(x(t)) \equiv 0 \\ 1, & \text{otherwise} \end{cases}, \quad i \neq j, \quad i,j = 1,2, \quad t \geq t_1^+, \tag{5.48}$$

and $\mathcal{C}_{(i,i)} = -\mathcal{C}_{(k,i)}$, $i \neq k$, $i,k = 1,2$, $\operatorname{rank} \mathcal{C} = 1$, and for $\mathcal{C}_{(i,j)} = 1$, $i \neq j$, $\phi(x(t)) = 0$ if and only if $V_{\rm p}(x(t)) = V_{\rm c}(x(t))$, $x(t) \notin \mathcal{Z}$, $t \geq t_1^+$.

Assumption 5. $\phi(x(t))(V_{\rm p}(x(t)) - V_{\rm c}(x(t))) \leq 0$, $x(t) \notin \mathcal{Z}$, $t \geq t_1^+$.

Furthermore, across resettings the energy difference between the plant and the controller must satisfy the following assumption (Axiom *iii*) of Section 3.3):

Assumption 6. $[V_{\rm p}(x + f_{\rm d}(x)) - V_{\rm c}(x + f_{\rm d}(x))][V_{\rm p}(x) - V_{\rm c}(x)] \geq 0$, $x \in \mathcal{Z}$.

The fact that $\phi(x(t)) = 0$ if and only if $V_{\rm p}(x(t)) = V_{\rm c}(x(t))$, $x(t) \notin \mathcal{Z}$, $t \geq t_1^+$, implies that the plant and the controller are *connected*; alternatively, $\phi(x(t)) \equiv 0$, $t \geq t_1^+$, implies that the plant and the controller are *disconnected*. Assumption 4 implies that if the energies in the plant and the controller are equal, then energy exchange between the plant and controller is not possible unless a resetting event occurs. This statement is consistent with the *zeroth law of thermodynamics*, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium of an isolated system. Assumption 5 implies that energy flows from a more energetic system to a less energetic system and is consistent with the *second law of thermodynamics*, which states that heat (energy) must flow in the direction of lower temperatures. Finally, Assumption 6 implies that the energy difference between the plant and the controller across resetting instants is monotonic, that is, $[V_{\rm p}(x(t_k^+)) - V_{\rm c}(x(t_k^+))][V_{\rm p}(x(t_k)) - V_{\rm c}(x(t_k))] \geq 0$ for all $V_{\rm p}(x) \neq V_{\rm c}(x)$, $x \in \mathcal{Z}$, $k \in \overline{\mathbb{Z}}_+$.

With the resetting law given by (5.47), it follows that the closed-loop dynamical system $\mathcal{G}$ satisfies Assumption 4–6 for all $t \geq t_1$. To see this, note that since $\phi(x) \not\equiv 0$, the connectivity matrix $\mathcal{C}$ is given by

$$\mathcal{C} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{5.49}$$

and hence, $\operatorname{rank}\mathcal{C} = 1$. The second condition in Assumption 4 need not be satisfied since the case where $\phi(x) = 0$ or $V_{\rm p}(x) = V_{\rm c}(x)$ corresponds to a resetting instant. Furthermore, it follows from the definition of the resetting set (5.47) that Assumption 5 is satisfied for the closed-loop system for all $t \geq t_1^+$. Finally, since $V_{\rm c}(x + f_{\rm d}(x)) = 0$ and $V_{\rm p}(x + f_{\rm d}(x)) = V_{\rm p}(x)$, $x \in \mathcal{Z}$, it follows from the definition of the resetting set that

$$[V_{\rm p}(x + f_{\rm d}(x)) - V_{\rm c}(x + f_{\rm d}(x))][V_{\rm p}(x) - V_{\rm c}(x)] = V_{\rm p}(x)[V_{\rm p}(x) - V_{\rm c}(x)] \geq 0, \quad x \in \mathcal{Z},$$

and hence, Assumption 6 is satisfied across resettings. Hence, the closed-loop system $\mathcal{G}$ is thermodynamically consistent after the first resetting event in the sense of [104] and Section 3.

Next, we give a hybrid definition of entropy for the closed-loop system $\mathcal{G}$ that generalizes the continuous-time and discrete-time entropy definitions established in [104] and Section 3.

Definition 5.2. For the impulsive closed-loop system $\mathcal{G}$ given by (5.19) and (5.20), a function $S : \overline{\mathbb{R}}_+^2 \to \mathbb{R}$ satisfying

$$S(E(x(T))) \geq S(E(x(t_1))) - \frac{1}{c} \sum_{k \in \mathbb{Z}_{[t_1,T)}} V_{\rm c}(x(t_k)), \quad T \geq t_1, \tag{5.50}$$

where $k \in \mathbb{Z}_{[t_1,T)} \triangleq \{k : t_1 \leq t_k < T\}$, $E \triangleq [V_{\rm p},\, V_{\rm c}]^{\rm T}$, $c > 0$, is called an *entropy* function of $\mathcal{G}$.

The next result gives necessary and sufficient conditions for establishing the existence of an entropy function of $\mathcal{G}$ over an interval $t \in (t_k, t_{k+1}]$ involving the consecutive resetting times t_k and t_{k+1}, $k \in \mathbb{Z}_+$.

Theorem 5.5. For the impulsive closed-loop system $\mathcal{G}$ given by (5.19) and (5.20), a function $S : \overline{\mathbb{R}}_+^2 \to \mathbb{R}$ is an entropy function of $\mathcal{G}$ if and only if

$$S(E(x(\hat{t}))) \geq S(E(x(t))), \quad t_k < t \leq \hat{t} \leq t_{k+1}, \tag{5.51}$$

$$S(E(x(t_k) + f_{\mathrm{d}}(x(t_k)))) \geq S(E(x(t_k))) - \frac{V_{\mathrm{c}}(x(t_k))}{c}, \quad k \in \mathbb{Z}_+. \tag{5.52}$$

Proof. Let $k \in \mathbb{Z}_+$ and suppose $S(E)$ is an entropy function of $\mathcal{G}$. Then, (5.50) holds. Now, since for $t_k < t \leq \hat{t} \leq t_{k+1}$, $\mathbb{Z}_{[t,\hat{t})} = \emptyset$, (5.51) is immediate. Next, note that

$$S(E(x(t_k^+))) \geq S(E(x(t_k))) - \frac{V_{\mathrm{c}}(x(t_k))}{c}, \tag{5.53}$$

which, since $\mathbb{Z}_{[t_k,t_k^+)} = k$, implies (5.52).

Conversely, suppose (5.51) and (5.52) hold, and let $\hat{t} \geq t \geq t_1$ and $\mathbb{Z}_{[t,\hat{t})} = \{i, i+1, \ldots, j\}$. (Note that if $\mathbb{Z}_{[t,\hat{t})} = \emptyset$ the converse result is a direct consequence of (5.51).) If $\mathbb{Z}_{[t,\hat{t})} \neq \emptyset$, it follows from (5.51) and (5.52) that

$$\begin{aligned} S(E(x(\hat{t}))) - S(E(x(t))) = {} & S(E(x(\hat{t}))) - S(E(x(t_j^+))) \\ & + \sum_{m=0}^{j-i} S(E(x(t_{j-m}) + f_{\mathrm{d}}(x(t_{j-m})))) - S(E(x(t_{j-m}))) \\ & + \sum_{m=0}^{j-i-1} S(E(x(t_{j-m}))) - S(E(x(t_{j-m-1}^+))) \\ & + S(E(x(t_i))) - S(E(x(t))) \\ \geq {} & -\frac{1}{c}\sum_{m=0}^{j-i} V_{\mathrm{c}}(x(t_{j-m})) = -\frac{1}{c}\sum_{k \in \mathbb{Z}_{[t,\hat{t})}} V_{\mathrm{c}}(x(t_k)), \end{aligned} \tag{5.54}$$

which implies that $S(E)$ is an entropy function of $\mathcal{G}$. □

The next theorem establishes the existence of an entropy function for the closed-loop system $\mathcal{G}$.

Theorem 5.6. Consider the impulsive closed-loop system $\mathcal{G}$ given by (5.19) and (5.20), with $\mathcal{Z}$ given by (5.47). Then the function $S : \overline{\mathbb{R}}_+^2 \to \mathbb{R}$ given by

$$S(E) = \log_e(c + V_{\mathrm{p}}) + \log_e(c + V_{\mathrm{c}}) - 2\log_e c, \quad E \in \overline{\mathbb{R}}_+^2, \tag{5.55}$$

where $c > 0$, is a continuously differentiable entropy function of $\mathcal{G}$. In addition,

$$\dot{S}(E(x(t))) > 0, \quad x(t) \notin \mathcal{Z}, \quad t_k < t \leq t_{k+1}, \tag{5.56}$$

$$-\frac{V_{\mathrm{c}}(x(t_k))}{c} < \Delta S(E(x(t_k))) < -\frac{V_{\mathrm{c}}(x(t_k))}{c + V_{\mathrm{c}}(x(t_k))}, \quad x(t_k) \in \mathcal{Z}, \quad k \in \mathbb{Z}_+. \tag{5.57}$$

Proof. Since $\dot{V}_{\mathrm{p}}(x(t)) = \phi(x(t))$ and $\dot{V}_{\mathrm{c}}(x(t)) = -\phi(x(t))$, $x(t) \notin \mathcal{Z}$, $t \in (t_k, t_{k+1}]$, $k \in \mathbb{Z}_+$, it follows that

$$\dot{S}(E(x(t))) = \frac{\phi(x(t))(V_{\mathrm{c}}(x(t)) - V_{\mathrm{p}}(x(t)))}{(c + V_{\mathrm{p}}(x(t)))(c + V_{\mathrm{c}}(x(t)))} > 0, \quad x(t) \notin \mathcal{Z}. \tag{5.58}$$

Furthermore, since $V_{\mathrm{c}}(x(t_k) + f_{\mathrm{d}}(x(t_k))) = 0$ and $V_{\mathrm{p}}(x(t_k) + f_{\mathrm{d}}(x(t_k))) = V_{\mathrm{p}}(x(t_k))$, $x(t_k) \in \mathcal{Z}$, $k \in \mathbb{Z}_+$, it follows that

$$\Delta S(E(x(t_k))) = \log_e\left[1 - \frac{V_{\mathrm{c}}(x(t_k))}{c + V_{\mathrm{c}}(x(t_k))}\right] > -\frac{V_{\mathrm{c}}(x(t_k))}{c}, \quad x(t_k) \in \mathcal{Z}, \quad k \in \mathbb{Z}_+, \tag{5.59}$$

and

$$\Delta S(E(x(t_k))) = \log_e\left[1 - \frac{V_{\mathrm{c}}(x(t_k))}{c + V_{\mathrm{c}}(x(t_k))}\right] < -\frac{V_{\mathrm{c}}(x(t_k))}{c + V_{\mathrm{c}}(x(t_k))}, \quad x(t_k) \in \mathcal{Z}, \quad k \in \mathbb{Z}_+, \tag{5.60}$$

where in (5.59) and (5.60) we used the fact that $\frac{x}{1+x} < \log_e(1 + x) < x$, $x > -1$, $x \neq 0$. The result is now an immediate consequence of Theorem 5.5. □

Remark 5.7. In the case where $\mathcal{G}_{\mathrm{p}}$ is dissipative the entropy function $S : \overline{\mathbb{R}}_+^2 \to \mathbb{R}$ of the impulsive closed-loop system $\mathcal{G}$ is such that

$$S(E(x(T))) \geq S(E(x(t_1))) - \int_{t_1}^{T} \frac{d(x_{\mathrm{p}}(t))}{c + V_{\mathrm{p}}(x(t))} \mathrm{d}t - \frac{1}{c} \sum_{k \in \mathbb{Z}_{[t_1, T)}} V_{\mathrm{c}}(x(t_k)), \quad T \geq t_1, \tag{5.61}$$

where $d : \mathcal{D}_{\mathrm{p}} \to \mathbb{R}$ is a continuous, nonnegative-definite dissipation rate function.

Note that it follows from (5.56) that the entropy of the closed-loop system strictly increases between resetting events, which is consistent with thermodynamic principles. This is not surprising since in this case the closed-loop system is *adiabatically isolated* (i.e., the system does not exchange energy (heat) with the environment) and the total energy of the closed-loop system is conserved between resetting events. Alternatively, it follows from (5.57) that the entropy of the closed-loop system strictly decreases across resetting events since the total energy strictly decreases at each resetting instant, and hence, energy is not conserved across resetting events.

Using Theorem 5.6, the resetting set $\mathcal{Z}$ given by (5.47) can be rewritten as

$$\mathcal{Z} \triangleq \left\{ x \in \mathcal{D} : \frac{\mathrm{d}}{\mathrm{d}t} S(E(x)) = 0 \text{ and } V_{\mathrm{c}}(x) > 0 \right\}, \tag{5.62}$$

where $\mathcal{X}(x) \triangleq \frac{\mathrm{d}}{\mathrm{d}t} S(E(x)) = \frac{\mathrm{d}}{\mathrm{d}t} S(E(\psi(t,x)))|_{t=0}$ is a continuously differentiable function that defines the resetting set as its zero level set. The resetting set (5.47) or, equivalently, (5.62) is motivated by thermodynamic principles and guarantees that the energy of the closed-loop system is always flowing from regions of higher to lower energies after the first resetting event, which is in accordance with the second law of thermodynamics. As shown in Theorem 5.6, this guarantees the existence of entropy function $S(E)$ for the closed-loop system that satisfies the Clausius-type inequality (5.56) between resetting events. If $\phi(x) = 0$ or $V_{\mathrm{p}}(x) = V_{\mathrm{c}}(x)$, then inequality (5.56) would be subverted, and hence, we reset the compensator states in order to ensure that the second law of thermodynamics is not violated. In this case, the hybrid controller (5.15)–(5.17), with resetting set (5.47), is a *thermodynamically stabilizing* compensator.

5.6. Energy Dissipating Hybrid Control Design

In this section, we apply the energy dissipating hybrid controller synthesis framework developed in Sections 5.4 and 5.5 to three examples. For the first example, consider the vector second-order nonlinear *Lienard* system given by

$$\ddot{q}(t)+f(q(t)) = u(t), \quad q(0)=q_0, \quad \dot{q}(0)=\dot{q}_0, \quad t\geq 0, \tag{5.63}$$

$$y(t) = \begin{bmatrix} C_1 q(t) \\ C_2 \dot{q}(t) \end{bmatrix}, \tag{5.64}$$

where $q\in\mathbb{R}^{\hat{n}_\mathrm{p}}$, $f:\mathbb{R}^{\hat{n}_\mathrm{p}}\to\mathbb{R}^{\hat{n}_\mathrm{p}}$ is infinitely differentiable, $f(q)=0$ if and only if $q=0$, $C_1\in\mathbb{R}^{l_1\times\hat{n}_\mathrm{p}}$, $C_2\in\mathbb{R}^{(l-l_1)\times\hat{n}_\mathrm{p}}$, and

$$\frac{\partial f_i}{\partial q_j} = \frac{\partial f_j}{\partial q_i}, \quad i,j=1,\ldots,\hat{n}_\mathrm{p}. \tag{5.65}$$

The plant energy of the system is given by

$$\begin{aligned} V_\mathrm{p}(q,\dot{q}) &= T(q,\dot{q})+U(q) \\ &= \frac{1}{2}\dot{q}^\mathrm{T}\dot{q} + \int_{0,\,\mathrm{path}}^{q} f^\mathrm{T}(\sigma)\mathrm{d}\sigma \\ &= \frac{1}{2}\dot{q}^\mathrm{T}\dot{q} + \int_{0,\,\mathrm{path}}^{q} \sum_{i=1}^{\hat{n}_\mathrm{p}} f_i(\sigma)\mathrm{d}\sigma_i \\ &= \frac{1}{2}\dot{q}^\mathrm{T}\dot{q} + \int_0^{q_1} f_1(\sigma_1,0,\ldots,0)\mathrm{d}\sigma_1 + \int_0^{q_2} f_2(q_1,\sigma_2,0,\ldots,0)\mathrm{d}\sigma_2 \\ &\quad + \cdots + \int_0^{q_{\hat{n}_\mathrm{p}}} f_{\hat{n}_\mathrm{p}}(q_1,q_2,\ldots,q_{\hat{n}_\mathrm{p}-1},\sigma_{\hat{n}_\mathrm{p}})\mathrm{d}\sigma_{\hat{n}_\mathrm{p}}, \end{aligned} \tag{5.66}$$

where $T(q,\dot{q})=\frac{1}{2}\dot{q}^\mathrm{T}\dot{q}$ and $U(q)=\int_{0,\,\mathrm{path}}^{q} f^\mathrm{T}(\sigma)\mathrm{d}\sigma$. Note that the path integral in (5.66) is taken over any path joining the origin to $q\in\mathbb{R}^{\hat{n}_\mathrm{p}}$. Furthermore, the path integral in (5.66) is well defined since $f(\cdot)$ is such that $\frac{\partial f}{\partial q}$ is symmetric, and hence, $f(\cdot)$ is a gradient of a real-valued function [6, Theorem 10-37]. Here, we assume that $U(0)=0$ and $U(q)>0$ for $q\neq 0$, $q\in\mathbb{R}^{\hat{n}_\mathrm{p}}$. Note that defining $p\triangleq\dot{q}$ and

$$\mathcal{H}(q,p) \triangleq \frac{1}{2}p^\mathrm{T}p + \int_{0,\,\mathrm{path}}^{q} f^\mathrm{T}(\sigma)\mathrm{d}\sigma, \tag{5.67}$$

it follows that (5.63) can be written in Hamiltonian form

$$\dot{q}(t) = \left[\frac{\partial\mathcal{H}}{\partial p}(q(t),p(t))\right]^\mathrm{T}, \quad q(0)=q_0, \quad t\geq 0, \tag{5.68}$$

$$\dot{p}(t) = -\left[\frac{\partial\mathcal{H}}{\partial q}(q(t),p(t))\right]^\mathrm{T} + u, \quad p(0)=p_0. \tag{5.69}$$

To design a state-dependent hybrid controller for the Lienard system (5.63), let $C_1 = C_2 = I_{\hat{n}_\mathrm{p}}$, let

$$T_\mathrm{c}(q_\mathrm{c}, \dot{q}_\mathrm{c}) = \frac{1}{2}\dot{q}_\mathrm{c}^\mathrm{T}\dot{q}_\mathrm{c}, \tag{5.70}$$

$$U_\mathrm{c}(q_\mathrm{c}, q) = \int_{0,\,\mathrm{path}}^{q_\mathrm{c}-q} g^\mathrm{T}(\sigma)\mathrm{d}\sigma, \tag{5.71}$$

where $q_\mathrm{c} \in \mathbb{R}^{\hat{n}_\mathrm{p}}$, $g : \mathbb{R}^{\hat{n}_\mathrm{p}} \to \mathbb{R}^{\hat{n}_\mathrm{p}}$ is infinitely differentiable, $g(x) = 0$ if and only if $x = 0$, and $g'(0)$ is positive definite, and let

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad i, j = 1, \ldots, \hat{n}_\mathrm{p}, \tag{5.72}$$

so that

$$\mathcal{L}_\mathrm{c}(q_\mathrm{c}, \dot{q}_\mathrm{c}, q) = \frac{1}{2}\dot{q}_\mathrm{c}^\mathrm{T}\dot{q}_\mathrm{c} - \int_{0,\,\mathrm{path}}^{q_\mathrm{c}-q} g^\mathrm{T}(\sigma)\mathrm{d}\sigma. \tag{5.73}$$

Here, we assume that $\int_{0,\,\mathrm{path}}^{x} g^\mathrm{T}(\sigma)\mathrm{d}\sigma > 0$ for all $x \neq 0$, $x \in \mathbb{R}^{\hat{n}_\mathrm{p}}$. In this case, the state-dependent hybrid controller has the form

$$\ddot{q}_\mathrm{c}(t) + g(q_\mathrm{c}(t) - q(t)) = 0, \quad (q(t), \dot{q}(t), q_\mathrm{c}(t), \dot{q}_\mathrm{c}(t)) \notin \mathcal{Z}, \quad t \geq 0, \tag{5.74}$$

$$\begin{bmatrix} \Delta q_\mathrm{c}(t) \\ \Delta \dot{q}_\mathrm{c}(t) \end{bmatrix} = \begin{bmatrix} q(t) - q_\mathrm{c}(t) \\ -\dot{q}_\mathrm{c}(t) \end{bmatrix}, \quad (q(t), \dot{q}(t), q_\mathrm{c}(t), \dot{q}_\mathrm{c}(t)) \in \mathcal{Z}, \quad t \geq 0, \tag{5.75}$$

$$u(t) = g(q_\mathrm{c}(t) - q(t)), \tag{5.76}$$

with the resetting set (5.46) taking the form

$$\mathcal{Z} = \left\{ (q, \dot{q}, q_\mathrm{c}, \dot{q}_\mathrm{c}) : [g(q_\mathrm{c} - q)]^\mathrm{T}\dot{q} = 0 \text{ and } \begin{bmatrix} q - q_\mathrm{c} \\ -\dot{q}_\mathrm{c} \end{bmatrix} \neq 0 \right\}. \tag{5.77}$$

Here, we consider the case where $\hat{n}_\mathrm{p} = \frac{n_\mathrm{p}}{2} = 1$. To show that Assumption 1 holds in this case, we show that upon reaching a nonequilibrium point $x(t) \triangleq [q(t), \dot{q}(t), q_\mathrm{c}(t), \dot{q}_\mathrm{c}(t)]^\mathrm{T} \notin \mathcal{Z}$ that is in the closure of $\mathcal{Z}$, the continuous-time dynamics $\dot{x} = f_\mathrm{c}(x)$ remove $x(t)$ from $\overline{\mathcal{Z}}$, and hence, necessarily move the trajectory a finite distance away from $\mathcal{Z}$. If $x(t) \notin \mathcal{Z}$ is an equilibrium point, then $x(s) \notin \mathcal{Z}$, $s \geq t$, which is also consistent with Assumption 1.

The closure of $\mathcal{Z}$ is given by

$$\overline{\mathcal{Z}} = \{ (q, \dot{q}, q_\mathrm{c}, \dot{q}_\mathrm{c}) : [g(q_\mathrm{c} - q)]\dot{q} \geq 0 \}. \tag{5.78}$$

Furthermore, the points x^* satisfying $[q^* - q_c^*, -\dot{q}_c^*]^T = 0$ have the form

$$x^* \triangleq [q, \dot{q}, q, 0]^T, \tag{5.79}$$

that is, $q_c = q$ and $\dot{q}_c = 0$. It follows that $x^* \notin \mathcal{Z}$, although $x^* \in \overline{\mathcal{Z}}$.

To show that the continuous-time dynamics $\dot{x} = f_c(x)$ remove x^* from $\overline{\mathcal{Z}}$, note that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_p(q, \dot{q}) = [g(q_c - q)]\dot{q} \tag{5.80}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} V_p(q, \dot{q}) = \ddot{q}[g(q_c - q)] + \dot{q}[g'(q_c - q)](\dot{q}_c - \dot{q}), \tag{5.81}$$

$$\begin{aligned} \frac{\mathrm{d}^3}{\mathrm{d}t^3} V_p(q, \dot{q}) = {} & q^{(3)}[g(q_c - q)] + [g'(q_c - q)](\dot{q}\ddot{q}_c + 2\dot{q}_c\ddot{q} - 3\dot{q}\ddot{q}) \\ & + [g''(q_c - q)](\dot{q}_c - \dot{q})^2\dot{q}, \end{aligned} \tag{5.82}$$

$$\begin{aligned} \frac{\mathrm{d}^4}{\mathrm{d}t^4} V_p(q, \dot{q}) = {} & q^{(4)}[g(q_c - q)] + [g'(q_c - q)](3\dot{q}_c q^{(3)} - 4\dot{q}q^{(3)} + 3\ddot{q}\ddot{q}_c + \dot{q}q_c^{(3)} - 3\ddot{q}^2) \\ & + [g''(q_c - q)](3\dot{q}\dot{q}_c\ddot{q}_c + 3\dot{q}_c^2\ddot{q} - 9\dot{q}\dot{q}_c\ddot{q} - 3\dot{q}^2\ddot{q}_c + 6\dot{q}^2\ddot{q}) \\ & + g^{(3)}(q_c - q)(\dot{q}_c - \dot{q})^3\dot{q}, \end{aligned} \tag{5.83}$$

where $g^{(n)}(t) \triangleq \frac{\mathrm{d}^n g(t)}{\mathrm{d}t^n}$. Since

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2} V_p(q, \dot{q})\right|_{x=x^*} = -g'(0)\dot{q}^2, \tag{5.84}$$

it follows that if $\dot{q} \neq 0$, then the continuous-time dynamics $\dot{x} = f_c(x)$ remove x^* from $\overline{\mathcal{Z}}$. If $\dot{q} = 0$, then it follows from (5.81)–(5.83) that

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2} V_p(q, \dot{q})\right|_{x=x^*, \dot{q}=0} = 0, \tag{5.85}$$

$$\left.\frac{\mathrm{d}^3}{\mathrm{d}t^3} V_p(q, \dot{q})\right|_{x=x^*, \dot{q}=0} = 0, \tag{5.86}$$

$$\left.\frac{\mathrm{d}^4}{\mathrm{d}t^4} V_p(q, \dot{q})\right|_{x=x^*, \dot{q}=0} = -3g'(0)\ddot{q}^2, \tag{5.87}$$

where in the evaluation of (5.86) and (5.87) we use the fact that if $q_c = q$ and $\dot{q}_c = 0$, then $\ddot{q}_c = 0$, which follows immediately from the continuous-time dynamics. Since if $\dot{q} = 0$ and

$\ddot{q} \neq 0$, then the lowest-order nonzero time derivative of $\dot{V}_{\rm p}(x_{\rm p})$ is negative, it follows that the continuous-time dynamics remove x^* from $\overline{\mathcal{Z}}$. However, if $\dot{q} = 0$ and $\ddot{q} = 0$, then it follows from the continuous-time dynamics that x^* is necessarily an equilibrium point, in which case the trajectory never again enters $\mathcal{Z}$. Therefore, we can conclude that Assumption 1 is indeed valid for this system. Also, since $f_{\rm d}(x + f_{\rm d}(x)) = 0$, it follows from (5.77) that if $x \in \mathcal{Z}$, then $x + f_{\rm d}(x) \notin \mathcal{Z}$, and thus Assumption 2 holds.

For thermodynamic stabilization, the resetting set (5.47) is given by

$$\mathcal{Z} = \Big\{(q, \dot{q}, q_{\rm c}, \dot{q}_{\rm c}) : \dot{q}^{\rm T}[g(q_{\rm c} - q)][V_{\rm p}(q, \dot{q}) - V_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, q)] = 0 \\ \text{and } \begin{bmatrix} q - q_{\rm c} \\ -\dot{q}_{\rm c} \end{bmatrix} \neq 0\Big\}. \tag{5.88}$$

Furthermore, the entropy function $S(E)$ is given by

$$S(E) = \log_e[1 + V_{\rm p}(q, \dot{q})] + \log_e[1 + V_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, q)]. \tag{5.89}$$

To illustrate the behavior of the closed-loop impulsive dynamical system, let $\hat{n}_{\rm p} = \frac{n_{\rm p}}{2} = 1$, $f(x) = x + x^3$, and $g(x) = 3x$ with initial conditions $q(0) = 0$, $\dot{q}(0) = 1$, $q_{\rm c}(0) = 0$, and $\dot{q}_{\rm c}(0) = 0$. For this system, the transversality condition is sufficiently complex that we have been unable to show it analytically. This condition was verified numerically, and hence, Assumption 3 holds. Figures 5.1 shows the controlled plant position and velocity states versus time, while 5.2 shows the virtual position and velocity compensator states versus time. Figure 5.3 shows the control force versus time. Note that the compensator states are the only states that reset. Furthermore, the control force versus time is discontinuous at the resetting times. A comparison of the plant energy, controller energy, and total energy is shown in Figure 5.4. Figures 5.5–5.8 show analogous representations for the thermodynamically stabilizing compensator. Finally, Figure 5.9 shows the closed-loop system entropy versus time. Note that the entropy of the closed-loop system strictly increases between resetting events.

As our next example, we consider the rotational/translational proof-mass actuator (RTAC) nonlinear system studied in [45]. The system (see Figure 5.10) involves an eccentric ro-

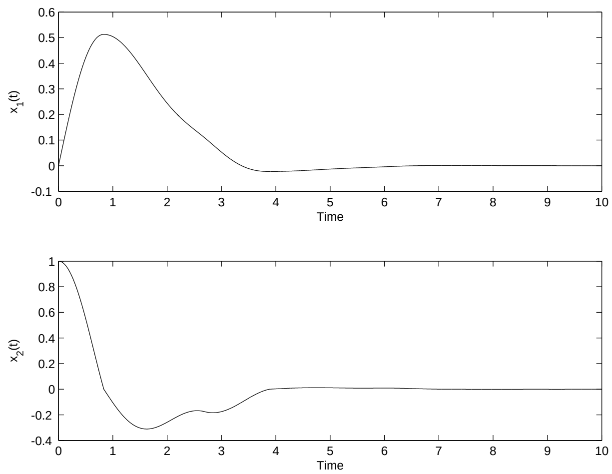

Figure 5.1: Plant position and velocity versus time

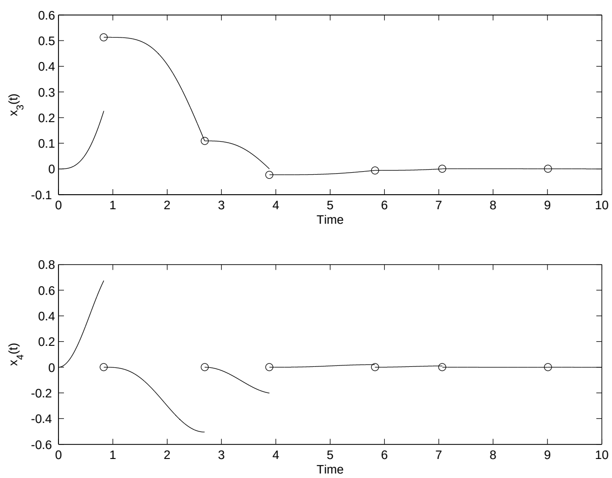

Figure 5.2: Controller position and velocity versus time

tational inertia, which acts as a proof-mass actuator mounted on a translational oscillator. The oscillator cart of mass M is connected to a fixed support via a linear spring of stiffness k. The cart is constrained to one-dimensional motion and the rotational proof-mass actuator consists of a mass m and mass moment of inertia I located a distance e from the center of mass of the cart. In Figure 5.10, N denotes the control torque applied to the proof mass. Since the motion is constrained to the horizontal plane the gravitational forces are not considered in the dynamic analysis.

Letting q, $\dot{q}$, θ, and $\dot{\theta}$ denote the translational position and velocity of the cart and the

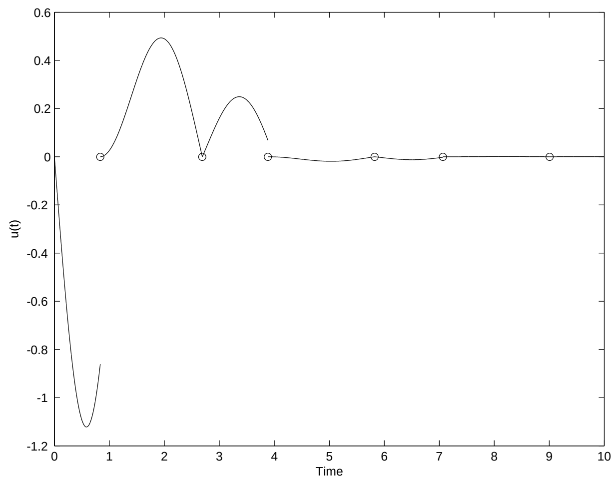

Figure 5.3: Control signal versus time

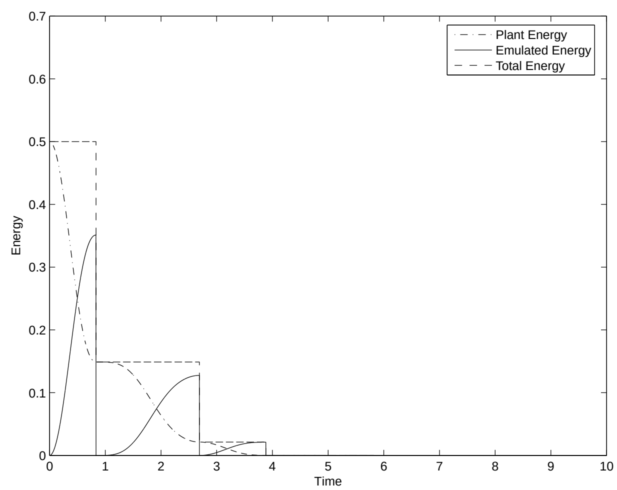

Figure 5.4: Plant, emulated, and total energy versus time

angular position and velocity of the rotational proof mass, respectively, and using the energy function

$$V_{\mathrm{s}}(q, \dot{q}, \theta, \dot{\theta}) = \frac{1}{2}[kq^2 + (M+m)\dot{q}^2 + (I + me^2)\dot{\theta}^2 + 2me\dot{q}\dot{\theta}\cos\theta], \tag{5.90}$$

the nonlinear dynamic equations of motion are given by

$$(M+m)\ddot{q} + kq = -me(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta), \tag{5.91}$$

$$(I + me^2)\ddot{\theta} = -me\ddot{q}\cos\theta + N, \tag{5.92}$$

with problem data given in Table 5.1 and output $y = [\theta, \dot{\theta}]^{\mathrm{T}}$. The physical configuration

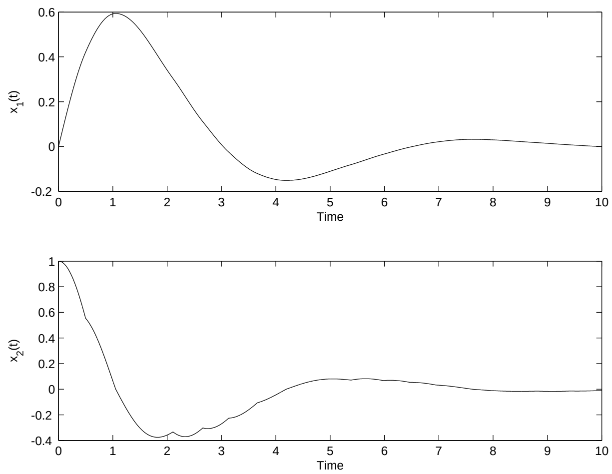

Figure 5.5: Plant position and velocity versus time for thermodynamic controller

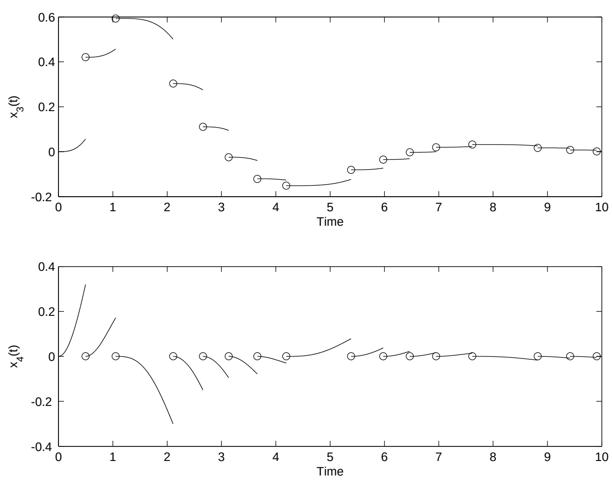

Figure 5.6: Controller position and velocity versus time for thermodynamic controller

of the system necessitates the constraint $|q| \leq 0.025\,\mathrm{m}$. In addition, the control torque is limited by $|N| \leq 0.100\,\mathrm{N\ m}$ [45]. With the normalization

$$\xi \triangleq \sqrt{\frac{M+m}{I+me^2}}q, \quad \tau \triangleq \sqrt{\frac{k}{M+m}}t, \quad u \triangleq \frac{M+m}{k(I+me^2)}N, \tag{5.93}$$

the equations of motion become

$$\ddot{\xi} + \xi = \varepsilon(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta), \tag{5.94}$$

$$\ddot{\theta} = -\varepsilon\ddot{\xi}\cos\theta + u, \tag{5.95}$$

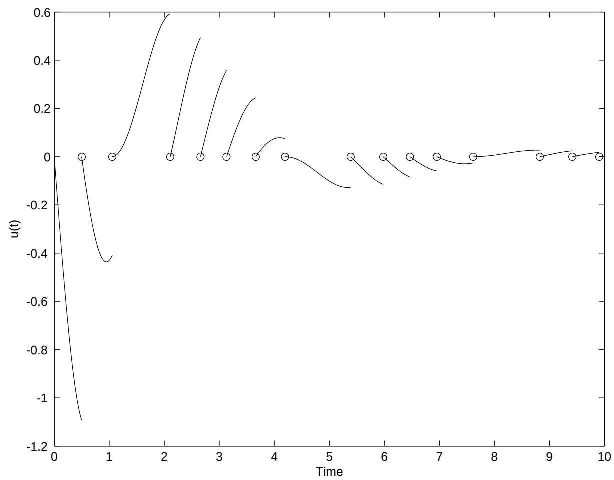

Figure 5.7: Control signal versus time for thermodynamic controller

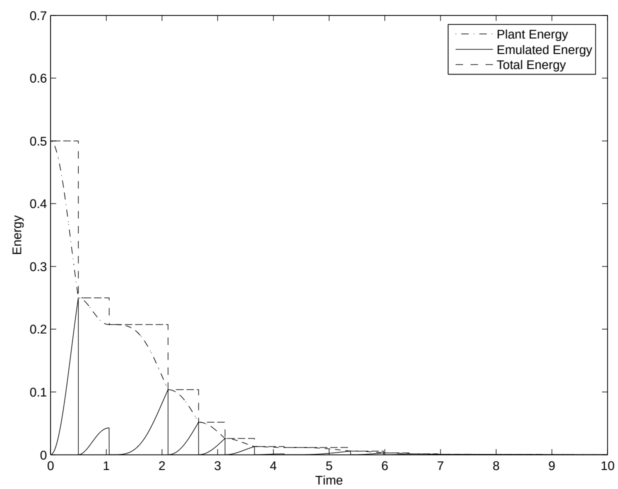

Figure 5.8: Plant, emulated, and total energy versus time for thermodynamic controller

where ξ is the normalized cart position and u represents the non-dimensionalized control torque. In the normalized equations (5.94) and (5.95), the symbol $\dot{(\cdot)}$ represents differentiation with respect to the normalized time τ and the parameter ε represents the coupling between the translational and rotational motions and is defined by

$$\varepsilon \triangleq \frac{me}{\sqrt{(I + me^2)(M + m)}}. \tag{5.96}$$

Since the plant energy function (5.90) is not positive definite in $\mathbb{R}^4$, we first design a control law $u = -k_\theta \theta + \hat{u}$, where $k_\theta > 0$, with associated positive definite normalized plant

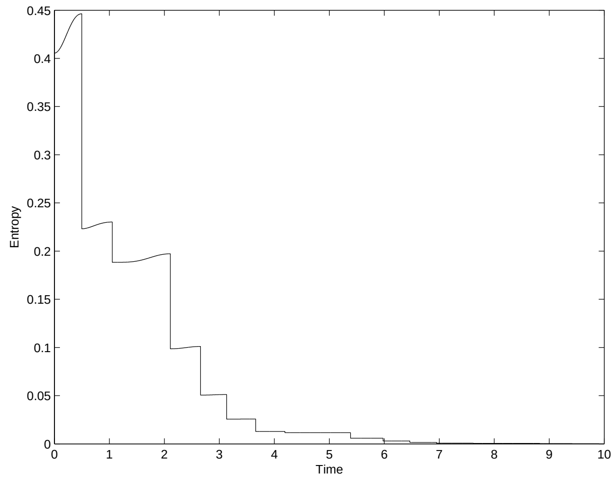

Figure 5.9: Closed-loop entropy versus time

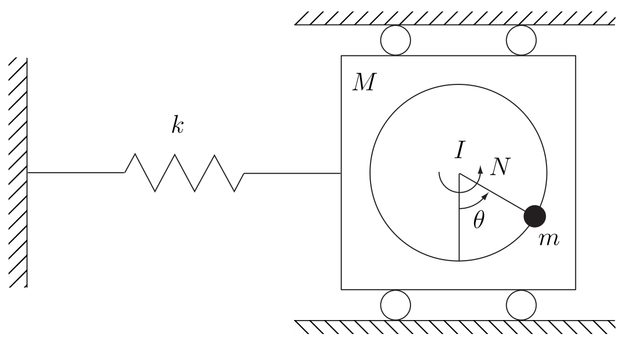

Figure 5.10: Rotational/translational proof-mass actuator

energy function given by

$$V_{\rm s}(\xi,\dot{\xi},\theta,\dot{\theta}) = \frac{1}{2}\xi^2 + \frac{1}{2}\dot{\xi}^2 + \frac{1}{2}k_\theta\theta^2 + \frac{1}{2}\dot{\theta}^2 + \varepsilon\dot{\xi}\dot{\theta}\cos\theta. \tag{5.97}$$

To design a state-dependent hybrid controller for (5.94) and (5.95), let $n_{\rm c} = 1$, $V_{\rm c}(\xi_{\rm c},\dot{\xi}_{\rm c},\theta)$ $= \frac{1}{2}m_{\rm c}\dot{\xi}_{\rm c}^2 + \frac{1}{2}k_{\rm c}(\xi_{\rm c}-\theta)^2$, $\mathcal{L}_{\rm c}(\xi_{\rm c},\dot{\xi}_{\rm c},\theta) = \frac{1}{2}m_{\rm c}\dot{\xi}_{\rm c}^2 - \frac{1}{2}k_{\rm c}(\xi_{\rm c}-\theta)^2$, $y_q = \theta$, and $\eta(y_q) = y_q$, where $m_{\rm c} > 0$ and $k_{\rm c} > 0$. Then the state-dependent hybrid controller has the form

$$m_{\rm c}\ddot{\xi}_{\rm c} + k_{\rm c}(\xi_{\rm c}-\theta) = 0, \quad (\xi_{\rm c},\dot{\xi}_{\rm c},\theta,\dot{\theta}) \notin \mathcal{Z}, \tag{5.98}$$

$$\begin{bmatrix} \Delta\xi_{\rm c} \\ \Delta\dot{\xi}_{\rm c} \end{bmatrix} = \begin{bmatrix} \theta-\xi_{\rm c} \\ -\dot{\xi}_{\rm c} \end{bmatrix}, \quad (\xi_{\rm c},\dot{\xi}_{\rm c},\theta,\dot{\theta}) \in \mathcal{Z}, \tag{5.99}$$

$$\hat{u} = k_{\rm c}(\xi_{\rm c}-\theta), \tag{5.100}$$

Description	Parameter	Value	Units
Cart mass	M	1.3608	kg
Arm mass	m	0.096	kg
Arm eccentricity	e	0.0592	m
Arm inertia	I	0.0002175	kg m^2
Spring stiffness	k	186.3	N/m
Coupling parameter	ε	0.200	—

Table 5.1: Problem data for the RTAC [45]

with the resetting set (5.46) taking the form

$$\mathcal{Z} = \left\{(\xi_{\rm c}, \dot{\xi}_{\rm c}, \theta, \dot{\theta}) \in \mathbb{R}^4 : k_{\rm c}\dot{\theta}(\xi_{\rm c} - \theta) = 0 \text{ and } \begin{bmatrix} \theta - \xi_{\rm c} \\ -\dot{\xi}_{\rm c} \end{bmatrix} \neq 0\right\}. \tag{5.101}$$

To show that Assumption 1 holds, we show that upon reaching a nonequilibrium point $x(\tau) \triangleq [\xi(\tau), \dot{\xi}(\tau), \theta(\tau), \dot{\theta}(\tau), \xi_{\rm c}(\tau), \dot{\xi}_{\rm c}(\tau)]^{\rm T} \notin \mathcal{Z}$ that is in the closure of $\mathcal{Z}$, the continuous-time dynamics $\dot{x} = f_{\rm c}(x)$ remove $x(\tau)$ from $\overline{\mathcal{Z}}$, and thus necessarily move the trajectory a finite distance away from $\mathcal{Z}$. If $x(\tau) \notin \mathcal{Z}$ is an equilibrium point, then $x(s) \notin \mathcal{Z}$, $s \geq \tau$, which is also consistent with Assumption 1.

The closure of $\mathcal{Z}$ is given by

$$\overline{\mathcal{Z}} = \left\{(\xi_{\rm c}, \dot{\xi}_{\rm c}, \theta, \dot{\theta}) : k_{\rm c}\dot{\theta}(\xi_{\rm c} - \theta) \geq 0\right\}. \tag{5.102}$$

Furthermore, the points x^* satisfying $[\theta^* - \xi_{\rm c}^*, -\dot{\xi}_{\rm c}^*]^{\rm T} = 0$ have the form

$$x^* \triangleq [\xi, \dot{\xi}, \theta, \dot{\theta}, \theta, 0]^{\rm T}, \tag{5.103}$$

that is, $\xi_{\rm c} = \theta$ and $\dot{\xi}_{\rm c} = 0$. It follows that $x^* \notin \mathcal{Z}$, although $x^* \in \overline{\mathcal{Z}}$.

To show that the continuous-time dynamics $\dot{x} = f_{\rm c}(x)$ remove x^* from $\overline{\mathcal{Z}}$, note that

$$\frac{\rm d}{{\rm d}\tau} V_{\rm s}(\xi, \dot{\xi}, \theta, \dot{\theta}) = k_{\rm c}\dot{\theta}(\xi_{\rm c} - \theta) \tag{5.104}$$

and

$$\frac{{\rm d}^2}{{\rm d}\tau^2} V_{\rm s}(\xi, \dot{\xi}, \theta, \dot{\theta}) = k_{\rm c}\ddot{\theta}(\xi_{\rm c} - \theta) + k_{\rm c}\dot{\theta}(\dot{\xi}_{\rm c} - \dot{\theta}), \tag{5.105}$$

$$\frac{{\rm d}^3}{{\rm d}\tau^3} V_{\rm s}(\xi, \dot{\xi}, \theta, \dot{\theta}) = k_{\rm c}\theta^{(3)}(\xi_{\rm c} - \theta) + 2k_{\rm c}\ddot{\theta}(\dot{\xi}_{\rm c} - \dot{\theta}) + k_{\rm c}\dot{\theta}(\ddot{\xi}_{\rm c} - \ddot{\theta}), \tag{5.106}$$

$$\frac{\mathrm{d}^4}{\mathrm{d}\tau^4}V_{\mathrm{s}}(\xi,\dot{\xi},\theta,\dot{\theta}) = k_{\mathrm{c}}\theta^{(4)}(\xi_{\mathrm{c}}-\theta)+3k_{\mathrm{c}}\theta^{(3)}(\dot{\xi}_{\mathrm{c}}-\dot{\theta})+3k_{\mathrm{c}}\ddot{\theta}(\ddot{\xi}_{\mathrm{c}}-\ddot{\theta}) +k_{\mathrm{c}}\dot{\theta}(\xi_{\mathrm{c}}^{(3)}-\theta^{(3)}), \tag{5.107}$$

where $g^{(n)}(\tau) \triangleq \frac{\mathrm{d}^n g(\tau)}{\mathrm{d}\tau^n}$. Since

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}V_{\mathrm{s}}(\xi,\dot{\xi},\theta,\dot{\theta})\right|_{x=x^*} = -k_{\mathrm{c}}\dot{\theta}^2, \tag{5.108}$$

it follows that if $\dot{\theta}\neq 0$, then the continuous-time dynamics $\dot{x}=f_{\mathrm{c}}(x)$ remove x^* from $\overline{\mathcal{Z}}$. If $\dot{\theta}=0$, then it follows from (5.105)–(5.107) that

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}V_{\mathrm{s}}(\xi,\dot{\xi},\theta,\dot{\theta})\right|_{x=x^*,\dot{\theta}=0} = 0, \tag{5.109}$$

$$\left.\frac{\mathrm{d}^3}{\mathrm{d}\tau^3}V_{\mathrm{s}}(\xi,\dot{\xi},\theta,\dot{\theta})\right|_{x=x^*,\dot{\theta}=0} = 0, \tag{5.110}$$

$$\left.\frac{\mathrm{d}^4}{\mathrm{d}\tau^4}V_{\mathrm{s}}(\xi,\dot{\xi},\theta,\dot{\theta})\right|_{x=x^*,\dot{\theta}=0} = -3k_{\mathrm{c}}\ddot{\theta}^2, \tag{5.111}$$

where in the evaluation of (5.110) and (5.111) we use the fact that if $\xi_{\mathrm{c}}=\theta$ and $\dot{\xi}_{\mathrm{c}}=0$, then $\ddot{\xi}_{\mathrm{c}}=0$, which follows immediately from the continuous-time dynamics. Since if $\dot{\theta}=0$ and $\ddot{\theta}\neq 0$, then the lowest-order nonzero time derivative of $\dot{V}_{\mathrm{s}}(\xi,\dot{\xi},\theta,\dot{\theta})$ is negative, it follows that the continuous-time dynamics remove x^* from $\overline{\mathcal{Z}}$. However, if $\dot{\theta}=0$ and $\ddot{\theta}=0$, then it follows from the continuous-time dynamics that x^* is necessarily an equilibrium point, in which case the trajectory never again enters $\mathcal{Z}$. Therefore, we can conclude that Assumption 1 is indeed valid for this system. Also, since $f_{\mathrm{d}}(x+f_{\mathrm{d}}(x))=0$, it follows from (5.101) that if $x\in\mathcal{Z}$, then $x+f_{\mathrm{d}}(x)\notin\mathcal{Z}$, and thus Assumption 2 holds.

For thermodynamic stabilization, the output y is modified as $y=[\xi,\dot{\xi},\theta,\dot{\theta}]^{\mathrm{T}}$ and the resetting set (5.47) is given by

$$\mathcal{Z} = \left\{(\xi,\dot{\xi},\theta,\dot{\theta},\xi_{\mathrm{c}},\dot{\xi}_{\mathrm{c}})\in\mathbb{R}^6 : k_{\mathrm{c}}\dot{\theta}(\xi_{\mathrm{c}}-\theta)[V_{\mathrm{s}}(\xi,\dot{\xi},\theta,\dot{\theta})-V_{\mathrm{c}}(\xi_{\mathrm{c}},\dot{\xi}_{\mathrm{c}},\theta)]=0 \text{ and } \begin{bmatrix}\theta-\xi_{\mathrm{c}}\\ -\dot{\xi}_{\mathrm{c}}\end{bmatrix}\neq 0\right\}. \tag{5.112}$$

Furthermore, the entropy function $S(E)$ is given by

$$S(E) = \log_e[1+V_{\mathrm{s}}(\xi,\dot{\xi},\theta,\dot{\theta})]+\log_e[1+V_{\mathrm{c}}(\xi_{\mathrm{c}},\dot{\xi}_{\mathrm{c}},\theta)]. \tag{5.113}$$

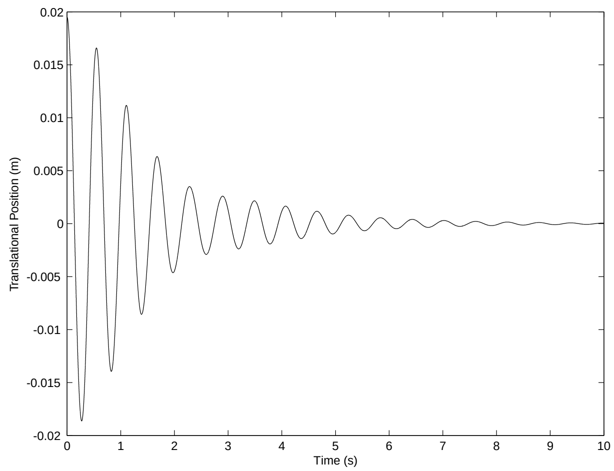

Figure 5.11: Translational position of the cart versus time

To illustrate the behavior of the closed-loop impulsive dynamical system, let $m_{\rm c} = 0.2$, $k_{\rm c} = 1$, and $k_\theta = 1$ with initial conditions $\xi(0) = 1$, $\dot{\xi}(0) = 0$, $\theta(0) = 0$, $\dot{\theta}(0) = 0$, $\xi_{\rm c}(0) = 0$, and $\dot{\xi}_{\rm c}(0) = 0$. For thermodynamic stabilization, the initial conditions are given by $\xi(0) = 0.6$, $\dot{\xi}(0) = 0$, $\theta(0) = 0$, $\dot{\theta}(0) = 0$, $\xi_{\rm c}(0) = 0.8$, and $\dot{\xi}_{\rm c}(0) = 0$. For this system, the transversality condition is sufficiently complex that we have been unable to show it analytically. This condition was verified numerically, and hence, Assumption 3 appears to hold. Figures 5.11 and 5.12 show the translational position of the cart and the angular position of the rotational proof mass versus time. Figure 5.13 shows the control torque versus time. Note that the compensator states are the only states that reset. Furthermore, the control torque versus time is discontinuous at the resetting times. A comparison of the plant energy, control energy, and total energy is shown in Figure 5.14. Figures 5.15–5.18 show analogous representations for the thermodynamically stabilizing compensator. Finally, Figure 5.19 shows the closed-loop system entropy versus time. Note that the entropy of the closed-loop system strictly increases between resetting events.

Our final example considers the design of a hybrid controller for a combustion system. High performance aeroengine afterburners and ramjets often experience combustion instabilities at some operating condition. Combustion in these high energy density engines is highly susceptible to flow disturbances, resulting in fluctuations to the instantaneous rate

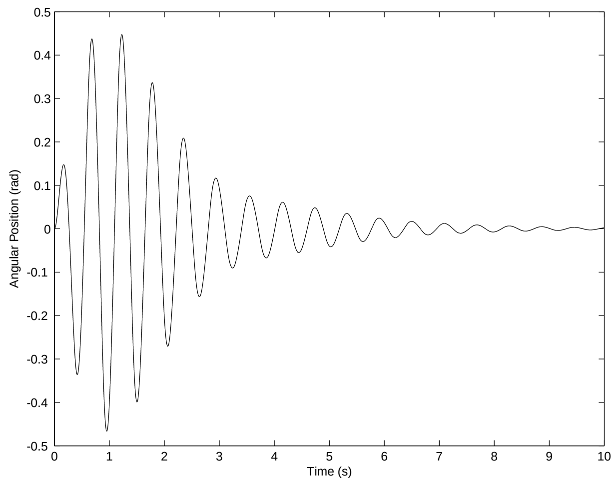

Figure 5.12: Angular position of the rotational proof mass versus time

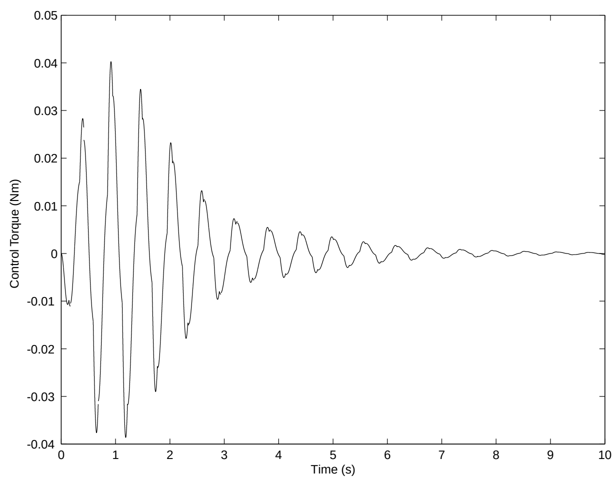

Figure 5.13: Control torque versus time

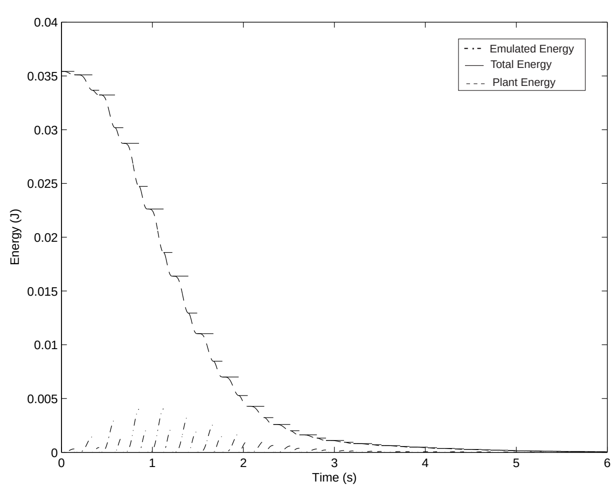

Figure 5.14: Plant, emulated, and total energy versus time

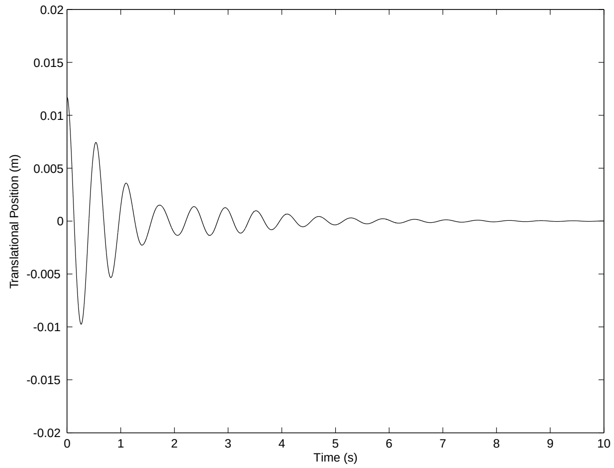

Figure 5.15: Translational position of the cart versus time for thermodynamic controller

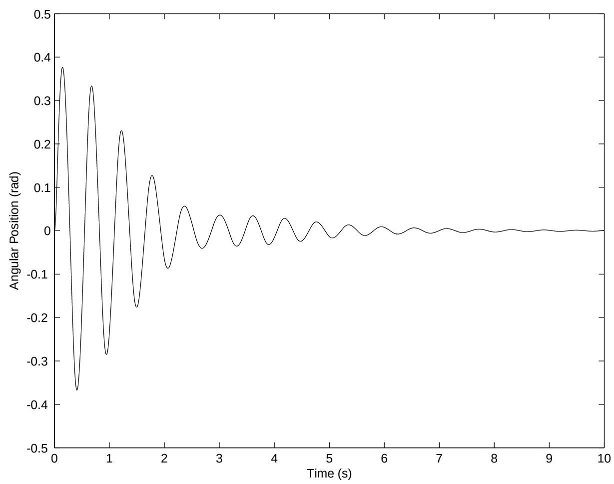

Figure 5.16: Angular position of the rotational proof mass versus time for thermodynamic controller

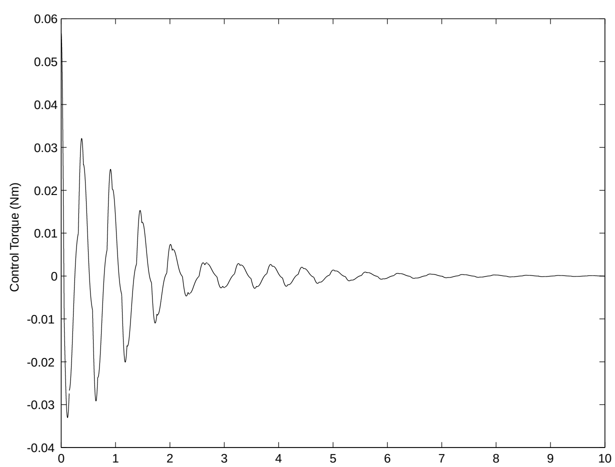

Figure 5.17: Control torque versus time for thermodynamic controller

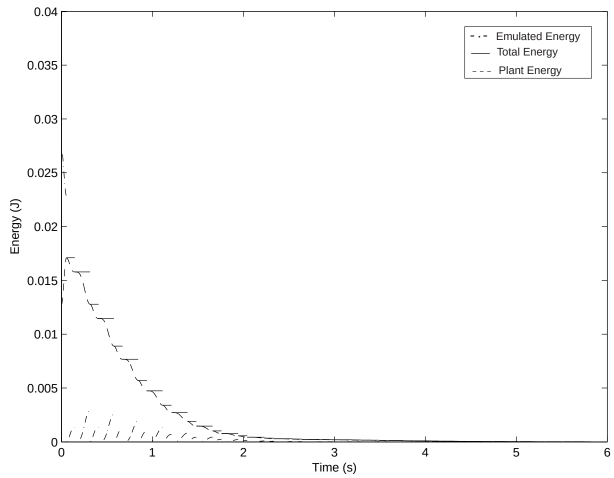

Figure 5.18: Plant, emulated, and total energy versus time for thermodynamic controller

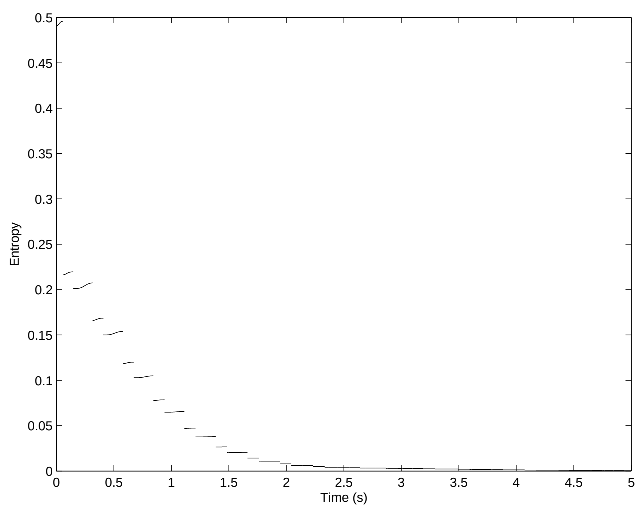

Figure 5.19: Closed-loop entropy versus time

of heat release in the combustor. This unsteady combustion provides an acoustic source resulting in self-excited oscillations. In particular, unsteady combustion generates acoustic pressure and velocity oscillations which in turn perturb the combustion even further [48,61]. These pressure oscillations, known as *thermoacoustic instabilities*, often lead to high vibration levels causing mechanical failures, high levels of acoustic noise, high burn rates, and even component melting. Hence, the need for active control to mitigate combustion induced pressure instabilities is crucial.

To design a hybrid controller for combustion systems we concentrate on a two-mode, nonlinear time-averaged combustion model with nonlinearities present due to the second-order gas dynamics. This model is developed in [62] and is given by

$$\dot{x}_1(t) = \alpha_1 x_1(t) + \theta_1 x_2(t) - \beta(x_1(t)x_3(t) + x_2(t)x_4(t)) + u_1(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{5.114}$$

$$\dot{x}_2(t) = -\theta_1 x_1(t) + \alpha_1 x_2(t) + \beta(x_2(t)x_3(t) - x_1(t)x_4(t)) + u_2(t), \quad x_2(0) = x_{20}, \tag{5.115}$$

$$\dot{x}_3(t) = \alpha_2 x_3(t) + \theta_2 x_4(t) + \beta(x_1^2(t) - x_2^2(t)) + u_3(t), \quad x_3(0) = x_{30}, \tag{5.116}$$

$$\dot{x}_4(t) = -\theta_2 x_3(t) + \alpha_2 x_4(t) + 2\beta x_1(t)x_2(t) + u_4(t), \quad x_4(0) = x_{40}, \tag{5.117}$$

where $x \triangleq [x_1, x_2, x_3, x_4]^{\mathrm{T}} \in \mathbb{R}^4$ is the plant state, $u \triangleq [u_1, u_2, u_3, u_4]^{\mathrm{T}} \in \mathbb{R}^4$ is the control input, $i = 1, \ldots, 4$, $\alpha_1, \alpha_2 \in \mathbb{R}$ represent growth/decay constants, $\theta_1, \theta_2 \in \mathbb{R}$ represent frequency shift constants, $\beta = ((\gamma + 1)/8\gamma)\omega_1$, where γ denotes the ratio of specific heats, ω_1 is frequency of the fundamental mode, and u_i, $i = 1, \ldots, 4$, are control input signals. For the data parameters $\alpha_1 = 5$, $\alpha_2 = -55$, $\theta_1 = 4$, $\theta_2 = 32$, $\gamma = 1.4$, $\omega_1 = 1$, and $x(0) = [1,\ 1,\ 1,\ 1]^{\mathrm{T}}$, the open-loop ($u_i(t) \equiv 0$, $i = 1, 2, 3, 4$) dynamics (5.114)–(5.117) result in a limit cycle instability. In addition, with the plant energy defined by $V_{\mathrm{p}}(x) \triangleq \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2)$, (5.114)–(5.117) is dissipative with respect to the supply rate $\hat{u}^{\mathrm{T}} y$, where $\hat{u} \triangleq [u_1 + \alpha_1 x_1, u_2 + \alpha_1 x_2, u_3, u_4]^{\mathrm{T}}$ and $y \triangleq x$.

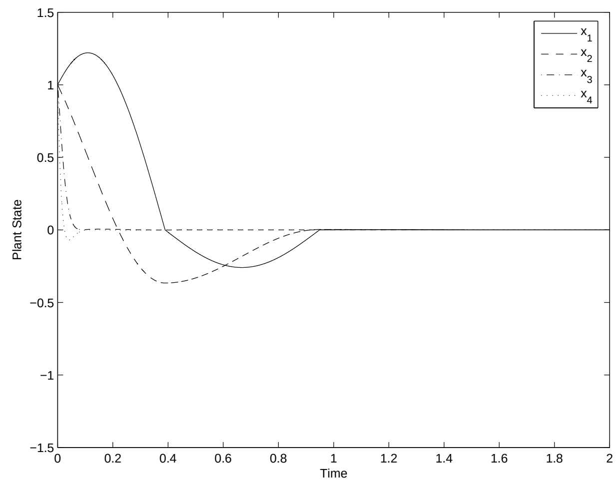

Figure 5.20: Plant state trajectories versus time

Next, consider the reduced-order dynamic compensator given by (5.15)–(5.17) with

$$f_{\mathrm{cc}}(x_{\mathrm{c}}, y) = A_{\mathrm{c}} x_{\mathrm{c}} + B_{\mathrm{c}} y, \quad \eta(y) = 0, \quad h_{\mathrm{cc}}(x_{\mathrm{c}}, y) = B_{\mathrm{c}}^{\mathrm{T}} x_{\mathrm{c}},$$

where $x_{\mathrm{c}} \triangleq [x_{\mathrm{c}1}, x_{\mathrm{c}2}]^{\mathrm{T}} \in \mathbb{R}^2$,

$$A_{\mathrm{c}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_{\mathrm{c}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}, \tag{5.118}$$

and controller energy given by $V_{\mathrm{c}}(x_{\mathrm{c}}) = \frac{1}{2} x_{\mathrm{c}}^{\mathrm{T}} x_{\mathrm{c}}$. Furthermore, the resetting set (5.23) is given by $\mathcal{Z} = \big\{(x, x_{\mathrm{c}}) : x_{\mathrm{c}}^{\mathrm{T}} B_{\mathrm{c}} x = 0,\ x_{\mathrm{c}} \neq 0\big\}$.

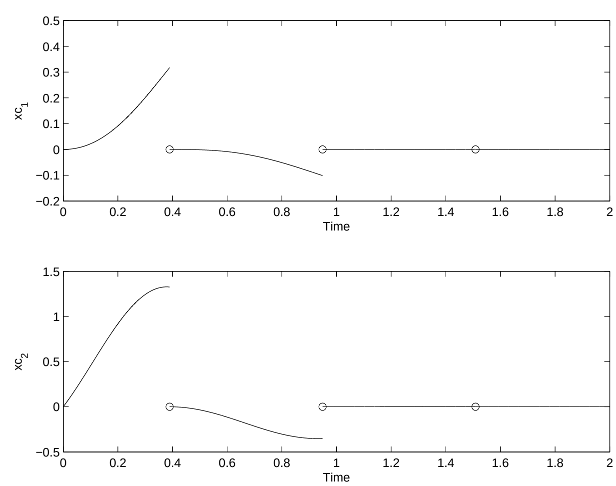

Figure 5.21: Compensator state trajectories versus time

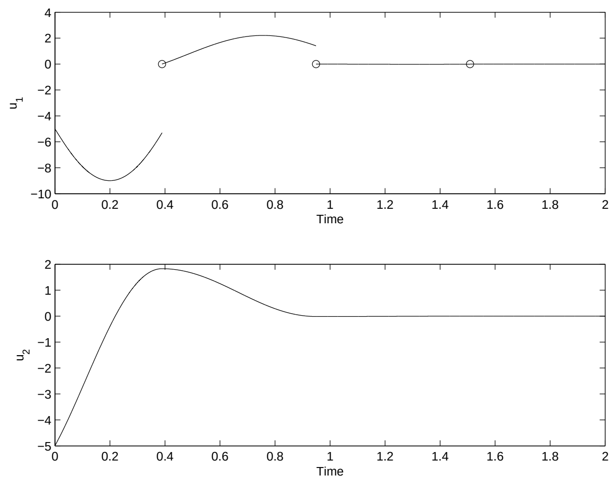

Figure 5.22: u_1 and u_2 versus time

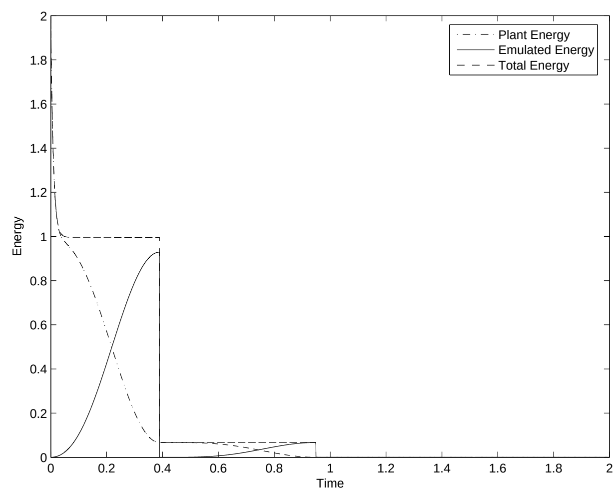

Figure 5.23: Plant, emulated, and total energy versus time

To illustrate the behavior of the closed-loop impulsive dynamical system, we choose the initial condition $x_c(0) = [0, 0]^T$. For this system a straightforward, but lengthy, calculation shows that Assumptions 1 and 2 hold. However, the k-transversality condition is sufficiently complex that we have been unable to show it analytically. This condition was verified numerically and Assumption 3 appears to hold. Figure 5.20 shows the state trajectories of the plant versus time, while Figure 5.21 shows the state trajectories of the compensator versus time. Figure 5.22 shows the control inputs u_1 and u_2 versus time. Note that the compensator states are the only states that reset. Furthermore, the control force versus time

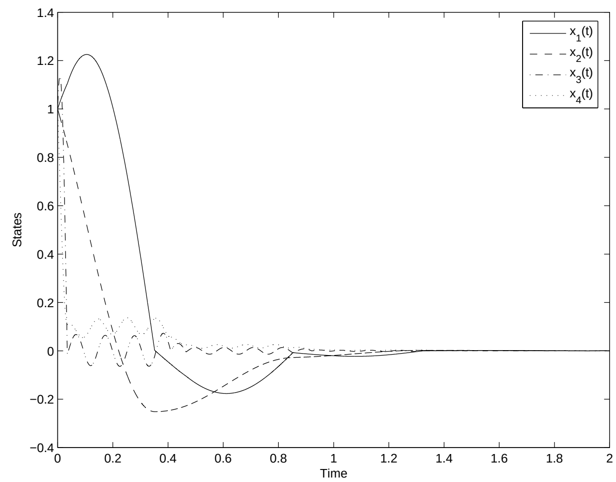

Figure 5.24: Plant state trajectories versus time

is discontinuous at the resetting times. A comparison of the plant energy, controller energy, and total energy is shown in Figure 5.23. Note that for the initial conditions chosen the proposed energy-based hybrid controller achieves finite-time stabilization.

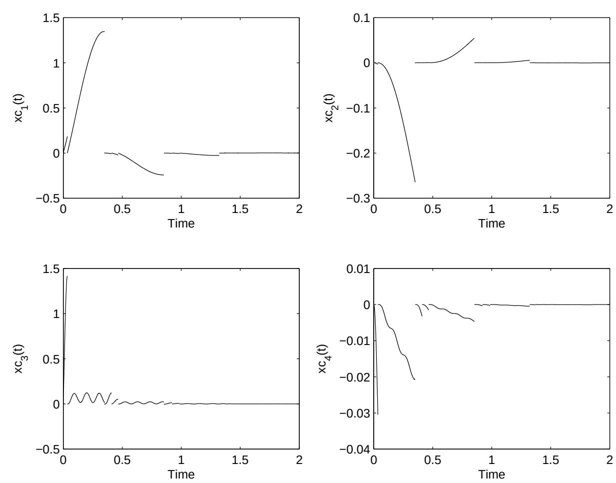

Figure 5.25: Compensator state trajectories versus time

Next, we consider the case where $\alpha_1 = 0$ and $\alpha_2 = 0$, that is, there is no decay or growth in the system. The other system parameters remain as before. In this case, the system is lossless with respect to the supply rate $u^{\mathrm{T}}y$. For this problem we consider an entropy-based hybrid dynamic compensator given by (5.15)–(5.17) with $f_{\mathrm{cc}}(x_{\mathrm{c}}, y) = A_{\mathrm{c}}x_{\mathrm{c}} + B_{\mathrm{c}}y$, $\eta(y) = 0$,

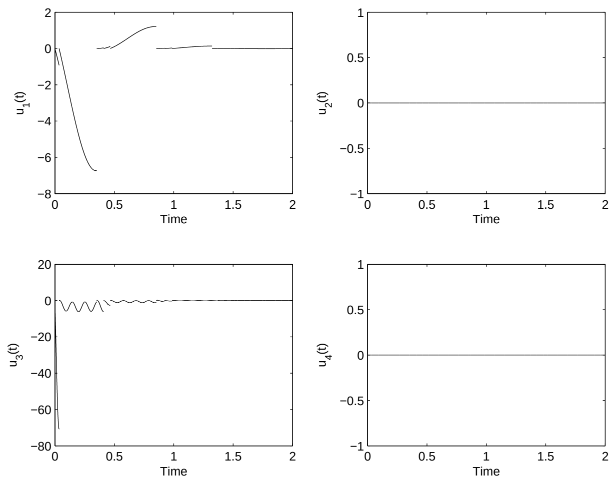

Figure 5.26: Control input versus time

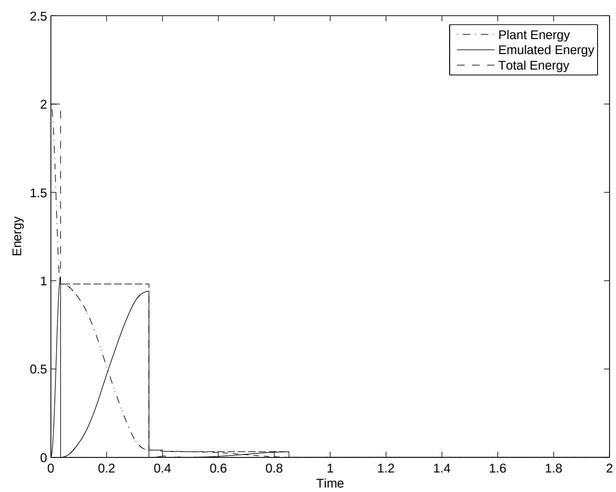

Figure 5.27: Plant, emulated, and total energy versus time

$h_{\mathrm{cc}}(x_{\mathrm{c}}, y) = B_{\mathrm{c}}^{\mathrm{T}} x_{\mathrm{c}}$, where $x_{\mathrm{c}} \triangleq [x_{\mathrm{c1}}, x_{\mathrm{c2}}, x_{\mathrm{c3}}, x_{\mathrm{c4}}]^{\mathrm{T}} \in \mathbb{R}^4$,

$$A_{\mathrm{c}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad B_{\mathrm{c}} = \begin{bmatrix} 0 & -30 & 0 & 0 \\ 30 & 0 & 0 & 0 \\ 0 & 0 & 60 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.119}$$

and controller energy given by $V_{\mathrm{c}}(x_{\mathrm{c}}) = \frac{1}{2} x_{\mathrm{c}}^{\mathrm{T}} x_{\mathrm{c}}$. Furthermore, the entropy function $S(E)$ is given by $S(E) = \log_e[1 + V_{\mathrm{p}}(x)] + \log_e[1 + V_{\mathrm{c}}(x_{\mathrm{c}})]$, and the resetting set (5.47) is given by $\mathcal{Z} = \big\{(x, x_{\mathrm{c}}) : x_{\mathrm{c}}^{\mathrm{T}} B_{\mathrm{c}} x [V_{\mathrm{c}}(x_{\mathrm{c}}) - V_{\mathrm{p}}(x)] = 0,\ x_{\mathrm{c}} \neq 0\big\}$.

To illustrate the behavior of the closed-loop impulsive dynamical system, we choose initial

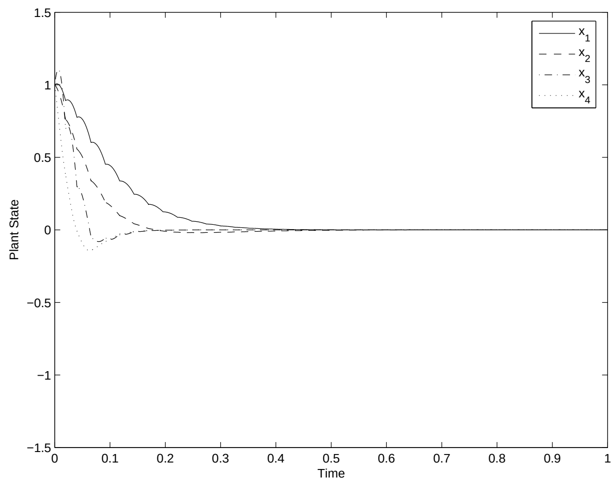

Figure 5.28: Plant state trajectories versus time for thermodynamic controller

condition $x_c(0) = [0, 0, 0, 0]^T$. Straightforward calculations show that Assumptions 1–3 hold. Figure 5.28 shows the state trajectories of the plant versus time, while Figure 5.29 shows the state trajectories of the compensator versus time. Figure 5.30 shows the control input versus time. Note that the compensator states are the only states that reset. Furthermore, the control force versus time is discontinuous at the resetting times. A comparison of the plant energy, controller energy, and total energy is shown in Figure 5.31. Finally, Figure 5.32 shows the closed-loop system entropy versus time. Note that the entropy of the closed-loop system strictly increases between resetting events.

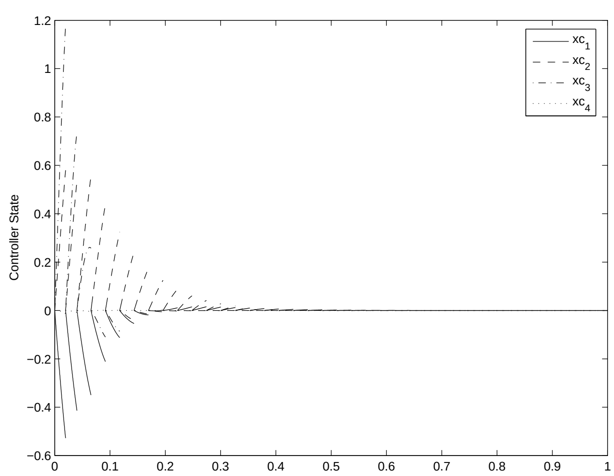

Figure 5.29: Compensator state trajectories versus time for thermodynamic controller

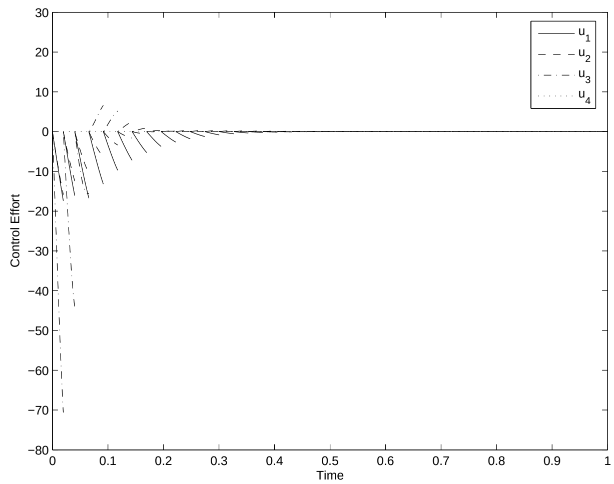

Figure 5.30: Control input versus time for thermodynamic controller

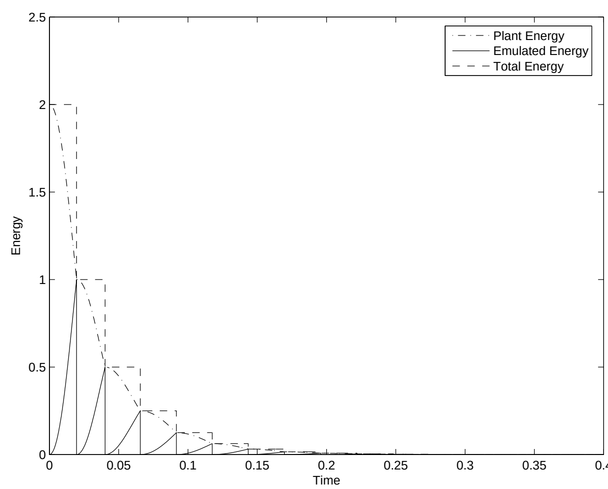

Figure 5.31: Plant, emulated, and total energy versus time for thermodynamic controller

5.7. Hybrid Control and Impulsive Dynamical Systems

In this section, we consider controlled impulsive dynamical systems of the form

$$\dot{x}_{\rm p}(t) = f_{\rm cp}(x_{\rm p}(t), u_{\rm c}(t)), \quad x_{\rm p}(0) = x_{\rm p0}, \quad (x_{\rm p}(t), u_{\rm c}(t)) \notin \mathcal{Z}_{\rm p}, \tag{5.120}$$

$$\Delta x_{\rm p}(t) = f_{\rm dp}(x_{\rm p}(t), u_{\rm d}(t)), \quad (x_{\rm p}(t), u_{\rm c}(t)) \in \mathcal{Z}_{\rm p}, \tag{5.121}$$

$$y(t) = h_{\rm p}(x_{\rm p}(t)), \tag{5.122}$$

where $t \geq 0$, $x_{\rm p}(t) \in \mathcal{D}_{\rm p} \subseteq \mathbb{R}^{n_{\rm p}}$, $\mathcal{D}_{\rm p}$ is an open set with $0 \in \mathcal{D}_{\rm p}$, $\Delta x_{\rm p}(t) \triangleq x_{\rm p}(t^+) - x_{\rm p}(t)$, $u_{\rm c}(t) \in \mathbb{R}^{m_{\rm c}}$, $u_{\rm d}(t) \in \mathbb{R}^{m_{\rm d}}$, $f_{\rm cp} : \mathcal{D}_{\rm p} \times \mathbb{R}^{m_{\rm c}} \to \mathbb{R}^{n_{\rm p}}$ is smooth (i.e., infinitely differentiable)

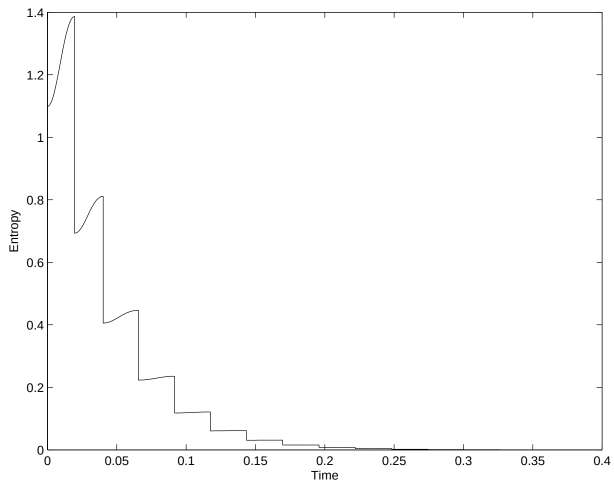

Figure 5.32: Closed-loop entropy versus time

on $\mathcal{D}_{\mathrm{p}}$ and satisfies $f_{\mathrm{cp}}(0,0) = 0$, $f_{\mathrm{dp}} : \mathcal{D}_{\mathrm{p}} \times \mathbb{R}^{m_{\mathrm{d}}} \to \mathbb{R}^{n_{\mathrm{p}}}$ is continuous, $h_{\mathrm{p}} : \mathcal{D}_{\mathrm{p}} \to \mathbb{R}^{l}$ is continuous and satisfies $h_{\mathrm{p}}(0) = 0$, and $\mathcal{Z}_{\mathrm{p}} \triangleq \mathcal{Z}_{x_{\mathrm{p}}} \times \mathcal{Z}_{u_{\mathrm{c}}} \subset \mathcal{D}_{\mathrm{p}} \times \mathbb{R}^{m_{\mathrm{c}}}$ is the *resetting set*. Furthermore, we consider hybrid (resetting) dynamic controllers of the form

$$\dot{x}_{\mathrm{c}}(t) = f_{\mathrm{cc}}(x_{\mathrm{c}}(t), y(t)), \quad x_{\mathrm{c}}(0) = x_{\mathrm{c}0}, \quad (x_{\mathrm{c}}(t), y(t)) \notin \mathcal{Z}_{\mathrm{c}}, \tag{5.123}$$

$$\Delta x_{\mathrm{c}}(t) = f_{\mathrm{dc}}(x_{\mathrm{c}}(t), y(t)), \quad (x_{\mathrm{c}}(t), y(t)) \in \mathcal{Z}_{\mathrm{c}}, \tag{5.124}$$

$$u_{\mathrm{c}}(t) = h_{\mathrm{cc}}(x_{\mathrm{c}}(t), y(t)), \tag{5.125}$$

$$u_{\mathrm{d}}(t) = h_{\mathrm{dc}}(x_{\mathrm{c}}(t), y(t)), \tag{5.126}$$

where $t \geq 0$, $x_{\mathrm{c}}(t) \in \mathcal{D}_{\mathrm{c}} \subseteq \mathbb{R}^{n_{\mathrm{c}}}$, $\mathcal{D}_{\mathrm{c}}$ is an open set with $0 \in \mathcal{D}_{\mathrm{c}}$, $\Delta x_{\mathrm{c}}(t) \triangleq x_{\mathrm{c}}(t^{+}) - x_{\mathrm{c}}(t)$, $f_{\mathrm{cc}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^{l} \to \mathbb{R}^{n_{\mathrm{c}}}$ is smooth on $\mathcal{D}_{\mathrm{c}}$ and satisfies $f_{\mathrm{cc}}(0,0) = 0$, $f_{\mathrm{dc}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^{l} \to \mathbb{R}^{n_{\mathrm{c}}}$ is continuous, $h_{\mathrm{cc}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^{l} \to \mathbb{R}^{m_{\mathrm{c}}}$ is continuous and satisfies $h_{\mathrm{cc}}(0,0) = 0$, $h_{\mathrm{dc}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^{l} \to \mathbb{R}^{m_{\mathrm{d}}}$ is continuous, and $\mathcal{Z}_{\mathrm{c}} \subset \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^{l}$ is the resetting set. Note that, for generality, we allow the hybrid dynamic controller to be of fixed dimension n_{c} which may be less than the plant order n_{p}.

The equations of motion for the closed-loop dynamical system (5.120)–(5.126) have the form

$$\dot{x}(t) = f_{\mathrm{c}}(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}, \tag{5.127}$$

$$\Delta x(t) = f_{\mathrm{d}}(x(t)), \quad x(t) \in \mathcal{Z}, \tag{5.128}$$

where

$$x \triangleq \begin{bmatrix} x_{\mathrm{p}} \\ x_{\mathrm{c}} \end{bmatrix} \in \mathbb{R}^n, \quad f_{\mathrm{c}}(x) \triangleq \begin{bmatrix} f_{\mathrm{cp}}(x_{\mathrm{p}}, h_{\mathrm{cc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}}))) \\ f_{\mathrm{cc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) \end{bmatrix}, \tag{5.129}$$

$$f_{\mathrm{d}}(x) \triangleq \begin{bmatrix} f_{\mathrm{dp}}(x_{\mathrm{p}}, h_{\mathrm{dc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})))\chi_{\mathcal{Z}_1}(x) \\ f_{\mathrm{dc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}}))\chi_{\mathcal{Z}_2}(x) \end{bmatrix}, \quad \chi_{\mathcal{Z}_i}(x) \triangleq \begin{cases} 1, & x \in \mathcal{Z}_i \\ 0, & x \notin \mathcal{Z}_i \end{cases}, \quad i = 1, 2, \tag{5.130}$$

and $\mathcal{Z} \triangleq \mathcal{Z}_1 \cup \mathcal{Z}_2$, $\mathcal{Z}_1 \triangleq \{x \in \mathcal{D} : (x_{\mathrm{p}}, h_{\mathrm{cc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}}))) \in \mathcal{Z}_{\mathrm{p}}\}$, $\mathcal{Z}_2 \triangleq \{x \in \mathcal{D} : (x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) \in \mathcal{Z}_{\mathrm{c}}\}$, with $n \triangleq n_{\mathrm{p}} + n_{\mathrm{c}}$ and $\mathcal{D} \triangleq \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}}$. We refer to the differential equation (5.127) as the *continuous-time dynamics*, and we refer to the difference equation (5.128) as the *resetting law*. A function $x : \mathcal{I}_{x_0} \to \mathcal{D}$ is a *solution* to the impulsive dynamical system (5.127) and (5.128) on the interval $\mathcal{I}_{x_0} \subseteq \mathbb{R}$ with initial condition $x(0) = x_0$ if $x(\cdot)$ is left-continuous and $x(t)$ satisfies (5.127) and (5.128) for all $t \in \mathcal{I}_{x_0}$. For further discussion on solutions to impulsive differential equations, see [14, 15, 41, 52, 98, 99, 147, 175, 215, 241]. For convenience, we use the notation $s(t, x_0)$ to denote the solution $x(t)$ of (5.127) and (5.128) at time $t \geq 0$ with initial condition $x(0) = x_0$.

For a particular closed-loop trajectory $x(t)$, we let $t_k \triangleq \tau_k(x_0)$ denote the kth instant of time at which $x(t)$ intersects $\mathcal{Z}$, and we call the times t_k the *resetting times*. Thus, the trajectory of the closed-loop system (5.127) and (5.128) from the initial condition $x(0) = x_0$ is given by $\psi(t, x_0)$ for $0 < t \leq t_1$, where $\psi(t, x_0)$ denotes the solution to the continuous-time dynamics (5.127). If and when the trajectory reaches a state $x_1 \triangleq x(t_1)$ satisfying $x_1 \in \mathcal{Z}$, then the state is instantaneously transferred to $x_1^+ \triangleq x_1 + f_{\mathrm{d}}(x_1)$ according to the resetting law (5.128). The trajectory $x(t)$, $t_1 < t \leq t_2$, is then given by $\psi(t - t_1, x_1^+)$, and so on. Note that the solution $x(t)$ of (5.127) and (5.128) is left continuous, that is, it is continuous everywhere except at the resetting times t_k, and

$$x_k \triangleq x(t_k) = \lim_{\varepsilon \to 0^+} x(t_k - \varepsilon), \tag{5.131}$$

$$x_k^+ \triangleq x(t_k) + f_{\mathrm{d}}(x(t_k)) = \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon), \tag{5.132}$$

for $k = 1, 2, \ldots$.

To ensure the well-posedness of the resetting times, we assume Assumptions 1 and 2 of Section 5.2 hold. It follows from Assumptions 1 and 2 that for a particular initial condition, the resetting times $t_k = \tau_k(x_0)$ are distinct and well defined [98]. Since the resetting set $\mathcal{Z}$ is a subset of the state space and is independent of time, impulsive dynamical systems of the form (5.127) and (5.128) are time-invariant systems. These systems are called *state-dependent impulsive dynamical systems* [98]. Since the resetting times are well defined and distinct, and since the solution to (5.127) exists and is unique, it follows that the solution of the impulsive dynamical system (5.127) and (5.128) also exists and is unique over a forward time interval. However, it is important to note that the analysis of impulsive dynamical systems can be quite involved. In particular, such systems can exhibit *Zenoness* and *beating*, as well as *confluence*, wherein solutions exhibit infinitely many resettings in a finite-time, encounter the same resetting surface a finite or infinite number of times in zero time, and coincide after a certain point in time [52, 98]. In this chapter we allow for the possibility of confluence and Zeno solutions, however, Assumption 2 precludes the possibility of beating. Furthermore, since *not* every bounded solution of an impulsive dynamical system over a forward time interval can be extended to infinity due to Zeno solutions, we assume that existence and uniqueness of solutions are satisfied in forward time. For details see [14, 15, 147, 215].

For the statement of the next result we assume Assumption 3 of Section 5.2 holds.

Proposition 5.3. Consider the impulsive dynamical system $\mathcal{G}$ given by (5.127) and (5.128). Assume that Assumptions 1 and 2 hold, $\tau_1(\cdot)$ is continuous at every $x \notin \overline{\mathcal{Z}}$ such that $0 < \tau_1(x) < \infty$, and if $x \in \mathcal{Z}$, then $x + f_\mathrm{d}(x) \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$. Furthermore, let $x_0 \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$ be such that $0 < \tau_1(x_0) < \infty$ and assume that the following statements hold:

i) If a sequence $\{x_i\}_{i=1}^{\infty} \in \mathcal{D}$ is such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} \tau_1(x_i)$ exists, then either both $f_\mathrm{d}(x_0) = 0$ and $\lim_{i\to\infty} \tau_1(x_i) = 0$, or $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$.

ii) If a sequence $\{x_i\}_{i=1}^{\infty} \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$ is such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} \tau_1(x_i)$ exists, then $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$.

Then $\mathcal{G}$ satisfies Assumption 3.

Proof. The proof is similar to the proof of Proposition 5.1 of Section 5.2 and, hence, is omitted. $\square$

The following result provides sufficient conditions for establishing continuity of $\tau_1(\cdot)$ at $x_0 \notin \overline{\mathcal{Z}}$ and *sequential continuity* of $\tau_1(\cdot)$ at $x_0 \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$, that is, $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$ for $\{x_i\}_{i=1}^{\infty} \notin \mathcal{Z}$ and $\lim_{i\to\infty} x_i = x_0$.

Definition 5.3. Let $\mathcal{M} \triangleq \{x \in \mathcal{D} : \ \mathcal{X}_{\rm p}(x) = 0\} \cup \{x \in \mathcal{D} : \ \mathcal{X}_{\rm c}(x) = 0\}$, where $\mathcal{X}_{\rm p} : \mathcal{D} \to \mathbb{R}$ and $\mathcal{X}_{\rm c} : \mathcal{D} \to \mathbb{R}$ are infinitely differentiable functions. A point $x \in \mathcal{M}$ such that $f_{\rm c}(x) \neq 0$ is *transversal* to (5.127) if there exist $k_{\rm p} \in \{1, 2, \ldots\}$ and $k_{\rm c} \in \{1, 2, \ldots\}$ such that

$$L_{f_{\rm c}}^r \mathcal{X}_{\rm p}(x) = 0, \quad r = 0, \ldots, 2k_{\rm p} - 2, \quad L_{f_{\rm c}}^{2k_{\rm p}-1} \mathcal{X}_{\rm p}(x) \neq 0, \tag{5.133}$$

$$L_{f_{\rm c}}^r \mathcal{X}_{\rm c}(x) = 0, \quad r = 0, \ldots, 2k_{\rm c} - 2, \quad L_{f_{\rm c}}^{2k_{\rm c}-1} \mathcal{X}_{\rm c}(x) \neq 0. \tag{5.134}$$

Proposition 5.4. Consider the impulsive dynamical system (5.127) and (5.128). Let $\mathcal{X}_{\rm p} : \mathcal{D} \to \mathbb{R}$ and $\mathcal{X}_{\rm c} : \mathcal{D} \to \mathbb{R}$ be infinitely differentiable functions such that $\overline{\mathcal{Z}} = \{x \in \mathcal{D} : \mathcal{X}_{\rm p}(x) = 0\} \cup \{x \in \mathcal{D} : \mathcal{X}_{\rm c}(x) = 0\}$, and assume every $x \in \overline{\mathcal{Z}}$ is transversal to (5.127). Then at every $x_0 \notin \overline{\mathcal{Z}}$ such that $0 < \tau_1(x_0) < \infty$, $\tau_1(\cdot)$ is continuous. Furthermore, if $x_0 \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$ is such that $\tau_1(x_0) \in (0, \infty)$ and $\{x_i\}_{i=1}^{\infty} \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$ or $\lim_{i\to\infty} \tau_1(x_i) > 0$, where $\{x_i\}_{i=1}^{\infty} \notin \overline{\mathcal{Z}}$ is such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} \tau_1(x_i)$ exists, then $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$.

Proof. Let $x_0 \notin \overline{\mathcal{Z}}$ be such that $0 < \tau_1(x_0) < \infty$. It follows from the definition of $\tau_1(\cdot)$ that $s(t, x_0) = \psi(t, x_0)$, $t \in [0, \tau_1(x_0)]$, $\mathcal{X}_{\rm p}(s(t, x_0))\mathcal{X}_{\rm c}(s(t, x_0)) \neq 0$, $t \in (0, \tau_1(x_0))$, and $\mathcal{X}_{\rm p}(s(\tau_1(x_0), x_0))\mathcal{X}_{\rm c}(s(\tau_1(x_0), x_0)) = 0$. Without loss of generality, let $\mathcal{X}_{\rm p}(s(t, x_0))\mathcal{X}_{\rm c}(s(t, x_0))$

> 0, $t \in (0, \tau_1(x_0))$. Since $\hat{x} \triangleq \psi(\tau_1(x_0), x_0) \in \overline{\mathcal{Z}}$ is transversal to (5.127), it follows that there exists $\theta > 0$ such that $\mathcal{X}_{\rm p}(\psi(t,\hat{x}))\mathcal{X}_{\rm c}(\psi(t,\hat{x})) > 0$, $t \in [-\theta, 0)$, and $\mathcal{X}_{\rm p}(\psi(t,\hat{x}))\mathcal{X}_{\rm c}(\psi(t,\hat{x})) < 0$, $t \in (0, \theta]$. (This fact can be easily shown by expanding $\mathcal{X}_{\rm p}(\psi(t,x))\mathcal{X}_{\rm c}(\psi(t,x))$ via a Taylor series expansion about $\hat{x}$ and using the fact that $\hat{x}$ is tranversal to (5.127).) Hence, $\mathcal{X}_{\rm p}(\psi(t,x_0))\mathcal{X}_{\rm c}(\psi(t,x_0)) > 0$, $t \in [\hat{t}_1, \tau_1(x_0))$, and $\mathcal{X}_{\rm p}(\psi(t,x_0))\mathcal{X}_{\rm c}(\psi(t,x_0)) < 0$, $t \in (\tau_1(x_0), \hat{t}_2]$, where $\hat{t}_1 \triangleq \tau_1(x_0) - \theta$ and $\hat{t}_2 \triangleq \tau_1(x_0) + \theta$.

Next, let $\varepsilon \triangleq \min\{|\mathcal{X}_{\rm p}(\psi(\hat{t}_1,x_0))\mathcal{X}_{\rm c}(\psi(\hat{t}_1,x_0))|, |\mathcal{X}_{\rm p}(\psi(\hat{t}_2,x_0))\mathcal{X}_{\rm c}(\psi(\hat{t}_2,x_0))|\}$. Now, it follows from the continuity of $\mathcal{X}_{\rm p}(\cdot)\mathcal{X}_{\rm c}(\cdot)$ and the continuous dependence of $\psi(\cdot,\cdot)$ on the system initial conditions that there exists $\delta > 0$ such that

$$\sup_{0 \le t \le \hat{t}_2} |\mathcal{X}_{\rm p}(\psi(t,x))\mathcal{X}_{\rm c}(\psi(t,x)) - \mathcal{X}_{\rm p}(\psi(t,x_0))\mathcal{X}_{\rm c}(\psi(t,x_0))| < \varepsilon, \quad x \in \mathcal{B}_\delta(x_0), \quad (5.135)$$

which implies that $\mathcal{X}_{\rm p}(\psi(\hat{t}_1,x))\mathcal{X}_{\rm c}(\psi(\hat{t}_1,x)) > 0$ and $\mathcal{X}_{\rm p}(\psi(\hat{t}_2,x))\mathcal{X}_{\rm c}(\psi(\hat{t}_2,x)) < 0$, $x \in \mathcal{B}_\delta(x_0)$. Hence, it follows that $\hat{t}_1 < \tau_1(x) < \hat{t}_2$, $x \in \mathcal{B}_\delta(x_0)$. The continuity of $\tau_1(\cdot)$ at x_0 now follows immediately by noting that θ can be chosen arbitrarily small. Finally, let $x_0 \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$ be such that $\lim_{i\to\infty} x_i = x_0$ for some sequence $\{x_i\}_{i=1}^\infty \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$. Then using similar arguments as above it can be shown that $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$. Alternatively, if $x_0 \in \overline{\mathcal{Z}}\backslash\mathcal{Z}$ is such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} \tau_1(x_i) > 0$ for some sequence $\{x_i\}_{i=1}^\infty \notin \mathcal{Z}$, then it follows that there exists sufficiently small $\hat{t} > 0$ and $I \in \mathbb{Z}_+$ such that $s(\hat{t}, x_i) = \psi(\hat{t}, x_i)$, $i = I, I+1, \ldots$, which implies that $\lim_{i\to\infty} s(\hat{t}, x_i) = s(\hat{t}, x_0)$. Next, define $y_i \triangleq \psi(\hat{t}, x_i)$, $i = 0, 1, \ldots$, so that $\lim_{i\to\infty} y_i = y_0$ and note that it follows from the transversality assumption that $y_0 \notin \overline{\mathcal{Z}}$, which implies that $\tau_1(\cdot)$ is continuous at y_0. Hence, $\lim_{i\to\infty} \tau_1(y_i) = \tau_1(y_0)$. The result now follows by noting that $\tau_1(x_i) = \hat{t} + \tau_1(y_i)$, $i = 1, 2, \ldots$. □

Remark 5.8. Let $x_0 \notin \mathcal{Z}$ be such that $\lim_{i\to\infty} \tau_1(x_i) \neq \tau_1(x_0)$ for some sequence $\{x_i\}_{i=1}^\infty \notin \mathcal{Z}$. Then it follows from Proposition 5.4 that $\lim_{i\to\infty} \tau_1(x_i) = 0$.

5.8. Hybrid Control Design for Lossless Impulsive Dynamical Systems

In this section, we present a hybrid controller design framework for lossless impulsive dynamical systems [98]. Specifically, we consider impulsive dynamical systems $\mathcal{G}_\mathrm{p}$ of the form given by (5.120)–(5.122) where $u(\cdot)$ satisfies sufficient regularity conditions such that (5.120) has a unique solution between the resetting times. Furthermore, we consider hybrid resetting dynamic controllers $\mathcal{G}_\mathrm{c}$ of the form

$$\dot{x}_\mathrm{c}(t) = f_\mathrm{cc}(x_\mathrm{c}(t), y(t)), \quad x_\mathrm{c}(0) = x_{\mathrm{c}0}, \quad (x_\mathrm{c}(t), y(t)) \notin \mathcal{Z}_\mathrm{c}, \tag{5.136}$$

$$\Delta x_\mathrm{c}(t) = \eta(y(t)) - x_\mathrm{c}(t), \quad (x_\mathrm{c}(t), y(t)) \in \mathcal{Z}_\mathrm{c}, \tag{5.137}$$

$$y_\mathrm{cc}(t) = h_\mathrm{cc}(x_\mathrm{c}(t), u_\mathrm{cc}(t)), \tag{5.138}$$

$$y_\mathrm{dc}(t) = h_\mathrm{dc}(x_\mathrm{c}(t), y(t)), \tag{5.139}$$

where $x_\mathrm{c}(t) \in \mathcal{D}_\mathrm{c} \subseteq \mathbb{R}^{n_\mathrm{c}}$, $\mathcal{D}_\mathrm{c}$ is an open set with $0 \in \mathcal{D}_\mathrm{c}$, $y(t) \in \mathbb{R}^l$, $y_\mathrm{cc}(t) \in \mathbb{R}^{m_\mathrm{c}}$, $y_\mathrm{dc}(t) \in \mathbb{R}^{m_\mathrm{d}}$, $f_\mathrm{cc} : \mathcal{D}_\mathrm{c} \times \mathbb{R}^l \to \mathbb{R}^{n_\mathrm{c}}$ is smooth on $\mathcal{D}_\mathrm{c}$ and satisfies $f_\mathrm{cc}(0,0) = 0$, $\eta : \mathbb{R}^l \to \mathcal{D}_\mathrm{c}$ is continuous and satisfies $\eta(0) = 0$, $h_\mathrm{cc} : \mathcal{D}_\mathrm{c} \times \mathbb{R}^l \to \mathbb{R}^{m_\mathrm{c}}$ is continuous and satisfies $h_\mathrm{cc}(0,0) = 0$, and $h_\mathrm{dc} : \mathcal{D}_\mathrm{c} \times \mathbb{R}^l \to \mathbb{R}^{m_\mathrm{d}}$ is continuous.

Recall that for the impulsive dynamical system $\mathcal{G}_\mathrm{p}$ given by (5.120)–(5.122), a function $(s_\mathrm{c}(u_\mathrm{c}, y), s_\mathrm{d}(u_\mathrm{d}, y))$, where $s_\mathrm{c} : \mathbb{R}^{m_\mathrm{c}} \times \mathbb{R}^l \to \mathbb{R}$ and $s_\mathrm{d} : \mathbb{R}^{m_\mathrm{d}} \times \mathbb{R}^l \to \mathbb{R}$ are such that $s_\mathrm{c}(0,0) = 0$ and $s_\mathrm{d}(0,0) = 0$, is called a *hybrid supply rate* [98] if it is locally integrable for all input-output pairs satisfying (5.120)–(5.122), that is, for all input-output pairs $u_\mathrm{c} \in \mathcal{U}_\mathrm{c}$ and $y \in \mathcal{Y}$ satisfying (5.120) and (5.122), $s_\mathrm{c}(\cdot,\cdot)$ satisfies $\int_t^{\hat{t}} |s_\mathrm{c}(u_\mathrm{c}(\sigma), y(\sigma))| \mathrm{d}\sigma < \infty$, $t, \hat{t} \geq 0$. Here, $\mathcal{U}_\mathrm{c}$ and $\mathcal{Y}$ are input and output spaces, respectively, that are assumed to be closed under the shift operator. Note that since all input-output pairs $u_\mathrm{d}(t_k) \in \mathcal{U}_\mathrm{d}$ and $y(t_k) \in \mathcal{Y}$ satisfying (5.121) and (5.122) are defined for discrete instants, $s_\mathrm{d}(\cdot,\cdot)$ satisfies $\sum_{k \in \mathbb{Z}_{[t,\hat{t})}} |s_\mathrm{d}(u_\mathrm{d}(t_k), y(t_k))| < \infty$, where $\mathcal{U}_\mathrm{d}$ is an input space and $\mathbb{Z}_{[t,\hat{t})} \triangleq \{k : t \leq t_k < \hat{t}\}$. Furthermore, we assume that $\mathcal{G}_\mathrm{p}$ is *lossless with respect to the hybrid supply rate* $(s_\mathrm{c}(u_\mathrm{c}, y), s_\mathrm{d}(u_\mathrm{d}, y))$, and hence, there exists a continuous, nonnegative-definite *storage function* $V_{\rm s} : \mathcal{D}_{\rm p} \to \overline{\mathbb{R}}_+$ such that $V_{\rm s}(0) = 0$ and

$$V_{\rm s}(x_{\rm p}(t)) = V_{\rm s}(x_{\rm p}(t_0)) + \int_{t_0}^{t} s_{\rm c}(u_{\rm c}(\sigma), y(\sigma)){\rm d}\sigma + \sum_{k\in\mathbb{Z}_{[t,t_0)}} s_{\rm d}(u_{\rm d}(t_k), y(t_k)), \quad t \ge t_0, \tag{5.140}$$

for all $t_0, t \ge 0$, where $x_{\rm p}(t)$, $t \ge t_0$, is the solution to (5.120) and (5.121) with $(u_{\rm c}, u_{\rm d}) \in \mathcal{U}_{\rm c} \times \mathcal{U}_{\rm d}$. Equivalently, over the interval $t \in (t_k, t_{k+1}]$, (5.140) can be written as ([98])

$$V_{\rm s}(x_{\rm p}(\hat{t})) - V_{\rm s}(x_{\rm p}(t)) = \int_{t}^{\hat{t}} s_{\rm c}(u_{\rm c}(\sigma), y(\sigma)){\rm d}\sigma, \quad t_k < t \le \hat{t} \le t_{k+1}, \quad k \in \overline{\mathbb{Z}}_+, \tag{5.141}$$

$$V_{\rm s}(x_{\rm p}(t_k) + f_{\rm dp}(x_{\rm p}(t_k), u_{\rm d}(t_k))) - V_{\rm s}(x_{\rm p}(t_k)) = s_{\rm d}(u_{\rm d}(t_k), y(t_k)). \tag{5.142}$$

In addition, we assume that the nonlinear impulsive dynamical system $\mathcal{G}_{\rm p}$ is *completely reachable* [98] and *zero-state observable* [98], and there exist functions $\kappa_{\rm c} : \mathbb{R}^l \to \mathbb{R}^{m_{\rm c}}$ and $\kappa_{\rm d} : \mathbb{R}^l \to \mathbb{R}^{m_{\rm d}}$ such that $\kappa_{\rm c}(0) = 0$, $\kappa_{\rm d}(0) = 0$, $s_{\rm c}(\kappa_{\rm c}(y), y) < 0$, $y \ne 0$, and $s_{\rm d}(\kappa_{\rm d}(y), y) < 0$, $y \ne 0$, so that all storage functions $V_{\rm s}(x_{\rm p})$, $x_{\rm p} \in \mathcal{D}_{\rm p}$, of $\mathcal{G}_{\rm p}$ are positive definite [98]. Finally, we assume that $V_{\rm s}(\cdot)$ is continuously differentiable.

Next, consider the negative feedback interconnection of $\mathcal{G}_{\rm p}$ and $\mathcal{G}_{\rm c}$ given by $y = u_{\rm cc}$ and $(u_{\rm c}, u_{\rm d}) = (-y_{\rm cc}, -y_{\rm dc})$. In this case, the closed-loop system $\mathcal{G}$ is given by

$$\dot{x}(t) = f_{\rm c}(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}, \quad t \ge 0, \tag{5.143}$$

$$\Delta x(t) = f_{\rm d}(x(t)), \quad x(t) \in \mathcal{Z}, \tag{5.144}$$

where $t \ge 0$, $x(t) \triangleq [x_{\rm p}^{\rm T}(t), x_{\rm c}^{\rm T}(t)]^{\rm T}$, $\mathcal{Z} \triangleq \mathcal{Z}_1 \cup \mathcal{Z}_2$, $\mathcal{Z}_1 \triangleq \{x \in \mathcal{D} : (x_{\rm p}, -h_{\rm cc}(x_{\rm c}, h_{\rm p}(x_{\rm p}))) \in \mathcal{Z}_{\rm p}\}$, $\mathcal{Z}_2 \triangleq \{x \in \mathcal{D} : (x_{\rm c}, h_{\rm p}(x_{\rm p})) \in \mathcal{Z}_{\rm c}\}$,

$$f_{\rm c}(x) \triangleq \begin{bmatrix} f_{\rm cp}(x_{\rm p}, -h_{\rm cc}(x_{\rm c}, h_{\rm p}(x_{\rm p}))) \\ f_{\rm cc}(x_{\rm c}, h_{\rm p}(x_{\rm p})) \end{bmatrix}, \quad f_{\rm d}(x) \triangleq \begin{bmatrix} f_{\rm dp}(x_{\rm p}, -h_{\rm dc}(x_{\rm c}, h_{\rm p}(x_{\rm p})))\chi_{\mathcal{Z}_1}(x) \\ (\eta(h_{\rm p}(x_{\rm p})) - x_{\rm c})\chi_{\mathcal{Z}_2}(x) \end{bmatrix}. \tag{5.145}$$

Assume that there exists an infinitely differentiable function $V_{\rm c} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \overline{\mathbb{R}}_+$ such that $V_{\rm c}(x_{\rm c}, y) \ge 0$, $x_{\rm c} \in \mathcal{D}_{\rm c}$, $y \in \mathbb{R}^l$, $V_{\rm c}(x_{\rm c}, y) = 0$ if and only if $x_{\rm c} = \eta(y)$, and

$$\dot{V}_{\rm c}(x_{\rm c}(t), y(t)) = s_{\rm cc}(u_{\rm cc}(t), y_{\rm cc}(t)), \quad (x_{\rm c}(t), y(t)) \notin \mathcal{Z}_{\rm c}, \quad t \ge 0, \tag{5.146}$$

where $s_{\rm cc} : \mathbb{R}^l \times \mathbb{R}^{m_{\rm c}} \to \mathbb{R}$ is such that $s_{\rm cc}(0, 0) = 0$.

We associate with the plant a positive-definite, continuously differentiable function $V_{\rm p}(x_{\rm p}) \triangleq V_{\rm s}(x_{\rm p})$, which we will refer to as the *plant energy*. Furthermore, we associate with the controller a nonnegative-definite, infinitely differentiable function $V_{\rm c}(x_{\rm c}, y)$ called the controller *emulated energy*. Finally, we associate with the closed-loop system the function

$$V(x) \triangleq V_{\rm p}(x_{\rm p}) + V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p})), \tag{5.147}$$

called the *total energy*.

Next, we construct the resetting set for $\mathcal{G}_{\rm c}$ in the following form

$$\begin{aligned} \mathcal{Z}_2 &= \{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c} : L_{f_{\rm c}} V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p})) = 0 \text{ and } V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p})) > 0\} \\ &= \{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c} : s_{\rm cc}(h_{\rm p}(x_{\rm p}), h_{\rm cc}(x_{\rm c}, h_{\rm p}(x_{\rm p}))) = 0 \text{ and } V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p})) > 0\}. \end{aligned} \tag{5.148}$$

The resetting set $\mathcal{Z}_2$ is thus defined to be the set of all points in the closed-loop state space that correspond to decreasing controller emulated energy. By resetting the controller states, the plant energy can never increase after the first resetting event. Furthermore, if the closed-loop system total energy is conserved between resetting events, then a decrease in plant energy is accompanied by a corresponding increase in emulated energy. Hence, this approach allows the plant energy to flow to the controller, where it increases the emulated energy but does not allow the emulated energy to flow back to the plant after the first resetting event. This energy dissipating hybrid controller effectively enforces a one-way energy transfer between the plant and the controller after the first resetting event. The next theorem gives sufficient conditions for asymptotic stability of the closed-loop system $\mathcal{G}$ using state-dependent hybrid controllers. For practical implementation, knowledge of $x_{\rm c}$ and y is sufficient to determine whether or not the closed-loop state vector is in the set $\mathcal{Z}_2$.

Theorem 5.7. Consider the closed-loop impulsive dynamical system $\mathcal{G}$ given by (5.143) and (5.144) with the resetting set $\mathcal{Z}_2$ given by (5.148). Assume that $\mathcal{D}_{\rm ci} \subset \mathcal{D}$ is a compact positively invariant set with respect to $\mathcal{G}$ such that $0 \in \overset{\circ}{\mathcal{D}}_{\mathrm{ci}}$, assume that if $x_0 \in \mathcal{Z}_1$ then $x_0 + f_{\mathrm{d}}(x_0) \in \overline{\mathcal{Z}}_1 \backslash \mathcal{Z}_1$, and if $x_0 \in \overline{\mathcal{Z}}_1 \backslash \mathcal{Z}_1$, then $f_{\mathrm{dp}}(x_{\mathrm{p}0}, -h_{\mathrm{dc}}(x_{\mathrm{c}0}, h_{\mathrm{p}}(x_{\mathrm{p}0}))) = 0$, where $\overline{\mathcal{Z}}_1 = \{x \in \mathcal{D} : \mathcal{X}_{\mathrm{p}}(x) = 0\}$ with an infinitely differentiable function $\mathcal{X}_{\mathrm{p}}(\cdot)$, and assume that $\mathcal{G}_{\mathrm{p}}$ is lossless with respect to the hybrid supply rate $(s_{\mathrm{c}}(u_{\mathrm{c}}, y), s_{\mathrm{d}}(u_{\mathrm{d}}, y))$ and with a positive-definite, continuously differentiable storage function $V_{\mathrm{p}}(x_{\mathrm{p}})$, $x_{\mathrm{p}} \in \mathcal{D}_{\mathrm{p}}$. In addition, assume there exists a smooth (i.e., infinitely differentiable) function $V_{\mathrm{c}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l \to \overline{\mathbb{R}}_+$ such that $V_{\mathrm{c}}(x_{\mathrm{c}}, y) \geq 0$, $x_{\mathrm{c}} \in \mathcal{D}_{\mathrm{c}}$, $y \in \mathbb{R}^l$, $V_{\mathrm{c}}(x_{\mathrm{c}}, y) = 0$ if and only if $x_{\mathrm{c}} = \eta(y)$, and (5.146) holds. Furthermore, assume that every $x_0 \in \overline{\mathcal{Z}}$ is transversal to (5.143) with $\mathcal{X}_{\mathrm{c}}(x) = \frac{\mathrm{d}}{\mathrm{d}t} V_{\mathrm{c}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}}))$, and

$$s_{\mathrm{c}}(u_{\mathrm{c}}, y) + s_{\mathrm{cc}}(u_{\mathrm{cc}}, y_{\mathrm{cc}}) = 0, \quad x \notin \mathcal{Z}, \tag{5.149}$$

$$s_{\mathrm{d}}(u_{\mathrm{d}}, y) < 0, \quad x \in \mathcal{Z}_1, \tag{5.150}$$

where $y = u_{\mathrm{cc}} = h_{\mathrm{p}}(x_{\mathrm{p}})$, $u_{\mathrm{c}} = -y_{\mathrm{cc}} = -h_{\mathrm{cc}}(x_{\mathrm{c}}, h_{\mathrm{c}}(x_{\mathrm{p}}))$, and $u_{\mathrm{d}} = -y_{\mathrm{dc}} = -h_{\mathrm{dc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}}))$. Then the zero solution $x(t) \equiv 0$ to the closed-loop system $\mathcal{G}$ is asymptotically stable. In addition, the total energy function $V(x)$ of $\mathcal{G}$ given by (5.147) is strictly decreasing across resetting events. Finally, if $\mathcal{D}_{\mathrm{p}} = \mathbb{R}^{n_{\mathrm{p}}}$, $\mathcal{D}_{\mathrm{c}} = \mathbb{R}^{n_{\mathrm{c}}}$, and $V(\cdot)$ is radially unbounded, then the zero solution $x(t) \equiv 0$ to $\mathcal{G}$ is globally asymptotically stable.

Proof. First, note that since $V_{\mathrm{c}}(x_{\mathrm{c}}, y) \geq 0$, $x_{\mathrm{c}} \in \mathcal{D}_{\mathrm{c}}$, $y \in \mathbb{R}^l$, it follows that

$$\begin{aligned}\overline{\mathcal{Z}} &= \overline{\mathcal{Z}}_1 \cup \{(x_{\mathrm{p}}, x_{\mathrm{c}}) \in \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}} : L_{f_{\mathrm{c}}} V_{\mathrm{c}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) = 0 \text{ and } V_{\mathrm{c}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) \geq 0\} \\ &= \overline{\mathcal{Z}}_1 \cup \{(x_{\mathrm{p}}, x_{\mathrm{c}}) \in \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}} : \mathcal{X}_{\mathrm{c}}(x) = 0\},\end{aligned} \tag{5.151}$$

where $\mathcal{X}_{\mathrm{c}}(x) = L_{f_{\mathrm{c}}} V_{\mathrm{c}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}}))$. Next, we show that if the transversality condition (5.133) holds, then Assumptions 1–3 hold and, for every $x_0 \in \mathcal{D}_{\mathrm{ci}}$, there exists $\tau \geq 0$ such that $x(\tau) \in \mathcal{Z}$. Note that if $x_0 \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$, that is, $\mathcal{X}_{\mathrm{p}}(x(0)) = 0$, or $V_{\mathrm{c}}(x_{\mathrm{c}}(0), h_{\mathrm{p}}(x(0))) = 0$ and $L_{f_{\mathrm{c}}} V_{\mathrm{c}}(x_{\mathrm{c}}(0), h_{\mathrm{p}}(x_{\mathrm{p}}(0))) = 0$, it follows from the transversality condition that there exists $\delta > 0$ such that for all $t \in (0, \delta]$, $\mathcal{X}_{\mathrm{p}}(x(t)) \neq 0$ and $L_{f_{\mathrm{c}}} V_{\mathrm{c}}(x_{\mathrm{c}}(t), h_{\mathrm{p}}(x_{\mathrm{p}}(t))) \neq 0$. Hence, since $V_{\mathrm{c}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) = V_{\mathrm{c}}(x_{\mathrm{c}}(0), h_{\mathrm{p}}(x_{\mathrm{p}}(0))) + t L_{f_{\mathrm{c}}} V_{\mathrm{c}}(x_{\mathrm{c}}(\tau), h_{\mathrm{p}}(x_{\mathrm{p}}(\tau)))$ for some $\tau \in (0, t]$ and

$V_{\rm c}(x_{\rm c},y)\geq 0$, $x_{\rm c}\in\mathcal{D}_{\rm c}$, $y\in\mathbb{R}^l$, it follows that $V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))>0$, $t\in(0,\delta]$, which implies that Assumption 1 is satisfied. Furthermore, if $x\in\mathcal{Z}$ then, since $V_{\rm c}(x_{\rm c},y)=0$ if and only if $x_{\rm c}=\eta(y)$, it follows from (5.144) that $x+f_{\rm d}(x)\in\overline{\mathcal{Z}}_2\backslash\mathcal{Z}_2$, and hence, $x+f_{\rm d}(x)\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$. Hence, Assumption 2 holds.

Next, consider the set $\mathcal{M}_\gamma\triangleq\{x\in\mathcal{D}_{\rm ci}:V_{\rm c}(x_{\rm c},h_{\rm p}(x_{\rm p}))=\gamma\}$, where $\gamma\geq 0$. It follows from the transversality condition that for every $\gamma\geq 0$, $\mathcal{M}_\gamma$ does not contain any nontrivial trajectory of $\mathcal{G}$. To see this, suppose, *ad absurdum*, there exists a nontrivial trajectory $x(t)\in\mathcal{M}_\gamma$, $t\geq 0$, for some $\gamma\geq 0$. In this case, it follows that $\frac{{\rm d}^k}{{\rm d}t^k}V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))=L_{f_{\rm c}}^kV_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))\equiv 0$, $k=1,2,\ldots$, which contradicts the transversality condition.

Next, we show that for every $x_0\notin\mathcal{Z}$, $x_0\neq 0$, there exists $\tau>0$ such that $x(\tau)\in\mathcal{Z}$. To see this, suppose, *ad absurdum*, $x(t)\notin\mathcal{Z}$, $t\geq 0$, which implies that

$$\frac{\rm d}{{\rm d}t}V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))\neq 0,\quad t\geq 0, \tag{5.152}$$

or

$$V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))=0,\quad t\geq 0. \tag{5.153}$$

If (5.152) holds, then it follows that $V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))$ is a (decreasing or increasing) monotonic function of time. Hence, $V_{\rm c}(x_{\rm c}(t),h_{\rm p}(x_{\rm p}(t)))\to\gamma$ as $t\to\infty$, where $\gamma\geq 0$ is a constant, which implies that the positive limit set of the closed-loop system is contained in $\mathcal{M}_\gamma$ for some $\gamma\geq 0$, and hence, is a contradiction. Similarly, if (5.153) holds then $\mathcal{M}_0$ contains a nontrivial trajectory of $\mathcal{G}$ also leading to a contradiction. Hence, for every $x_0\notin\mathcal{Z}$, there exists $\tau>0$ such that $x(\tau)\in\mathcal{Z}$. Thus, it follows that for every $x_0\notin\mathcal{Z}$, $0<\tau_1(x_0)<\infty$. Now, it follows from Proposition 5.4 that $\tau_1(\cdot)$ is continuous at $x_0\notin\overline{\mathcal{Z}}$. Furthermore, for all $x_0\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$ and for every sequence $\{x_i\}_{i=1}^\infty\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$ converging to $x_0\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$, it follows from the transversality condition and Proposition 5.4 that $\lim_{i\to\infty}\tau_1(x_i)=\tau_1(x_0)$. Next, let $x_0\in\overline{\mathcal{Z}}\backslash\mathcal{Z}$ and let $\{x_i\}_{i=1}^\infty\in\mathcal{D}_{\rm ci}$ be such that $\lim_{i\to\infty}x_i=x_0$ and $\lim_{i\to\infty}\tau_1(x_i)$ exists. In this case, it follows from Proposition 5.4 that either $\lim_{i\to\infty}\tau_1(x_i)=0$ or $\lim_{i\to\infty}\tau_1(x_i)=\tau_1(x_0)$.

Furthermore, since $x_0 \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$ corresponds to the case where $f_{\rm dp}(x_{\rm p0}, -h_{\rm dc}(x_{\rm c0}, h_{\rm p}(x_{\rm p0}))) = 0$ or $V_{\rm c}(x_{\rm c0}, h_{\rm p}(x_{\rm p0})) = 0$, if $V_{\rm c}(x_{\rm c0}, h_{\rm p}(x_{\rm p0})) = 0$, then it follows that $x_{\rm c0} = \eta(h_{\rm p}(x_{\rm p0}))$, and hence, $f_{\rm d}(x_0) = 0$. Now, it follows from Proposition 5.3 that Assumption 3 holds.

To show that the zero solution $x(t) \equiv 0$ to $\mathcal{G}$ is asymptotically stable, consider the Lyapunov function candidate corresponding to the total energy function $V(x)$ given by (5.147). Since $\mathcal{G}_{\rm p}$ is lossless with respect to the hybrid supply rate $(s_{\rm c}(u_{\rm c}, y),\ s_{\rm d}(u_{\rm d}, y))$ and (5.146) and (5.149) hold, it follows that

$$\dot{V}(x(t)) = s_{\rm c}(u_{\rm c}(t), y(t)) + s_{\rm cc}(u_{\rm cc}(t), y_{\rm cc}(t)) = 0, \quad x(t) \notin \mathcal{Z}. \tag{5.154}$$

Furthermore, it follows from (5.142), (5.145), and (5.148) that

$$\begin{aligned}
\Delta V(x(t_k)) = {} & V_{\rm p}(x_{\rm p}(t_k^+)) - V_{\rm p}(x_{\rm p}(t_k)) \\
& + V_{\rm c}(x_{\rm c}(t_k^+), h_{\rm p}(x_{\rm p}(t_k^+))) - V_{\rm c}(x_{\rm c}(t_k), h_{\rm p}(x_{\rm p}(t_k))) \\
= {} & s_{\rm d}(u_{\rm d}(t_k), y(t_k))\chi_{\mathcal{Z}_1}(x(t_k)) \\
& + [V_{\rm c}(\eta(h_{\rm p}(x_{\rm p}(t_k))), h_{\rm p}(x_{\rm p}(t_k))) - V_{\rm c}(x_{\rm c}(t_k), h_{\rm p}(x_{\rm p}(t_k)))]\chi_{\mathcal{Z}_2}(x(t_k)) \\
= {} & s_{\rm d}(u_{\rm d}(t_k), y(t_k))\chi_{\mathcal{Z}_1}(x(t_k)) - V_{\rm c}(x_{\rm c}(t_k), h_{\rm p}(x_{\rm p}(t_k)))\chi_{\mathcal{Z}_2}(x(t_k)) \\
< {} & 0, \quad x(t_k) \in \mathcal{Z}, \quad k \in \overline{\mathbb{Z}}_+.
\end{aligned} \tag{5.155}$$

Thus, it follows from Theorem 5.2 that the zero solution $x(t) \equiv 0$ to $\mathcal{G}$ is asymptotically stable. Finally, if $\mathcal{D}_{\rm p} = \mathbb{R}^{n_{\rm p}}$, $\mathcal{D}_{\rm c} = \mathbb{R}^{n_{\rm c}}$, and $V(\cdot)$ is radially unbounded, then global asymptotic stability is immediate. $\square$

Remark 5.9. If $V_{\rm c} = V_{\rm c}(x_{\rm c}, y)$ is only a function of $x_{\rm c}$ and $V_{\rm c}(x_{\rm c})$ is a positive-definite function, then we can choose $\eta(y) \equiv 0$. In this case, $V_{\rm c}(x_{\rm c}) = 0$ if and only if $x_{\rm c} = 0$, and hence, Theorem 5.7 specializes to the case of a negative feedback interconnection of two hybrid lossless dynamical systems $\mathcal{G}_{\rm p}$ and $\mathcal{G}_{\rm c}$ [99].

Remark 5.10. In the proof of Theorem 5.7, we assume that $x_0 \notin \mathcal{Z}$ for $x_0 \neq 0$. This proviso is necessary since it may be possible to reset the states of the closed-loop system to the origin, in which case $x(s) = 0$ for a finite value of s. In this case, for $t > s$, we have $V(x(t)) = V(x(s)) = V(0) = 0$. This situation does not present a problem, however, since reaching the origin in finite time is a stronger condition than reaching the origin as $t \to \infty$.

Remark 5.11. Theorem 5.7 can be trivially generalized to the case where $\mathcal{G}_{\rm p}$ is *dissipative* with respect to the hybrid supply rate $(s_{\rm c}(u_{\rm c}, y), s_{\rm d}(u_{\rm d}, y))$ in the sense of ([98])

$$V_{\rm s}(x_{\rm p}(\hat{t})) = V_{\rm s}(x_{\rm p}(t)) + \int_t^{\hat{t}} s_{\rm c}(u_{\rm c}(\sigma), y(\sigma)){\rm d}\sigma, \quad t_k < t \le \hat{t} \le t_{k+1}, \tag{5.156}$$

$$V_{\rm s}(x_{\rm p}(t_k) + f_{\rm dp}(x_{\rm p}(t_k), u_{\rm d}(t_k))) \le V_{\rm s}(x_{\rm p}(t_k)) + s_{\rm d}(u_{\rm d}(t_k), y(t_k)), \quad k \in \overline{\mathbb{Z}}_+. \tag{5.157}$$

In this case, the dissipation rate function inherent in (5.157) does not add any additional complexity to the hybrid stabilization process. Similar remarks hold for impulsive port-controlled Hamiltonian systems considered below.

Finally, we specialize the hybrid controller design framework just presented to *impulsive port-controlled Hamiltonian systems* [109]. Specifically, consider the state-dependent impulsive port-controlled Hamiltonian system given by

$$\dot{x}_{\rm p}(t) = \mathcal{J}_{\rm cp}(x_{\rm p}(t)) \left(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}(t)) \right)^{\rm T} + G_{\rm p}(x_{\rm p}(t))u_{\rm c}(t), \quad x_{\rm p}(0) = x_{\rm p0}, \quad (x_{\rm p}(t), u_{\rm c}(t)) \notin \mathcal{Z}_{\rm p}, \tag{5.158}$$

$$\Delta x_{\rm p}(t) = \mathcal{J}_{\rm dp}(x_{\rm p}(t)) \left(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}(t)) \right)^{\rm T} + G_{\rm p}(x_{\rm p}(t))u_{\rm d}(t), \quad (x_{\rm p}(t), u_{\rm c}(t)) \in \mathcal{Z}_{\rm p}, \tag{5.159}$$

$$y(t) = G_{\rm p}^{\rm T}(x_{\rm p}(t)) \left(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}(t)) \right)^{\rm T}, \tag{5.160}$$

where $t \ge 0$, $x_{\rm p}(t) \in \mathcal{D}_{\rm p} \subseteq \mathbb{R}^{n_{\rm p}}$, $\mathcal{D}_{\rm p}$ is an open set with $0 \in \mathcal{D}_{\rm p}$, $u_{\rm c}(t) \in \mathbb{R}^m$, $u_{\rm d}(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, $\mathcal{H}_{\rm p} : \mathcal{D}_{\rm p} \to \mathbb{R}$ is an infinitely differentiable Hamiltonian function for the system (5.158)–(5.160), $\mathcal{J}_{\rm cp} : \mathcal{D}_{\rm p} \to \mathbb{R}^{n_{\rm p} \times n_{\rm p}}$ is such that $\mathcal{J}_{\rm cp}(x_{\rm p}) = -\mathcal{J}_{\rm cp}^{\rm T}(x_{\rm p})$, $x_{\rm p} \in \mathcal{D}_{\rm p}$, $\mathcal{J}_{\rm cp}(x_{\rm p})(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}))^{\rm T}$, $x_{\rm p} \in \mathcal{D}_{\rm p}$, is smooth on $\mathcal{D}_{\rm p}$, $G_{\rm p} : \mathcal{D}_{\rm p} \to \mathbb{R}^{n_{\rm p} \times m}$, $\mathcal{J}_{\rm dp} : \mathcal{D}_{\rm p} \to \mathbb{R}^{n_{\rm p} \times n_{\rm p}}$ is such that $\mathcal{J}_{\rm dp}(x_{\rm p}) = -\mathcal{J}_{\rm dp}^{\rm T}(x_{\rm p})$, $x_{\rm p} \in \mathcal{D}_{\rm p}$, $\mathcal{J}_{\rm dp}(x_{\rm p})(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}))^{\rm T}$, $x_{\rm p} \in \mathcal{D}_{\rm p}$, is smooth on $\mathcal{D}_{\rm p}$, and $\mathcal{Z}_{\rm p} \triangleq \mathcal{Z}_{x_{\rm p}} \times \mathcal{Z}_{u_{\rm c}} \subset \mathcal{D}_{\rm p} \times \mathbb{R}^m$ is the resetting set. The skew-symmetric matrix functions $\mathcal{J}_{\rm cp}(x_{\rm p})$

and $\mathcal{J}_{\mathrm{dp}}(x_{\mathrm{p}})$, $x_{\mathrm{p}} \in \mathcal{D}_{\mathrm{p}}$, capture the internal hybrid system interconnection structure and the input matrix function $G_{\mathrm{p}}(x_{\mathrm{p}})$, $x_{\mathrm{p}} \in \mathcal{D}_{\mathrm{p}}$, captures interconnections with the environment. Furthermore, we assume $\mathcal{H}_{\mathrm{p}}(\cdot)$ is such that

$$\mathcal{H}_{\mathrm{p}}\left(x_{\mathrm{p}} + \mathcal{J}_{\mathrm{dp}}(x_{\mathrm{p}})\left(\frac{\partial \mathcal{H}_{\mathrm{p}}}{\partial x_{\mathrm{p}}}(x_{\mathrm{p}})\right)^{\mathrm{T}} + G_{\mathrm{p}}(x_{\mathrm{p}})u_{\mathrm{d}}\right) = \mathcal{H}_{\mathrm{p}}(x_{\mathrm{p}}) + \frac{\partial \mathcal{H}_{\mathrm{p}}}{\partial x_{\mathrm{p}}}(x_{\mathrm{p}})G_{\mathrm{p}}(x_{\mathrm{p}})u_{\mathrm{d}},$$
$$x_{\mathrm{p}} \in \mathcal{D}_{\mathrm{p}}, \quad u_{\mathrm{d}} \in \mathbb{R}^m. \tag{5.161}$$

Finally, we assume that $\mathcal{H}_{\mathrm{p}}(0) = 0$ and $\mathcal{H}_{\mathrm{p}}(x_{\mathrm{p}}) > 0$ for all $x_{\mathrm{p}} \neq 0$ and $x_{\mathrm{p}} \in \mathcal{D}_{\mathrm{p}}$.

Next, consider the fixed-order, energy-based hybrid controller

$$\dot{x}_{\mathrm{c}}(t) = \mathcal{J}_{\mathrm{cc}}(x_{\mathrm{c}}(t))\left(\frac{\partial \mathcal{H}_{\mathrm{c}}}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}}(t))\right)^{\mathrm{T}} + G_{\mathrm{cc}}(x_{\mathrm{c}}(t))y(t), \quad x_{\mathrm{c}}(0) = x_{\mathrm{c}0}, \quad (x_{\mathrm{c}}(t), y(t)) \notin \mathcal{Z}_{\mathrm{c}}, \tag{5.162}$$

$$\Delta x_{\mathrm{c}}(t) = -x_{\mathrm{c}}(t), \quad (x_{\mathrm{c}}(t), y(t)) \in \mathcal{Z}_{\mathrm{c}}, \tag{5.163}$$

$$u_{\mathrm{c}}(t) = -G_{\mathrm{cc}}^{\mathrm{T}}(x_{\mathrm{c}}(t))\left(\frac{\partial \mathcal{H}_{\mathrm{c}}}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}}(t))\right)^{\mathrm{T}}, \tag{5.164}$$

$$u_{\mathrm{d}}(t) = -G_{\mathrm{p}}^{\mathrm{T}}(x_{\mathrm{p}}(t))\left(\frac{\partial \mathcal{H}_{\mathrm{p}}}{\partial x_{\mathrm{p}}}(x_{\mathrm{p}}(t))\right)^{\mathrm{T}}, \tag{5.165}$$

where $t \geq 0$, $x_{\mathrm{c}}(t) \in \mathcal{D}_{\mathrm{c}} \subseteq \mathbb{R}^{n_{\mathrm{c}}}$, $\mathcal{D}_{\mathrm{c}}$ is an open set with $0 \in \mathcal{D}_{\mathrm{c}}$, $\Delta x_{\mathrm{c}}(t) \triangleq x_{\mathrm{c}}(t^+) - x_{\mathrm{c}}(t)$, $\mathcal{H}_{\mathrm{c}} : \mathcal{D}_{\mathrm{c}} \to \mathbb{R}$ is an infinitely differentiable Hamiltonian function for (5.162), $\mathcal{J}_{\mathrm{cc}} : \mathcal{D}_{\mathrm{c}} \to \mathbb{R}^{n_{\mathrm{c}} \times n_{\mathrm{c}}}$ is such that $\mathcal{J}_{\mathrm{cc}}(x_{\mathrm{c}}) = -\mathcal{J}_{\mathrm{cc}}^{\mathrm{T}}(x_{\mathrm{c}})$, $x_{\mathrm{c}} \in \mathcal{D}_{\mathrm{c}}$, $\mathcal{J}_{\mathrm{cc}}(x_{\mathrm{c}})(\frac{\partial \mathcal{H}_{\mathrm{c}}}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}}))^{\mathrm{T}}$, $x_{\mathrm{c}} \in \mathcal{D}_{\mathrm{c}}$, is smooth on $\mathcal{D}_{\mathrm{c}}$, $G_{\mathrm{cc}} : \mathcal{D}_{\mathrm{c}} \to \mathbb{R}^{n_{\mathrm{c}} \times m}$, and resetting set $\mathcal{Z}_{\mathrm{c}} \subset \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}}$ given by

$$\mathcal{Z}_{\mathrm{c}} \triangleq \left\{(x_{\mathrm{p}}, x_{\mathrm{c}}) \in \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}} : \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}) = 0 \text{ and } \mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}) > 0\right\}, \tag{5.166}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}(t)) \triangleq \lim_{\tau \to t^-} \frac{1}{t-\tau}[\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}(t)) - \mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}(\tau))]$ whenever limit on the right-hand side exists. Here, we assume that $\mathcal{H}_{\mathrm{c}}(0) = 0$ and $\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}) > 0$ for all $x_{\mathrm{c}} \neq 0$ and $x_{\mathrm{c}} \in \mathcal{D}_{\mathrm{c}}$.

Note that $\mathcal{H}_{\mathrm{p}}(x_{\mathrm{p}})$, $x_{\mathrm{p}} \in \mathcal{D}_{\mathrm{p}}$, is the plant energy and $\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}})$, $x_{\mathrm{c}} \in \mathcal{D}_{\mathrm{c}}$, is the controller emulated energy. Furthermore, the closed-loop system energy is given by $\mathcal{H}(x_{\mathrm{p}}, x_{\mathrm{c}}) \triangleq \mathcal{H}_{\mathrm{p}}(x_{\mathrm{p}}) + \mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}})$. The resetting set $\mathcal{Z}$ is given by $\mathcal{Z} \triangleq \mathcal{Z}_1 \cup \mathcal{Z}_2$, where

$$\mathcal{Z}_1 \triangleq \left\{(x_{\mathrm{p}}, x_{\mathrm{c}}) \in \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}} : \left(x_{\mathrm{p}}, -G_{\mathrm{cc}}^{\mathrm{T}}(x_{\mathrm{c}})\left(\frac{\partial \mathcal{H}_{\mathrm{c}}}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}})\right)^{\mathrm{T}}\right) \in \mathcal{Z}_{\mathrm{p}}\right\}, \tag{5.167}$$

$$\mathcal{Z}_2 \triangleq \left\{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c} : \left(x_{\rm c}, G_{\rm p}^{\rm T}(x_{\rm p}) \left(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p})\right)^{\rm T}\right) \in \mathcal{Z}_{\rm c}\right\}. \tag{5.168}$$

Here, we assume that $\overline{\mathcal{Z}}_1 = \{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c} : \mathcal{X}_1(x_{\rm p}, x_{\rm c}) = 0\}$. Furthermore, if $(x_{\rm p}, x_{\rm c}) \in \mathcal{Z}_1$ then $x_{\rm p} + \mathcal{J}_{\rm dp}(x_{\rm p})(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}))^{\rm T} - G_{\rm p}(x_{\rm p})G_{\rm p}^{\rm T}(x_{\rm p})(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}))^{\rm T} \in \overline{\mathcal{Z}}_1 \backslash \mathcal{Z}_1$, and if $(x_{\rm p}, x_{\rm c}) \in \overline{\mathcal{Z}}_1 \backslash \mathcal{Z}_1$ then $\mathcal{J}_{\rm dp}(x_{\rm p})(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}))^{\rm T} - G_{\rm p}(x_{\rm p})G_{\rm p}^{\rm T}(x_{\rm p})(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}))^{\rm T} = 0$. Finally, we assume that

$$\mathcal{Z}_1 \cap \left\{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c} : G_{\rm p}^{\rm T}(x_{\rm p}) \left(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p})\right)^{\rm T} = 0\right\} = \emptyset. \tag{5.169}$$

Next, note that total energy function $\mathcal{H}(x_{\rm p}, x_{\rm c})$ along the trajectories of the closed-loop dynamics (5.158)–(5.168) satisfies

$$\frac{\rm d}{{\rm d}t}\mathcal{H}(x_{\rm p}(t), x_{\rm c}(t)) = 0, \quad (x_{\rm p}(t), x_{\rm c}(t)) \notin \mathcal{Z}, \tag{5.170}$$

$$\begin{aligned}\Delta\mathcal{H}(x_{\rm p}(t_k), x_{\rm c}(t_k)) = {} & -\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}(t_k))G_{\rm p}(x_{\rm p}(t_k))G_{\rm p}^{\rm T}(x_{\rm p}(t_k)) \left(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}(t_k))\right)^{\rm T} \\ & \cdot\chi_{\mathcal{Z}_1}(x_{\rm p}(t_k), x_{\rm c}(t_k)) - \mathcal{H}_{\rm c}(x_{\rm c}(t_k))\chi_{\mathcal{Z}_2}(x_{\rm p}(t_k), x_{\rm c}(t_k)), \\ & (x_{\rm p}(t_k), x_{\rm c}(t_k)) \in \mathcal{Z}, \quad k \in \overline{\mathbb{Z}}_+. \end{aligned} \tag{5.171}$$

Here, we assume that every $(x_{\rm p0}, x_{\rm c0}) \in \overline{\mathcal{Z}}$ is transversal to the closed-loop dynamical system given by (5.158)–(5.168) with $\mathcal{X}_{\rm p}(x_{\rm p}, x_{\rm c}) = \mathcal{X}_1(x_{\rm p}, x_{\rm c})$ and $\mathcal{X}_{\rm c}(x_{\rm p}, x_{\rm c}) = \frac{\rm d}{{\rm d}t}\mathcal{H}_{\rm c}(x_{\rm c})$. Furthermore, we assume $\mathcal{D}_{\rm ci} \subset \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c}$ is a compact positively invariant set with respect to the closed-loop dynamical system (5.158)–(5.168), such that $0 \in \overset{\circ}{\mathcal{D}}_{\rm ci}$. In this case, it follows from Theorem 5.7, with $V_{\rm s}(x_{\rm p}) = \mathcal{H}_{\rm p}(x_{\rm p})$, $V_{\rm c}(x_{\rm c}, y) = \mathcal{H}_{\rm c}(x_{\rm c})$, $s_{\rm c}(u_{\rm c}, y) = u_{\rm c}^{\rm T}y$, $s_{\rm d}(u_{\rm d}, y) = u_{\rm d}^{\rm T}y$, and $s_{\rm cc}(u_{\rm cc}, y_{\rm cc}) = u_{\rm cc}^{\rm T}y_{\rm cc}$, that the zero solution $(x_{\rm p}(t), x_{\rm c}(t)) \equiv (0, 0)$ to the closed-loop system (5.158)–(5.168) is asymptotically stable.

5.9. Hybrid Control Design for Nonsmooth Euler-Lagrange Systems

In this section, we present a hybrid feedback control framework for nonsmooth Euler-Lagrange dynamical systems. Consider the governing equations of motion of an $\hat{n}_{\rm p}$ degree-of-freedom dynamical system given by the *hybrid Euler-Lagrange* equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial\dot{q}}(q(t),\dot{q}(t))\right]^{\mathrm{T}}-\left[\frac{\partial\mathcal{L}}{\partial q}(q(t),\dot{q}(t))\right]^{\mathrm{T}}=u_{\mathrm{c}}(t),\quad q(0)=q_0,\quad \dot{q}(0)=\dot{q}_0,$$

$$(q(t),\dot{q}(t))\notin\mathcal{Z}_{\mathrm{p}},\tag{5.172}$$

$$\begin{bmatrix}\Delta q(t)\\ \Delta\dot{q}(t)\end{bmatrix}=\begin{bmatrix}P(q(t))-q(t)\\ Q(\dot{q}(t))-\dot{q}(t)\end{bmatrix},\quad (q(t),\dot{q}(t))\in\mathcal{Z}_{\mathrm{p}},\tag{5.173}$$

with outputs

$$y=\begin{bmatrix}h_1(q)\\ h_2(\dot{q})\end{bmatrix},\tag{5.174}$$

where $t\geq 0$, $q\in\mathbb{R}^{\hat{n}_{\mathrm{p}}}$ represents the generalized system positions, $\dot{q}\in\mathbb{R}^{\hat{n}_{\mathrm{p}}}$ represents the generalized system velocities, $\mathcal{L}:\mathbb{R}^{\hat{n}_{\mathrm{p}}}\times\mathbb{R}^{\hat{n}_{\mathrm{p}}}\to\mathbb{R}$ denotes the system Lagrangian given by $\mathcal{L}(q,\dot{q})=T(q,\dot{q})-U(q)$, where $T:\mathbb{R}^{\hat{n}_{\mathrm{p}}}\times\mathbb{R}^{\hat{n}_{\mathrm{p}}}\to\mathbb{R}$ is the system kinetic energy and $U:\mathbb{R}^{\hat{n}_{\mathrm{p}}}\to\mathbb{R}$ is the system potential energy, $u_{\mathrm{c}}\in\mathbb{R}^{\hat{n}_{\mathrm{p}}}$ is the vector of generalized control forces acting on the system, $\mathcal{Z}_{\mathrm{p}}\subset\mathbb{R}^{\hat{n}_{\mathrm{p}}}\times\mathbb{R}^{\hat{n}_{\mathrm{p}}}$ is the resetting set such that the closure of $\mathcal{Z}_{\mathrm{p}}$ is given by

$$\overline{\mathcal{Z}}_{\mathrm{p}}\triangleq\{(q,\dot{q}):H(q,\dot{q})=0\},\tag{5.175}$$

where $H:\mathbb{R}^{\hat{n}_{\mathrm{p}}}\times\mathbb{R}^{\hat{n}_{\mathrm{p}}}\to\mathbb{R}$ is an infinitely differentiable function, $\Delta q(t)\triangleq q(t^+)-q(t)$, $\Delta\dot{q}(t)\triangleq\dot{q}(t^+)-\dot{q}(t)$, $P:\mathbb{R}^{\hat{n}_{\mathrm{p}}}\to\mathbb{R}^{\hat{n}_{\mathrm{p}}}$ and $Q:\mathbb{R}^{\hat{n}_{\mathrm{p}}}\to\mathbb{R}^{\hat{n}_{\mathrm{p}}}$ are smooth functions such that if $(q,\dot{q})\in\mathcal{Z}_{\mathrm{p}}$, then $(P(q),Q(\dot{q}))\in\overline{\mathcal{Z}}_{\mathrm{p}}\backslash\mathcal{Z}_{\mathrm{p}}$, and if $(q,\dot{q})\in\overline{\mathcal{Z}}_{\mathrm{p}}\backslash\mathcal{Z}_{\mathrm{p}}$, then $(P(q),Q(\dot{q}))=(q,\dot{q})$, $T(P(q),Q(\dot{q}))+U(P(q))<T(q,\dot{q})+U(q)$, $(q,\dot{q})\in\mathcal{Z}_{\mathrm{p}}$, $h_1:\mathbb{R}^{\hat{n}_{\mathrm{p}}}\to\mathbb{R}^{l_1}$ and $h_2:\mathbb{R}^{\hat{n}_{\mathrm{p}}}\to\mathbb{R}^{l-l_1}$ are smooth functions, $h_1(0)=0$, $h_2(0)=0$, and $h_1(q)\not\equiv 0$. We assume that the system kinetic energy is such that $T(q,\dot{q})=\frac{1}{2}\dot{q}^{\mathrm{T}}[\frac{\partial T}{\partial\dot{q}}(q,\dot{q})]^{\mathrm{T}}$, $T(q,0)=0$, and $T(q,\dot{q})>0$, $\dot{q}\neq 0$, $\dot{q}\in\mathbb{R}^{\hat{n}_{\mathrm{p}}}$.

Furthermore, let $\mathcal{H}:\mathbb{R}^{\hat{n}_{\mathrm{p}}}\times\mathbb{R}^{\hat{n}_{\mathrm{p}}}\to\mathbb{R}$ denote the *Legendre transformation* of the Lagrangian function $\mathcal{L}(q,\dot{q})$ with respect to the generalized velocity $\dot{q}$ defined by $\mathcal{H}(q,p)\triangleq\dot{q}^{\mathrm{T}}p-\mathcal{L}(q,\dot{q})$, where p denotes the vector of generalized momenta given by

$$p(q,\dot{q})=\left[\frac{\partial\mathcal{L}}{\partial\dot{q}}(q,\dot{q})\right]^{\mathrm{T}},\tag{5.176}$$

where the map from the generalized velocities $\dot{q}$ to the generalized momenta p is assumed to be *bijective* (i.e., one-to-one and onto). Now, if $\mathcal{H}(q,p)$ is lower bounded, then we can always shift $\mathcal{H}(q,p)$ so that, with a minor abuse of notation, $\mathcal{H}(q,p) \geq 0$, $(q,p) \in \mathbb{R}^{\hat{n}_{\rm p}} \times \mathbb{R}^{\hat{n}_{\rm p}}$. In this case, using (5.172) and the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{L}(q,\dot{q})] = \frac{\partial \mathcal{L}}{\partial q}(q,\dot{q})\dot{q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}(q,\dot{q})\ddot{q}, \quad (q,\dot{q}) \notin \mathcal{Z}_{\rm p}, \tag{5.177}$$

it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(q,p) = u_{\rm c}^{\rm T}\dot{q}$, $(q,\dot{q}) \notin \mathcal{Z}_{\rm p}$. We also assume that the system potential energy $U(\cdot)$ is such that $U(0) = 0$ and $U(q) > 0$, $q \neq 0$, $q \in \mathcal{D}_{\rm q} \subseteq \mathbb{R}^{\hat{n}_{\rm p}}$, which implies that $\mathcal{H}(q,p) = T(q,\dot{q}) + U(q) > 0$, $(q,\dot{q}) \neq 0$, $(q,\dot{q}) \in \mathcal{D}_{\rm q} \times \mathbb{R}^{\hat{n}_{\rm p}}$.

Next, consider the energy-based hybrid controller

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{L}_{\rm c}}{\partial \dot{q}_{\rm c}}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\right]^{\rm T} - \left[\frac{\partial \mathcal{L}_{\rm c}}{\partial q_{\rm c}}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\right]^{\rm T} = 0, \quad q_{\rm c}(0) = q_{\rm c0}, \quad \dot{q}_{\rm c}(0) = \dot{q}_{\rm c0},$$

$$(q_{\rm c}(t),\dot{q}_{\rm c}(t),y(t)) \notin \mathcal{Z}_{\rm c}, \tag{5.178}$$

$$\begin{bmatrix} \Delta q_{\rm c}(t) \\ \Delta \dot{q}_{\rm c}(t) \end{bmatrix} = \begin{bmatrix} \eta(y_q(t)) - q_{\rm c}(t) \\ -\dot{q}_{\rm c}(t) \end{bmatrix}, \quad (q_{\rm c}(t),\dot{q}_{\rm c}(t),y(t)) \in \mathcal{Z}_{\rm c}, \tag{5.179}$$

$$u_{\rm c}(t) = \left[\frac{\partial \mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\right]^{\rm T}, \tag{5.180}$$

where $t \geq 0$, $q_{\rm c} \in \mathbb{R}^{\hat{n}_{\rm c}}$ represents virtual controller positions, $\dot{q}_{\rm c} \in \mathbb{R}^{\hat{n}_{\rm c}}$ represents virtual controller velocities, $y_q \triangleq h_1(q)$, $\mathcal{L}_{\rm c} : \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{l_1} \to \mathbb{R}$ denotes the controller Lagrangian given by $\mathcal{L}_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q) \triangleq T_{\rm c}(q_{\rm c},\dot{q}_{\rm c}) - U_{\rm c}(q_{\rm c},y_q)$, where $T_{\rm c} : \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{\hat{n}_{\rm c}} \to \mathbb{R}$ is the controller kinetic energy, $U_{\rm c} : \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{l_1} \to \mathbb{R}$ is the controller potential energy, $\eta(\cdot)$ is a continuously differentiable function such that $\eta(0) = 0$, $\mathcal{Z}_{\rm c} \subset \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{l}$ is the resetting set, $\Delta q_{\rm c}(t) \triangleq q_{\rm c}(t^+) - q_{\rm c}(t)$, and $\Delta \dot{q}_{\rm c}(t) \triangleq \dot{q}_{\rm c}(t^+) - \dot{q}_{\rm c}(t)$. We assume that the controller kinetic energy $T_{\rm c}(q_{\rm c},\dot{q}_{\rm c})$ is such that $T_{\rm c}(q_{\rm c},\dot{q}_{\rm c}) = \frac{1}{2}\dot{q}_{\rm c}^{\rm T}[\frac{\partial T_{\rm c}}{\partial \dot{q}_{\rm c}}(q_{\rm c},\dot{q}_{\rm c})]^{\rm T}$, with $T_{\rm c}(q_{\rm c},0) = 0$ and $T_{\rm c}(q_{\rm c},\dot{q}_{\rm c}) > 0$, $\dot{q}_{\rm c} \neq 0$, $\dot{q}_{\rm c} \in \mathbb{R}^{\hat{n}_{\rm c}}$. Furthermore, we assume that $U_{\rm c}(\eta(y_q),y_q) = 0$ and $U_{\rm c}(q_{\rm c},y_q) > 0$ for $q_{\rm c} \neq \eta(y_q)$, $q_{\rm c} \in \mathcal{D}_{\rm q_c} \subseteq \mathbb{R}^{\hat{n}_{\rm c}}$.

As in Section 5.8, note that $V_{\rm p}(q,\dot{q}) \triangleq T(q,\dot{q}) + U(q)$ is the plant energy and $V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q) \triangleq T_{\rm c}(q_{\rm c},\dot{q}_{\rm c}) + U_{\rm c}(q_{\rm c},y_q)$ is the controller emulated energy. Furthermore, $V(q,\dot{q},q_{\rm c},\dot{q}_{\rm c}) \triangleq V_{\rm p}(q,\dot{q}) + V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q)$ is the total energy of the closed-loop system. It is important to note that the

Lagrangian dynamical system (5.172) is *not* lossless with outputs y_q or y. Next, we study the behavior of the total energy function $V(q,\dot{q},q_{\rm c},\dot{q}_{\rm c})$ along the trajectories of the closed-loop system dynamics. For the closed-loop system, we define our resetting set as $\mathcal{Z} \triangleq \mathcal{Z}_1 \cup \mathcal{Z}_2$, where $\mathcal{Z}_1 \triangleq \{(q,\dot{q},q_{\rm c},\dot{q}_{\rm c}) : (q,\dot{q}) \in \mathcal{Z}_{\rm p}\}$ and $\mathcal{Z}_2 \triangleq \{(q,\dot{q},q_{\rm c},\dot{q}_{\rm c}) : (q_{\rm c},\dot{q}_{\rm c},y) \in \mathcal{Z}_{\rm c}\}$. Note that

$$\frac{\rm d}{{\rm d}t}V_{\rm p}(q,\dot{q}) = \frac{\rm d}{{\rm d}t}\mathcal{H}(q,p) = u_{\rm c}^{\rm T}\dot{q}, \quad (q,\dot{q},q_{\rm c},\dot{q}_{\rm c}) \notin \mathcal{Z}. \tag{5.181}$$

To obtain an expression for $\frac{\rm d}{{\rm d}t}V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q)$ when $(q,\dot{q},q_{\rm c},\dot{q}_{\rm c}) \notin \mathcal{Z}$, define the controller Hamiltonian by

$$\mathcal{H}_{\rm c}(q_{\rm c},\dot{q}_{\rm c},p_{\rm c},y_q) \triangleq \dot{q}_{\rm c}^{\rm T}p_{\rm c} - \mathcal{L}_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q), \tag{5.182}$$

where the virtual controller momentum $p_{\rm c}$ is given by $p_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q) = \left[\frac{\partial\mathcal{L}_{\rm c}}{\partial\dot{q}_{\rm c}}(q_{\rm c},\dot{q}_{\rm c},y_q)\right]^{\rm T}$. Then $\mathcal{H}_{\rm c}(q_{\rm c},\dot{q}_{\rm c},p_{\rm c},y_q) = T_{\rm c}(q_{\rm c},\dot{q}_{\rm c}) + U_{\rm c}(q_{\rm c},y_q)$. Now, it follows from (5.178) and the structure of $T_{\rm c}(q_{\rm c},\dot{q}_{\rm c})$ that, for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned}
0 &= \frac{\rm d}{{\rm d}t}\left[p_{\rm c}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\right]^{\rm T}\dot{q}_{\rm c}(t) - \frac{\partial\mathcal{L}_{\rm c}}{\partial q_{\rm c}}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}_{\rm c}(t)\\
&= \frac{\rm d}{{\rm d}t}\left[p_{\rm c}^{\rm T}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}_{\rm c}(t)\right] - p_{\rm c}^{\rm T}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\ddot{q}_{\rm c}(t) + \frac{\partial\mathcal{L}_{\rm c}}{\partial\dot{q}_{\rm c}}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\ddot{q}_{\rm c}(t)\\
&\quad + \frac{\partial\mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}(t) - \frac{\rm d}{{\rm d}t}\mathcal{L}_{\rm c}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\\
&= \frac{\rm d}{{\rm d}t}[p_{\rm c}^{\rm T}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}_{\rm c}(t) - \mathcal{L}_{\rm c}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))] + \frac{\partial\mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}(t)\\
&= \frac{\rm d}{{\rm d}t}\mathcal{H}_{\rm c}(q_{\rm c}(t),\dot{q}_{\rm c}(t),p_{\rm c}(t),y_q(t)) + \frac{\partial\mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}(t)\\
&= \frac{\rm d}{{\rm d}t}V_{\rm c}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t)) + \frac{\partial\mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}(t), \quad (q(t),\dot{q}(t),q_{\rm c}(t),\dot{q}_{\rm c}(t)) \notin \mathcal{Z}.
\end{aligned} \tag{5.183}$$

Hence,

$$\begin{aligned}
\frac{\rm d}{{\rm d}t}V(q(t),\dot{q}(t),q_{\rm c}(t),\dot{q}_{\rm c}(t)) &= u_{\rm c}^{\rm T}(t)\dot{q}(t) - \frac{\partial\mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}(t)\\
&= 0, \quad (q(t),\dot{q}(t),q_{\rm c}(t),\dot{q}_{\rm c}(t)) \notin \mathcal{Z}, \quad t_k < t \le t_{k+1},
\end{aligned} \tag{5.184}$$

which implies that the total energy of the closed-loop system between resetting events is conserved.

The total energy difference across resetting events is given by

$$\begin{aligned}\Delta V(q(t_k),\dot q(t_k),q_{\mathrm c}(t_k),\dot q_{\mathrm c}(t_k)) &= V_{\mathrm p}(q(t_k^+),\dot q(t_k^+)) - V_{\mathrm p}(q(t_k),\dot q(t_k))\\ &\quad +T_{\mathrm c}(q_{\mathrm c}(t_k^+),\dot q_{\mathrm c}(t_k^+)) + U_{\mathrm c}(q_{\mathrm c}(t_k^+),y_q(t_k))\\ &\quad -V_{\mathrm c}(q_{\mathrm c}(t_k),\dot q_{\mathrm c}(t_k),y_q(t_k))\\ &= [V_{\mathrm p}(P(q(t_k)),Q(\dot q(t_k))) - V_{\mathrm p}(q(t_k),\dot q(t_k))]\\ &\quad \cdot\chi_{\mathcal Z_1}(q(t_k),\dot q(t_k),q_{\mathrm c}(t_k),\dot q_{\mathrm c}(t_k)) - V_{\mathrm c}(q_{\mathrm c}(t_k),\dot q_{\mathrm c}(t_k),y_q(t_k))\\ &\quad \cdot\chi_{\mathcal Z_2}(q(t_k),\dot q(t_k),q_{\mathrm c}(t_k),\dot q_{\mathrm c}(t_k))\\ &< 0,\quad (q(t_k),\dot q(t_k),q_{\mathrm c}(t_k),\dot q_{\mathrm c}(t_k))\in\mathcal Z,\quad k\in\overline{\mathbb Z}_+,\end{aligned}\tag{5.185}$$

which implies that the resetting law (5.179) ensures the total energy decrease across resetting events.

Here, we concentrate on an energy dissipating state-dependent resetting controller that affects a one-way energy transfer between the plant and the controller. Specifically, consider the closed-loop system (5.172)–(5.180), where $\mathcal Z_{\mathrm c}$ is defined by

$$\mathcal Z_{\mathrm c}\triangleq\left\{(q,\dot q,q_{\mathrm c},\dot q_{\mathrm c}):\frac{\mathrm d}{\mathrm dt}V_{\mathrm c}(q_{\mathrm c},\dot q_{\mathrm c},y_q)=0\text{ and }V_{\mathrm c}(q_{\mathrm c},\dot q_{\mathrm c},y_q)>0\right\}.\tag{5.186}$$

Since $y_q=h_1(q)$ and

$$\frac{\mathrm d}{\mathrm dt}V_{\mathrm c}(q_{\mathrm c},\dot q_{\mathrm c},y_q) = -\left[\frac{\partial\mathcal L_{\mathrm c}}{\partial q}(q_{\mathrm c},\dot q_{\mathrm c},y_q)\right]\dot q = \left[\frac{\partial U_{\mathrm c}}{\partial q}(q_{\mathrm c},y_q)\right]\dot q,\quad (q_{\mathrm c},\dot q_{\mathrm c},y)\notin\mathcal Z_{\mathrm c},\tag{5.187}$$

it follows that (5.186) can be equivalently rewritten as

$$\mathcal Z_{\mathrm c}=\left\{(q,\dot q,q_{\mathrm c},\dot q_{\mathrm c}):\left[\frac{\partial U_{\mathrm c}}{\partial q}(q_{\mathrm c},h_1(q))\right]\dot q=0\text{ and }V_{\mathrm c}(q_{\mathrm c},\dot q_{\mathrm c},h_1(q))>0\right\}.\tag{5.188}$$

Once again, for practical implementation, knowledge of $q_{\mathrm c}$, $\dot q_{\mathrm c}$, and y is often sufficient to determine whether or not the closed-loop state vector is in the set $\mathcal Z_{\mathrm c}$.

The next theorem gives sufficient conditions for stabilization of nonsmooth Euler-Lagrange dynamical systems using state-dependent hybrid controllers. For this result define the closed-loop system states $x\triangleq[q^{\mathrm T},\dot q^{\mathrm T},q_{\mathrm c}^{\mathrm T},\dot q_{\mathrm c}^{\mathrm T}]^{\mathrm T}$.

Theorem 5.8. Consider the closed-loop dynamical system $\mathcal{G}$ given by (5.172)–(5.180), with the resetting set $\mathcal{Z}_{\rm c}$ given by (5.186). Assume that $\mathcal{D}_{\rm ci} \subset \mathcal{D}_{\rm q} \times \mathbb{R}^{\hat{n}_{\rm p}} \times \mathcal{D}_{\rm q_c} \times \mathbb{R}^{\hat{n}_{\rm c}}$ is a compact positively invariant set with respect to $\mathcal{G}$ such that $0 \in \overset{\circ}{\mathcal{D}}_{\rm ci}$. Furthermore, assume that the transversality condition (5.133) and (5.134) holds with $\mathcal{X}_{\rm p}(x) = H(q,\dot{q})$ and $\mathcal{X}_{\rm c}(x) = \frac{\rm d}{{\rm d}t}V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q)$. Then the zero solution $x(t) \equiv 0$ to $\mathcal{G}$ is asymptotically stable. In addition, the total energy function $V(x)$ of $\mathcal{G}$ is strictly decreasing across resetting events. Finally, if $\mathcal{D}_{\rm q} = \mathbb{R}^{\hat{n}_{\rm p}}$, $\mathcal{D}_{\rm q_c} = \mathbb{R}^{\hat{n}_{\rm c}}$, and the total energy function $V(x)$ is radially unbounded, then the zero solution $x(t) \equiv 0$ to $\mathcal{G}$ is globally asymptotically stable.

Proof. The proof is similar to the proof of Theorem 5.7 with $V_{\rm p}(x_{\rm p}) = V_{\rm p}(q,\dot{q})$, $V_{\rm c}(x_{\rm c},y) = V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q)$, $y = u_{\rm cc} = x_{\rm p}$, $u_{\rm c} = -y_{\rm cc} = \frac{\partial \mathcal{L}_{\rm c}}{\partial q}$, $s_{\rm c}(u_{\rm c},y) = u_{\rm c}^{\rm T}\rho(y)$, $s_{\rm d}(u_{\rm d},y) = 0$, $V_{\rm p}(P(q),Q(\dot{q})) - V_{\rm p}(q,\dot{q}) < 0$, $(q,\dot{q}) \in \mathcal{Z}_{\rm p}$, $s_{\rm cc}(u_{\rm cc},y_{\rm cc}) = y_{\rm cc}^{\rm T}\rho(u_{\rm c})$, where $\rho(y) = \rho\left(\begin{bmatrix} q \\ \dot{q} \end{bmatrix}\right) = \dot{q}$, $\eta(y)$ replaced by $\begin{bmatrix} \eta(y_q) \\ 0 \end{bmatrix}$, and noting that (5.184) and (5.185) hold. □

5.10. Hybrid Control Design for Impact Mechanics

In this section, we apply the energy dissipating hybrid controller synthesis framework to the constrained inverted pendulum shown in Figure 5.33, where $m = 1$ kg and $L = 1$ m. In the case where $|\theta(t)| < \theta_{\rm c} \leq \frac{\pi}{2}$, the system is governed by the dynamic equation of motion

$$\ddot{\theta}(t) - g\sin\theta(t) = u_{\rm c}(t), \quad \theta(0) = \theta_0, \quad \dot{\theta}(0) = \dot{\theta}_0, \quad t \geq 0, \tag{5.189}$$

where g denotes the gravitational acceleration and $u_{\rm c}(\cdot)$ is a (thruster) control force. At the instant of collision with the vertical constraint $|\theta(t)| = \theta_{\rm c}$, the system resets according to the resetting law

$$\theta(t_k^+) = \theta(t_k), \quad \dot{\theta}(t_k^+) = -e\dot{\theta}(t_k), \tag{5.190}$$

where $e \in [0,1)$ is the coefficient of restitution. Defining $q = \theta$ and $\dot{q} = \dot{\theta}$, we can rewrite the continuous-time dynamics (5.189) and resetting dynamics (5.190) in Lagrangian form

(5.172) and (5.173) with $\mathcal{L}(q,\dot{q}) = \frac{1}{2}\dot{q}^2 - g\cos q$, $P(q) = q$, $Q(\dot{q}) = -e\dot{q}$, and $\mathcal{Z}_{\rm p} = \{(q,\dot{q}) \in \mathbb{R}^2 : q = \theta_{\rm c}, \dot{q} > 0\} \cup \{(q,\dot{q}) \in \mathbb{R}^2 : q = -\theta_{\rm c}, \dot{q} < 0\}$.

Next, to stabilize the equilibrium point $(q_{\rm e}, \dot{q}_{\rm e}) = (0,0)$, consider the hybrid dynamic compensator

$$\ddot{q}_{\rm c}(t) + k_{\rm c}q_{\rm c}(t) = k_{\rm c}q(t), \quad q_{\rm c}(0) = q_{\rm c0}, \quad (q(t),\dot{q}(t),q_{\rm c}(t),\dot{q}_{\rm c}(t)) \notin \mathcal{Z}_{\rm c}, \quad t \geq 0, \tag{5.191}$$

$$\begin{bmatrix} \Delta q_{\rm c}(t) \\ \Delta\dot{q}_{\rm c}(t) \end{bmatrix} = \begin{bmatrix} q(t) - q_{\rm c}(t) \\ -\dot{q}_{\rm c}(t) \end{bmatrix}, \quad (q(t),\dot{q}(t),q_{\rm c}(t),\dot{q}_{\rm c}(t)) \in \mathcal{Z}_{\rm c}, \tag{5.192}$$

$$u_{\rm c}(t) = -k_{\rm p}q + k_{\rm c}(q_{\rm c}(t) - q(t)), \tag{5.193}$$

where $k_{\rm p} > g$ and $k_{\rm c} > 0$, with the resetting set (5.186) taking the form

$$\mathcal{Z}_{\rm c} = \left\{ (q,\dot{q},q_{\rm c},\dot{q}_{\rm c}) : k_{\rm c}(q_{\rm c} - q)\dot{q} = 0 \text{ and } \begin{bmatrix} q - q_{\rm c} \\ -\dot{q}_{\rm c} \end{bmatrix} \neq 0 \right\}. \tag{5.194}$$

To illustrate the behavior of the closed-loop impulsive dynamical system, let $\theta_{\rm c} = \frac{\pi}{6}$, $g = 9.8$, $e = 0.5$, $k_{\rm p} = 9.9$, and $k_{\rm c} = 2$ with initial conditions $q(0) = 0$, $\dot{q}(0) = 1$, $q_{\rm c}(0) = 0$, and $\dot{q}_{\rm c}(0) = 0$. For this system a straightforward, but lengthy, calculation shows that Assumptions 1 and 2 hold. However, the transversality condition is sufficiently complex that we have been unable to show it analytically. This condition was verified numerically, and hence, Assumption 3 holds. Figure 5.34 shows the phase portrait of the closed-loop impulsive dynamical system with $x_1 = q$ and $x_2 = \dot{q}$. Figure 5.35 shows the controlled plant position and velocity states versus time, while Figure 5.36 shows the controller position and velocity versus time. Figure 5.37 shows the control force versus time. Note that for this example the plant velocity and the controller velocity are the only states that reset. Furthermore, in this case, the control force is continuous since the plant position and the controller position are continuous functions of time.

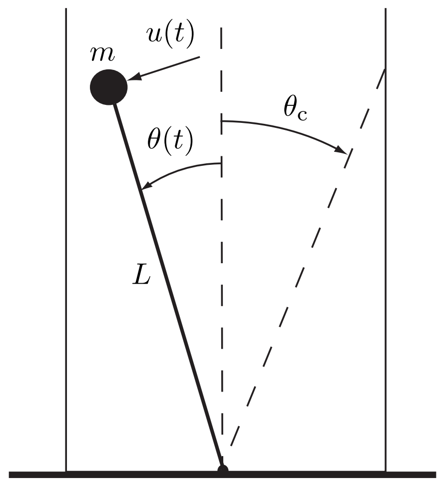

Figure 5.33: Constrained inverted pendulum

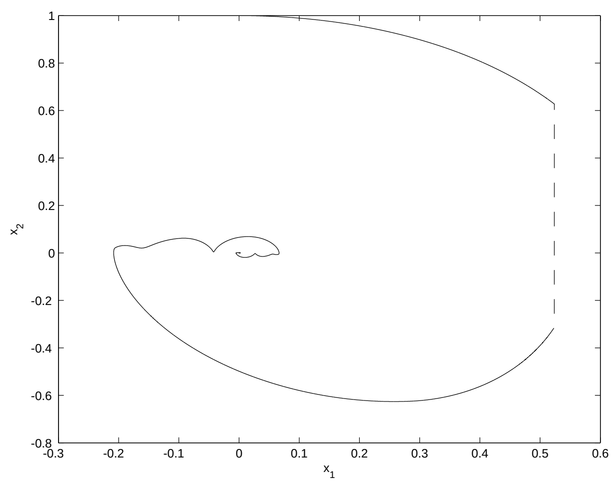

Figure 5.34: Phase portrait of the constraint inverted pendulum

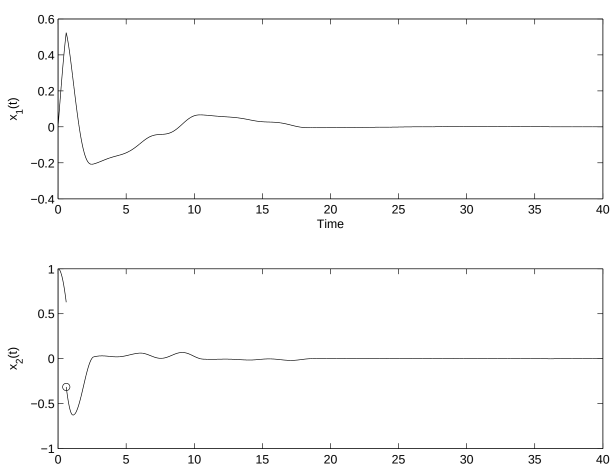

Figure 5.35: Plant position and velocity versus time

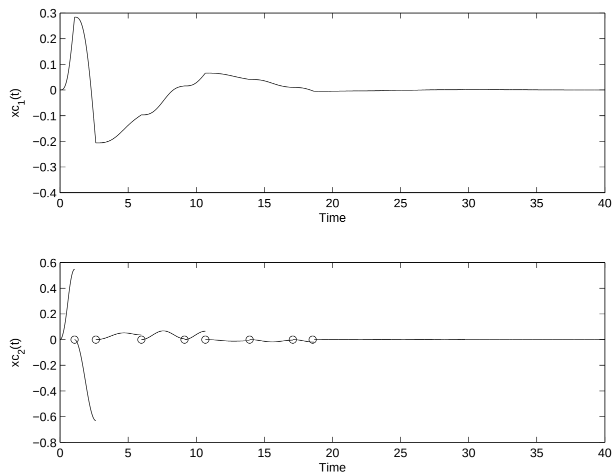

Figure 5.36: Controller position and velocity versus time

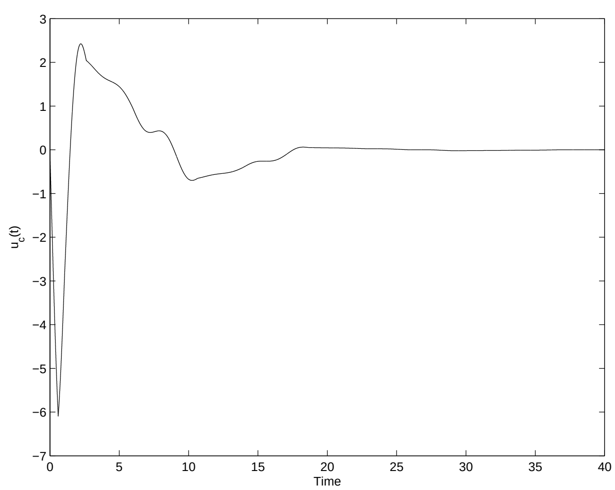

Figure 5.37: Control signal versus time

Chapter 6

Hybrid Decentralized Maximum Entropy Control for Large-Scale Dynamical Systems

6.1. Introduction

Modern complex dynamical systems[5] are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication network constraints. The sheer size (i.e., dimensionality) and complexity of these large-scale dynamical systems often necessitates a decentralized architecture for analyzing and controlling these systems. Specifically, in the control-system design of complex large-scale dynamical systems it is often desirable to treat the overall system as a collection of interconnected subsystems. The behavior of the composite (i.e., large-scale) system can then be predicted from the behaviors of the individual subsystems and their interconnections. The need for decentralized control design of large-scale systems is a direct consequence of the physical size and complexity of the dynamical model. In particular, computational complexity may be too large for model analysis while severe constraints on communication links between system sensors, actuators, and processors may render centralized control architectures impractical. Moreover, even when communication constraints do not exist, decentralized processing may be more economical.

The complexity of modern controlled large-scale dynamical systems is further exacerbated by the use of hierarchial embedded control subsystems within the feedback control system, that is, abstract decision-making units performing logical checks that identity system mode operation and specify the continuous-variable subcontroller to be activated. Such systems typically possess a multiechelon hierarchical hybrid decentralized control architec-

[5] Here we have in mind large flexible space structures, aerospace systems, electric power systems, network systems, economic systems, and ecological systems, to cite but a few examples.

ture characterized by continuous-time dynamics at the lower levels of the hierarchy and discrete-time dynamics at the higher levels of the hierarchy. The lower-level units directly interact with the dynamical system to be controlled while the higher-level units receive information from the lower-level units as inputs and provide (possibly discrete) output commands which serve to coordinate and reconcile the (sometimes competing) actions of the lower-level units. The hierarchical controller organization reduces processor cost and controller complexity by breaking up the processing task into relatively small pieces and decomposing the fast and slow control functions. Typically, the higher-level units perform logical checks that determine system mode operation, while the lower-level units execute continuous-variable commands for a given system mode of operation.

Since implementation constraints, cost, and reliability considerations often require decentralized controller architectures for controlling large-scale systems, decentralized control has received considerable attention in the literature [21, 27, 50, 51, 64, 128–131, 137, 156, 159, 192, 204, 214, 219, 222]. A straightforward decentralized control design technique is that of *sequential optimization* [21, 64, 137], wherein a sequential centralized subcontroller design procedure is applied to an augmented closed-loop plant composed of the actual plant and the remaining subcontrollers. Clearly, a key difficulty with decentralized control predicated on sequential optimization is that of dimensionality. An alternative approach to sequential optimization for decentralized control is based on *subsystem decomposition* with centralized design procedures applied to the individual subsystems of the large-scale system [50, 51, 128–131, 156, 159, 192, 204, 214, 219]. Decomposition techniques exploit subsystem interconnection data and in many cases, such as in the presence of very high system dimensionality, is absolutely essential for designing decentralized controllers.

In this chapter, we develop a novel energy-based hybrid decentralized control framework for lossless and dissipative large-scale dynamical systems [236] based on subsystem decomposition. The notion of energy here refers to abstract energy notions for which a physical system energy interpretation is not necessary. These dynamical systems cover a very broad spectrum of applications including mechanical systems, fluid systems, electromechanical systems, electrical systems, combustion systems, structural vibration systems, biological systems, physiological systems, power systems, telecommunications systems, and economic systems, to cite but a few examples. The concept of an energy-based hybrid decentralized controller can be viewed as a feedback control technique that exploits the coupling between a physical large-scale dynamical system and an energy-based decentralized controller to efficiently remove energy from the physical large-scale system. Specifically, if a dissipative or lossless large-scale system is at high energy level, and a lossless feedback decentralized controller at a low energy level is attached to it, then subsystem energy will generally tend to flow from each subsystem into the corresponding subcontroller, decreasing the subsystem energy and increasing the subcontroller energy [142]. Of course, emulated energy, and not physical energy, is accumulated by each subcontroller. Conversely, if each attached subcontroller is at a high energy level and the corresponding subsystem is at a low energy level, then energy can flow from each subcontroller to each corresponding subsystem, since each subcontroller can generate real, physical energy to effect the required energy flow. Hence, if and when the subcontroller states coincide with a high emulated energy level, then we can *reset* these states to remove the emulated energy so that the emulated energy is not returned to the plant. In this case, the overall closed-loop system consisting of the plant and the controller possesses discontinuous flows since it combines logical switchings with continuous dynamics, leading to impulsive differential equations [14, 15, 52, 98, 99, 147, 215].

6.2. Hybrid Decentralized Control and Large-Scale Impulsive Dynamical Systems

In this chapter, we consider continuous-time nonlinear dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = F(x(t), u(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{6.1}$$

$$y(t) = H(x(t)), \tag{6.2}$$

where $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^l$, $F : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}^n$, $H : \mathcal{D} \to \mathbb{R}^l$, and $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$. Here, we assume that $\mathcal{G}$ represents a large-scale dynamical system composed of q interconnected controlled subsystems $\mathcal{G}_i$ so that, for all $i = 1, ..., q$,

$$F_i(x, u) = f_i(x_i) + \mathcal{I}_i(x) + G_i(x_i)u_i, \tag{6.3}$$

$$H_i(x) = h_i(x_i), \tag{6.4}$$

where $x_i \in \mathcal{D}_i \subseteq \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \triangleq h_i(x_i) \in \mathbb{R}^{l_i}$, (u_i, y_i) is the input-output pair for the ith subsystem, $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ and $\mathcal{I}_i : \mathcal{D} \to \mathbb{R}^{n_i}$ are smooth (i.e., infinitely differentiable) and satisfy $f_i(0) = 0$ and $\mathcal{I}_i(0) = 0$, $G_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i \times m_i}$ is smooth, $h_i : \mathbb{R}^{n_i} \to \mathbb{R}^{l_i}$ and satisfies $h_i(0) = 0$, $\sum_{i=1}^q n_i = n$, $\sum_{i=1}^q m_i = m$, and $\sum_{i=1}^q l_i = l$. Here, $f_i : \mathcal{D}_i \subseteq \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ defines the vector field of each isolated subsystem of (6.1) and $\mathcal{I}_i : \mathcal{D} \to \mathbb{R}^{n_i}$ defines the structure of the interconnection dynamics of the ith subsystem with all other subsystems. Furthermore, for the large-scale dynamical system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied, that is, for every $i \in \{1, ..., q\}$, $u_i(\cdot)$ satisfies sufficient regularity conditions such that the system (6.1) has a unique solution forward in time. We define the composite input and composite output for the large-scale system $\mathcal{G}$ as $u \triangleq [u_1^{\mathrm{T}}, ..., u_q^{\mathrm{T}}]^{\mathrm{T}}$ and $y \triangleq [y_1^{\mathrm{T}}, ..., y_q^{\mathrm{T}}]^{\mathrm{T}}$, respectively.

Next, we consider state-dependent hybrid (resetting) decentralized dynamic controllers $\mathcal{G}_{\mathrm{c}i}$, $i = 1, \ldots, q$, of the form

$$\dot{x}_{\mathrm{c}i}(t) = f_{\mathrm{c}i}(x_{\mathrm{c}i}(t), y_i(t)), \quad x_{\mathrm{c}i}(0) = x_{\mathrm{c}i0}, \quad (x_{\mathrm{c}i}(t), y_i(t)) \notin \mathcal{Z}_{\mathrm{c}i}, \quad t \geq 0, \tag{6.5}$$

$$\Delta x_{\mathrm{c}i}(t) = f_{\mathrm{d}i}(x_{\mathrm{c}i}(t), y_i(t)), \quad (x_{\mathrm{c}i}(t), y_i(t)) \in \mathcal{Z}_{\mathrm{c}i}, \tag{6.6}$$

$$u_i(t) = h_{\mathrm{c}i}(x_{\mathrm{c}i}(t), y_i(t)), \tag{6.7}$$

where $x_{\mathrm{c}i} \in \mathcal{D}_{\mathrm{c}i} \subseteq \mathbb{R}^{n_{\mathrm{c}i}}$, $\mathcal{D}_{\mathrm{c}i}$ is an open set with $0 \in \mathcal{D}_{\mathrm{c}i}$, $y_{\mathrm{c}i} \triangleq h_{\mathrm{c}i}(x_{\mathrm{c}i}, y_i) \in \mathbb{R}^{m_i}$, $f_{\mathrm{c}i} : \mathcal{D}_{\mathrm{c}i} \times \mathbb{R}^{l_i} \to \mathbb{R}^{n_{\mathrm{c}i}}$ is smooth and satisfies $f_{\mathrm{c}i}(0, 0) = 0$, $f_{\mathrm{d}i} : \mathcal{D}_{\mathrm{c}i} \times \mathbb{R}^{l_i} \to \mathbb{R}^{n_{\mathrm{c}i}}$ is continuous, $h_{\mathrm{c}i} : \mathcal{D}_{\mathrm{c}i} \times \mathbb{R}^{l_i} \to \mathbb{R}^{m_i}$ is smooth and satisfies $h_{\mathrm{c}i}(0, 0) = 0$, $\Delta x_{\mathrm{c}i}(t) \triangleq x_{\mathrm{c}i}(t^+) - x_{\mathrm{c}i}(t)$, $\mathcal{Z}_{\mathrm{c}i} \subset \mathcal{D}_{\mathrm{c}i} \times \mathbb{R}^{l_i}$ is the resetting set, and $\sum_{i=1}^q n_{\mathrm{c}i} = n_{\mathrm{c}}$. Note that the hybrid decentralized controller (6.5)–(6.7) represents an impulsive dynamical system $\mathcal{G}_{\mathrm{c}}$ composed of q impulsive subsystems $\mathcal{G}_{\mathrm{c}i}$

involving multiple hybrid processors operating independently, with each processor receiving a subset of the available system measurements and updating a subset of the system actuators. Furthermore, for generality, we allow the hybrid decentralized dynamic controller to be of fixed dimension n_{c} which may be less than the plant order n. In addition, we define the composite input and composite output for the impulsive decentralized dynamic compensator $\mathcal{G}_{\mathrm{c}}$ as $u_{\mathrm{c}} \triangleq y = [u_{\mathrm{c}1}^{\mathrm{T}}, ..., u_{\mathrm{c}q}^{\mathrm{T}}]^{\mathrm{T}}$ and $y_{\mathrm{c}} \triangleq u = [y_{\mathrm{c}1}^{\mathrm{T}}, ..., y_{\mathrm{c}q}^{\mathrm{T}}]^{\mathrm{T}}$, respectively.

The equations of motion for each closed-loop dynamical subsystem $\tilde{\mathcal{G}}_i$, $i = 1, \ldots, q$, have the form

$$\dot{\tilde{x}}_i(t) = \tilde{f}_{\mathrm{c}i}(\tilde{x}_i(t)) + \tilde{\mathcal{I}}_i(x), \quad \tilde{x}_i(0) = \tilde{x}_{i0}, \quad \tilde{x}_i(t) \notin \tilde{\mathcal{Z}}_i, \quad t \geq 0, \tag{6.8}$$

$$\Delta\tilde{x}_i(t) = \tilde{f}_{\mathrm{d}i}(\tilde{x}_i(t)), \quad \tilde{x}_i(t) \in \tilde{\mathcal{Z}}_i, \tag{6.9}$$

where

$$\tilde{x}_i \triangleq \begin{bmatrix} x_i \\ x_{\mathrm{c}i} \end{bmatrix} \in \mathbb{R}^{\tilde{n}_i}, \quad \tilde{f}_{\mathrm{c}i}(\tilde{x}_i) \triangleq \begin{bmatrix} f_i(x_i) + G_i(x_i)h_{\mathrm{c}i}(x_{\mathrm{c}i}, h_i(x_i)) \\ f_{\mathrm{c}i}(x_{\mathrm{c}i}, h_i(x_i)) \end{bmatrix}, \tag{6.10}$$

$$\tilde{\mathcal{I}}_i(x) \triangleq \begin{bmatrix} \mathcal{I}_i(x) \\ 0 \end{bmatrix}, \quad \tilde{f}_{\mathrm{d}i}(\tilde{x}_i) \triangleq \begin{bmatrix} 0 \\ f_{\mathrm{d}i}(x_{\mathrm{c}i}, h_i(x_i)) \end{bmatrix}, \tag{6.11}$$

and $\tilde{\mathcal{Z}}_i \triangleq \{\tilde{x}_i \in \tilde{\mathcal{D}}_i : (x_{\mathrm{c}i}, h_i(x_i)) \in \mathcal{Z}_{\mathrm{c}i}\}$, with $\tilde{n}_i \triangleq n_i + n_{\mathrm{c}i}$ and $\tilde{\mathcal{D}}_i \triangleq \mathcal{D}_i \times \mathcal{D}_{\mathrm{c}i}$, $i = 1, \ldots, q$. Hence, the equations of motion for the closed-loop dynamical system $\tilde{\mathcal{G}}$ have the form

$$\dot{\tilde{x}}(t) = \tilde{f}_{\mathrm{c}}(\tilde{x}(t)), \quad \tilde{x}(0) = \tilde{x}_0, \quad \tilde{x}(t) \notin \tilde{\mathcal{Z}}, \quad t \geq 0, \tag{6.12}$$

$$\Delta\tilde{x}(t) = \tilde{f}_{\mathrm{d}}(\tilde{x}(t)), \quad \tilde{x}(t) \in \tilde{\mathcal{Z}}, \tag{6.13}$$

where $\tilde{x}(t) = [\tilde{x}_1^{\mathrm{T}}(t), \ldots, \tilde{x}_q^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\tilde{f}_{\mathrm{c}}(\tilde{x}) \triangleq [\tilde{f}_{\mathrm{c}1}^{\mathrm{T}}(\tilde{x}_1) + \tilde{\mathcal{I}}_1^{\mathrm{T}}(x), \ldots, \tilde{f}_{\mathrm{c}q}^{\mathrm{T}}(\tilde{x}_q) + \tilde{\mathcal{I}}_q^{\mathrm{T}}(x)]^{\mathrm{T}}$, $\tilde{\mathcal{Z}} \triangleq \cup_{i=1}^{q}\{\tilde{x} \in \tilde{\mathcal{D}} : \tilde{x}_i \in \tilde{\mathcal{Z}}_i\}$, $\tilde{\mathcal{D}} \triangleq \cup_{i=1}^{q}\tilde{\mathcal{D}}_i$, and

$$\tilde{f}_{\mathrm{d}}(\tilde{x}) \triangleq \begin{bmatrix} \tilde{f}_{\mathrm{d}1}(\tilde{x}_1)\chi_{\tilde{\mathcal{Z}}_1}(\tilde{x}_1) \\ \vdots \\ \tilde{f}_{\mathrm{d}q}(\tilde{x}_q)\chi_{\tilde{\mathcal{Z}}_q}(\tilde{x}_q) \end{bmatrix}, \quad \chi_{\tilde{\mathcal{Z}}_i}(\tilde{x}_i) = \begin{cases} 1, & \tilde{x}_i \in \tilde{\mathcal{Z}}_i \\ 0, & \tilde{x}_i \notin \tilde{\mathcal{Z}}_i \end{cases}, \quad i = 1, \ldots, q. \tag{6.14}$$

We refer to the differential equation (6.12) as the *continuous-time dynamics*, and we refer to the difference equation (6.13) as the *resetting law*. Note that although the closed-loop state vector consists of plant states and controller states, it is clear from (6.11) that only those states associated with the controller are reset. A function $\tilde{x} : \mathcal{I}_{\tilde{x}_0} \to \tilde{\mathcal{D}}$ is a *solution* to the impulsive dynamical system (6.12) and (6.13) on the interval $\mathcal{I}_{\tilde{x}_0} \subseteq \mathbb{R}$ with initial condition $\tilde{x}(0) = \tilde{x}_0$ if $\tilde{x}(\cdot)$ is left-continuous and $\tilde{x}(t)$ satisfies (6.12) and (6.13) for all $t \in \mathcal{I}_{\tilde{x}_0}$. For further discussion on solutions to impulsive differential equations, see [14,15,41,52,98,99,147,175,215,241]. For convenience, we use the notation $\tilde{s}(t, \tilde{x}_0)$ to denote the solution $\tilde{x}(t)$ of (6.12) and (6.13) at time $t \geq 0$ with initial condition $\tilde{x}(0) = \tilde{x}_0$.

For a particular closed-loop trajectory $\tilde{x}(t)$, we let $t_k \triangleq \tau_k(\tilde{x}_0)$ denote the kth instant of time at which $\tilde{x}(t)$ intersects $\tilde{\mathcal{Z}}$, and we call the times t_k the *resetting times*. Thus, the trajectory of the closed-loop system $\tilde{\mathcal{G}}$ from the initial condition $\tilde{x}(0) = \tilde{x}_0$ is given by $\tilde{\psi}(t, \tilde{x}_0)$ for $0 < t \leq t_1$, where $\tilde{\psi}(t, \tilde{x}_0)$ denotes the solution to the continuous-time dynamics of the closed-loop system $\tilde{\mathcal{G}}$. If and when the trajectory reaches a state $\tilde{x}(t_1)$ satisfying $\tilde{x}(t_1) \in \tilde{\mathcal{Z}}$, then the state is instantaneously transferred to $\tilde{x}(t_1^+) \triangleq \tilde{x}(t_1) + \tilde{f}_{\mathrm{d}}(\tilde{x}(t_1))$ according to the resetting law (6.13). The trajectory $\tilde{x}(t)$, $t_1 < t \leq t_2$, is then given by $\tilde{\psi}(t - t_1, \tilde{x}(t_1^+))$, and so on. Our convention here is that the solution $\tilde{x}(t)$ of $\tilde{\mathcal{G}}$ is left-continuous, that is, it is continuous everywhere except at the resetting times t_k, and

$$\tilde{x}_k \triangleq \tilde{x}(t_k) = \lim_{\varepsilon \to 0^+} \tilde{x}(t_k - \varepsilon), \tag{6.15}$$

$$\tilde{x}_k^+ \triangleq \tilde{x}(t_k) + \tilde{f}_{\mathrm{d}}(\tilde{x}(t_k)) = \lim_{\varepsilon \to 0^+} \tilde{x}(t_k + \varepsilon), \tag{6.16}$$

for $k = 1, 2, \ldots$.

To ensure the well-posedness of the resetting times, we make the following additional assumptions (see Assumptions 1 and 2 of Section 5.2):

Assumption 1. If $\tilde{x} \in \overline{\tilde{\mathcal{Z}}} \setminus \tilde{\mathcal{Z}}$, then there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$, $\tilde{\psi}(\delta, \tilde{x}) \notin \tilde{\mathcal{Z}}$.

Assumption 2. If $\tilde{x} \in \tilde{\mathcal{Z}}$, then $\tilde{x} + \tilde{f}_{\mathrm{d}}(\tilde{x}) \notin \tilde{\mathcal{Z}}$.

For the statement of the next result the following key assumption is needed.

Assumption 3. Consider the closed-loop impulsive dynamical system $\tilde{\mathcal{G}}$. Then for every $\tilde{x}_0 \notin \tilde{\mathcal{Z}}$ and every $\varepsilon > 0$ and $t \neq t_k$, there exists $\delta(\varepsilon, \tilde{x}_0, t) > 0$ such that if $\|\tilde{x}_0 - y\| < \delta(\varepsilon, \tilde{x}_0, t)$, $y \in \tilde{\mathcal{D}}$, then $\|\tilde{s}(t, \tilde{x}_0) - \tilde{s}(t, y)\| < \varepsilon$.

As discussed in Section 5, Assumption 3 is a weakened version of the quasi-continuous dependence assumption given in [52, 98], and is a generalization of the standard continuous dependence property for dynamical systems with continuous flows to dynamical systems with left-continuous flows.

Proposition 6.1. Consider the large-scale impulsive dynamical system $\tilde{\mathcal{G}}$ given by the feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$. Assume that Assumptions 1 and 2 hold, $\tau_1(\cdot)$ is continuous at every $\tilde{x} \notin \tilde{\mathcal{Z}}$ such that $0 < \tau_1(\tilde{x}) < \infty$, and if $\tilde{x} \in \tilde{\mathcal{Z}}$, then $\tilde{x} + \tilde{f}_{\rm d}(\tilde{x}) \in \overline{\tilde{\mathcal{Z}}} \backslash \tilde{\mathcal{Z}}$. Furthermore, for every $\tilde{x} \in \overline{\tilde{\mathcal{Z}}} \backslash \tilde{\mathcal{Z}}$ such that $0 < \tau_1(\tilde{x}) < \infty$, assume that the following statements hold:

i) If a sequence $\{\tilde{x}_{(i)}\}_{i=1}^{\infty} \in \tilde{\mathcal{D}}$ is such that $\lim_{i\to\infty} \tilde{x}_{(i)} = \tilde{x}$ and $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)})$ exists, then either $\tilde{f}_{\rm d}(\tilde{x}) = 0$ and $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)}) = 0$, or $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)}) = \tau_1(\tilde{x})$.

ii) If a sequence $\{\tilde{x}_{(i)}\}_{i=1}^{\infty} \in \overline{\tilde{\mathcal{Z}}} \backslash \tilde{\mathcal{Z}}$ is such that $\lim_{i\to\infty} \tilde{x}_{(i)} = \tilde{x}$ and $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)})$ exists, then $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)}) = \tau_1(\tilde{x})$.

Then $\tilde{\mathcal{G}}$ satisfies Assumption 3.

Proof. The proof is similar to the proof of Proposition 5.1 of Section 5.2 and, hence, is omitted. □

The following result provides sufficient conditions for establishing continuity of $\tau_1(\cdot)$ at $\tilde{x}_0 \notin \overline{\tilde{\mathcal{Z}}}$ and *sequential continuity* of $\tau_1(\cdot)$ at $\tilde{x}_0 \in \overline{\tilde{\mathcal{Z}}} \backslash \tilde{\mathcal{Z}}$, that is, $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)}) = \tau_1(\tilde{x}_0)$ for $\{\tilde{x}_{(i)}\}_{i=1}^{\infty} \notin \tilde{\mathcal{Z}}$ and $\lim_{i\to\infty} \tilde{x}_{(i)} = \tilde{x}_0$. For this result, the following definition is needed. First, however, recall that the *Lie derivative* of a smooth function $\mathcal{X} : \tilde{\mathcal{D}} \to \mathbb{R}$ along the vector field of the continuous-time dynamics $\tilde{f}_{\rm c}(\tilde{x})$ is given by $L_{\tilde{f}_{\rm c}}\mathcal{X}(\tilde{x}) \triangleq \frac{\rm d}{{\rm d}t}\mathcal{X}(\tilde{\psi}(t, \tilde{x}))|_{t=0} = \frac{\partial \mathcal{X}(\tilde{x})}{\partial \tilde{x}} \tilde{f}_{\rm c}(\tilde{x})$, and the *zeroth* and *higher-order Lie derivatives* are, respectively, defined by $L^0_{\tilde{f}_c}\mathcal{X}(\tilde{x}) \triangleq \mathcal{X}(\tilde{x})$ and $L^k_{\tilde{f}_c}\mathcal{X}(\tilde{x}) \triangleq L_{\tilde{f}_c}(L^{k-1}_{\tilde{f}_c}\mathcal{X}(\tilde{x}))$, where $k \geq 1$.

Definition 6.1. Let $\mathcal{M} \triangleq \cup_{i=1}^q \{\tilde{x} \in \tilde{\mathcal{D}} : \ \mathcal{X}_i(\tilde{x}) = 0\}$, where $\mathcal{X}_i : \tilde{\mathcal{D}} \to \mathbb{R}$, $i = 1, \ldots, q$, are infinitely differentiable functions. A point $\tilde{x} \in \mathcal{M}$ such that $\tilde{f}_c(\tilde{x}) \neq 0$ is *transversal* to (6.12) if there exist $k_i \in \{1, 2, \ldots\}$, $i = 1, \ldots, q$, such that

$$L^r_{\tilde{f}_c}\mathcal{X}_i(\tilde{x}) = 0, \quad r = 0, \ldots, 2k_i - 2, \quad L^{2k_i-1}_{\tilde{f}_c}\mathcal{X}_i(\tilde{x}) \neq 0, \quad i = 1, \ldots, q. \tag{6.17}$$

Proposition 6.2. Consider the large-scale impulsive dynamical system $\tilde{\mathcal{G}}$ given by the feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$. Let $\mathcal{X}_i : \tilde{\mathcal{D}} \to \mathbb{R}$, $i = 1, \ldots, q$, be infinitely differentiable functions such that $\overline{\tilde{\mathcal{Z}}} = \cup_{i=1}^q \{\tilde{x} \in \tilde{\mathcal{D}} : \mathcal{X}_i(\tilde{x}) = 0\}$, and assume that every $\tilde{x} \in \overline{\tilde{\mathcal{Z}}}$ is transversal to (6.12). Then at every $\tilde{x}_0 \notin \overline{\tilde{\mathcal{Z}}}$ such that $0 < \tau_1(\tilde{x}_0) < \infty$, $\tau_1(\cdot)$ is continuous. Furthermore, if $\tilde{x}_0 \in \overline{\tilde{\mathcal{Z}}}\backslash\tilde{\mathcal{Z}}$ is such that $\tau_1(\tilde{x}_0) \in (0, \infty)$ and *i*) $\{\tilde{x}_{(i)}\}_{i=1}^\infty \in \overline{\tilde{\mathcal{Z}}}\backslash\tilde{\mathcal{Z}}$ or *ii*) $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)}) > 0$, where $\{\tilde{x}_{(i)}\}_{i=1}^\infty \notin \overline{\tilde{\mathcal{Z}}}$ is such that $\lim_{i\to\infty} \tilde{x}_{(i)} = \tilde{x}_0$ and $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)})$ exists, then $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)}) = \tau_1(\tilde{x}_0)$.

Proof. The proof is similar to the proof of Proposition 5.4 of Section 5.7 and, hence, is omitted. □

Remark 6.1. Let $\tilde{x}_0 \notin \tilde{\mathcal{Z}}$ be such that $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)}) \neq \tau_1(\tilde{x}_0)$ for some sequence $\{\tilde{x}_{(i)}\}_{i=1}^\infty \notin \tilde{\mathcal{Z}}$. Then it follows from Proposition 6.2 that $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)}) = 0$.

Remark 6.2. Proposition 6.2 is a nontrivial generalization of Proposition 4.2 of [52] and Lemma 3 of [84]. Specifically, Proposition 6.2 establishes the continuity of $\tau_1(\cdot)$ in the case where the resetting set $\tilde{\mathcal{Z}}$ is not a closed set. In addition, the transversality condition given in Definition 6.1 is also a generalization of the conditions given in [52] and [84] by considering higher-order derivatives of the function $\mathcal{X}_i(\cdot)$ rather than simply considering the first-order derivative as in [52,84]. This condition guarantees that the solution of the closed-loop system

(6.8) and (6.9) is not tangent to the closure of the resetting set $\tilde{\mathcal{Z}}$ at the intersection with $\overline{\tilde{\mathcal{Z}}}$.

The next result characterizes impulsive dynamical system limit sets in terms of continuously differentiable functions. In particular, we show that the system trajectories of a state-dependent impulsive dynamical system converge to an invariant set contained in a union of level surfaces characterized by the continuous-time system dynamics and the resetting system dynamics. For the next result assume that $\tilde{f}_{\rm c}(\cdot)$, $\tilde{f}_{\rm d}(\cdot)$, $\tilde{\mathcal{I}}(\cdot)$, and $\tilde{\mathcal{Z}}$ are such that the dynamical system $\tilde{\mathcal{G}}$ given by (6.12) and (6.13) satisfies Assumptions 1–3. Note that for addressing the stability of the zero solution of an impulsive dynamical system the usual stability definitions are valid. For details, see [14, 15, 52, 98, 99, 147, 215].

Theorem 6.1. Consider the impulsive dynamical system (6.12) and (6.13) and assume Assumptions 1–3 hold. Assume $\tilde{\mathcal{D}}_{\rm ci} \subset \tilde{\mathcal{D}}$ is a compact positively invariant set with respect to (6.12) and (6.13), assume that if $\tilde{x}_0 \in \tilde{\mathcal{Z}}$ then $\tilde{x}_0 + \tilde{f}_{\rm d}(\tilde{x}_0) \in \overline{\tilde{\mathcal{Z}}}\backslash\tilde{\mathcal{Z}}$, and assume that there exist a continuously differentiable function $V : \tilde{\mathcal{D}}_{\rm ci} \to \mathbb{R}$ such that

$$V'(\tilde{x})\tilde{f}_{\rm c}(\tilde{x}) \leq 0, \quad \tilde{x} \in \tilde{\mathcal{D}}_{\rm ci}, \quad \tilde{x} \notin \tilde{\mathcal{Z}}, \tag{6.18}$$

$$V(\tilde{x} + \tilde{f}_{\rm d}(\tilde{x})) \leq V(\tilde{x}), \quad \tilde{x} \in \tilde{\mathcal{D}}_{\rm ci}, \quad \tilde{x} \in \tilde{\mathcal{Z}}. \tag{6.19}$$

Let $\mathcal{R} \triangleq \{\tilde{x} \in \tilde{\mathcal{D}}_{\rm ci} : \tilde{x} \notin \tilde{\mathcal{Z}}, V(\tilde{x})\tilde{f}_{\rm c}(\tilde{x}) = 0\} \cup \{\tilde{x} \in \tilde{\mathcal{D}}_{\rm ci} : \tilde{x} \in \tilde{\mathcal{Z}}, V(\tilde{x} + \tilde{f}_{\rm d}(\tilde{x})) - V(\tilde{x}) = 0\}$ and let $\mathcal{M}$ denote the largest invariant set contained in $\mathcal{R}$. If $\tilde{x}_0 \in \tilde{\mathcal{D}}_{\rm ci}$, then $\tilde{x}(t) \to \mathcal{M}$ as $t \to \infty$. Furthermore, if $0 \in \overset{\circ}{\tilde{\mathcal{D}}}_{\rm ci}$, $V(0) = 0$, $V(\tilde{x}) > 0$, $\tilde{x} \neq 0$, and the set $\mathcal{R}$ contains no invariant set other than the set $\{0\}$, then the zero solution $\tilde{x}(t) \equiv 0$ to (6.12) and (6.13) is asymptotically stable and $\tilde{\mathcal{D}}_{\rm ci}$ is a subset of the domain of attraction of (6.12) and (6.13).

Proof. The proof is similar to the proof of Corollary 5.1 given in [52] and, hence, is omitted. $\square$

Remark 6.3. Setting $\tilde{\mathcal{D}} = \mathbb{R}^n$ and requiring $V(\tilde{x}) \to \infty$ as $\|\tilde{x}\| \to \infty$ in Theorem 6.1, it follows that the zero solution $\tilde{x}(t) \equiv 0$ to (6.12) and (6.13) is globally asymptotically stable. A similar remark holds for Theorem 6.2 below.

Theorem 6.2. Consider the impulsive dynamical system $\tilde{\mathcal{G}}$ (6.12) and (6.13) and assume Assumptions 1–3 hold. Assume $\tilde{\mathcal{D}}_{\mathrm{ci}} \subset \tilde{\mathcal{D}}$ is a compact positively invariant set with respect to (6.12) and (6.13) such that $0 \in \overset{\circ}{\tilde{\mathcal{D}}}_{\mathrm{ci}}$, assume that if $\tilde{x}_0 \in \tilde{\mathcal{Z}}$ then $\tilde{x}_0 + \tilde{f}_{\mathrm{d}}(\tilde{x}_0) \in \overline{\tilde{\mathcal{Z}}} \backslash \tilde{\mathcal{Z}}$, and assume that for all $\tilde{x}_0 \in \tilde{\mathcal{D}}_{\mathrm{ci}}$, $\tilde{x}_0 \neq 0$, there exists $\tau \geq 0$ such that $\tilde{x}(\tau) \in \tilde{\mathcal{Z}}$, where $\tilde{x}(t)$, $t \geq 0$, denotes the solution to (6.12) and (6.13) with the initial condition $\tilde{x}_0$. Furthermore, assume that there exist a continuously differentiable vector function $V = [v_1, \ldots, v_q]^{\mathrm{T}} : \tilde{\mathcal{D}} \to \overline{\mathbb{R}}_+^q$ and a positive vector $p \in \mathbb{R}_+^q$ such that $V(0) = 0$, the scalar function $v : \tilde{\mathcal{D}} \to \overline{\mathbb{R}}_+$ defined by $v(\tilde{x}) \triangleq p^{\mathrm{T}} V(\tilde{x})$, $\tilde{x} \in \tilde{\mathcal{D}}$, is such that $v(\tilde{x}) > 0$, $\tilde{x} \in \tilde{\mathcal{D}}$, $\tilde{x} \neq 0$, and

$$v'(\tilde{x}) \tilde{f}_{\mathrm{c}}(\tilde{x}) \leq 0, \quad \tilde{x} \in \tilde{\mathcal{D}}_{\mathrm{ci}}, \quad \tilde{x} \notin \tilde{\mathcal{Z}}, \tag{6.20}$$

$$v(\tilde{x} + \tilde{f}_{\mathrm{d}}(\tilde{x})) < v(\tilde{x}), \quad \tilde{x} \in \tilde{\mathcal{D}}_{\mathrm{ci}}, \quad \tilde{x} \in \tilde{\mathcal{Z}}. \tag{6.21}$$

Then the zero solution $\tilde{x}(t) \equiv 0$ to (6.12) and (6.13) is asymptotically stable and $\tilde{\mathcal{D}}_{\mathrm{ci}}$ is a subset of the domain of attraction of (6.12) and (6.13).

Proof. It follows from (6.21) that $\mathcal{R} = \{\tilde{x} \in \tilde{\mathcal{D}}_{\mathrm{ci}} : \tilde{x} \notin \tilde{\mathcal{Z}}, v'(\tilde{x}) \tilde{f}_{\mathrm{c}}(\tilde{x}) = 0\}$. Since for all $\tilde{x}_0 \in \tilde{\mathcal{D}}_{\mathrm{ci}}$, $\tilde{x}_0 \neq 0$, there exists $\tau \geq 0$ such that $\tilde{x}(\tau) \in \tilde{\mathcal{Z}}$, it follows that the largest invariant set contained in $\mathcal{R}$ is $\{0\}$. Now, the result is a direct consequence of Theorem 6.1. $\square$

6.3. Hybrid Decentralized Control for Large-Scale Dynamical Systems

In this section, we present a hybrid decentralized controller design framework for large-scale dynamical systems. Specifically, we consider nonlinear large-scale dynamical systems $\mathcal{G}$ of the form given by (6.1) and (6.2) where $u(\cdot)$ satisfies sufficient regularity conditions such that (6.1) has a unique solution forward in time. Furthermore, we consider hybrid decentralized dynamic controllers $\mathcal{G}_{ci}$, $i = 1, \ldots, q$, of the form

$$\dot{x}_{ci}(t) = f_{ci}(x_{ci}(t), y_i(t)), \quad x_{ci}(0) = x_{c0i}, \quad (x_{ci}(t), y_i(t)) \notin \mathcal{Z}_{ci}, \tag{6.22}$$

$$\Delta x_{ci}(t) = \eta_i(y_i(t)) - x_{ci}(t), \quad (x_{ci}(t), y_i(t)) \in \mathcal{Z}_{ci}, \tag{6.23}$$

$$y_{ci}(t) = h_{ci}(x_{ci}(t), y_i(t)), \tag{6.24}$$

where $x_{ci}(t) \in \mathcal{D}_{ci} \subseteq \mathbb{R}^{n_{ci}}$, $\mathcal{D}_{ci}$ is an open set with $0 \in \mathcal{D}_{ci}$, $y_i(t) \in \mathbb{R}^{l_i}$, $y_{ci}(t) \in \mathbb{R}^{m_i}$, $f_{ci} : \mathcal{D}_{ci} \times \mathbb{R}^{l_i} \to \mathbb{R}^{n_{ci}}$ is smooth on $\mathcal{D}_{ci}$ and satisfies $f_{ci}(0, 0) = 0$, $\eta_i : \mathbb{R}^{l_i} \to \mathcal{D}_{ci}$ is continuous and satisfies $\eta_i(0) = 0$, $h_{ci} : \mathcal{D}_{ci} \times \mathbb{R}^{l_i} \to \mathbb{R}^{m_i}$ is smooth and satisfies $h_{ci}(0, 0) = 0$, $\sum_{i=1}^{q} l_i = l$, and $\sum_{i=1}^{q} m_i = m$.

Recall that for the dynamical system $\mathcal{G}$ given by (6.1) and (6.2), a vector function $S(u, y) \triangleq [s_1(u_1, y_1), \ldots, s_q(u_q, y_q)]^{\mathrm{T}}$, where $S : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}^q$ is such that $S(0, 0) = 0$, is called a *vector supply rate* [102, 103] if it is componentwise locally integrable for all input-output pairs satisfying (6.1) and (6.2), that is, for every $i \in \{1, \ldots, q\}$ and for all input-output pairs $(u_i, y_i) \in \mathcal{U}_i \times \mathcal{Y}_i$ satisfying (6.1) and (6.2), $s_i(\cdot, \cdot)$ satisfies $\int_{t_1}^{t_2} |s_i(u_i(\sigma), y_i(\sigma))| \mathrm{d}\sigma < \infty$, $t_2 \geq t_1 \geq 0$. Here, $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_q$ and $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{U}_q$ are input and output spaces, respectively, that are assumed to be closed under the shift operator. Furthermore, we assume that $\mathcal{G}$ is *vector lossless with respect to the vector supply rate* $S(u, y)$, and hence, there exist a continuous, nonnegative definite *vector storage function* $V_{\mathrm{s}} = [v_{\mathrm{s}1}, \ldots, v_{\mathrm{s}q}]^{\mathrm{T}} : \mathcal{D} \to \overline{\mathbb{R}}_+^q$ and a *Kamke function* $w : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ such that $V_{\mathrm{s}}(0) = 0$, $w(0) = 0$, the zero solution $z(t) \equiv 0$ to the comparison system

$$\dot{z}(t) = w(z(t)), \quad z(0) = z_0, \quad t \geq 0, \tag{6.25}$$

is Lyapunov stable, and the *vector dissipation equality*

$$V_{\mathrm{s}}(x(t)) = V_{\mathrm{s}}(x(t_0)) + \int_{t_0}^{t} w(V_{\mathrm{s}}(x(\sigma))) \mathrm{d}\sigma + \int_{t_0}^{t} S(u(\sigma), y(\sigma)) \mathrm{d}\sigma, \tag{6.26}$$

is satisfied for all $t \geq t_0 \geq 0$, where $x(t)$, $t \geq t_0$, is the solution to $\mathcal{G}$ with $u \in \mathcal{U}$.

In this case, it follows from Theorem 3.2 of [102] that there exists a nonnegative vector $p \in \overline{\mathbb{R}}_+^q$, $p \neq 0$, such that $\mathcal{G}$ is lossless with respect to the supply rate $p^{\mathrm{T}} S(u, y)$ and with the storage function $v_{\mathrm{s}}(x) = p^{\mathrm{T}}V_{\mathrm{s}}(x)$, $x \in \mathcal{D}$. In addition, we assume that the nonlinear large-scale dynamical system $\mathcal{G}$ is *completely reachable* [236] and *zero-state observable* [236], and there exist functions $\kappa_i : \mathcal{Y}_i \to \mathcal{U}_i$ such that $\kappa_i(0) = 0$ and $s_i(\kappa_i(y_i), y_i) < 0$, $y_i \neq 0$, for all $i = 1, \ldots, q$, so that all storage functions $v_{\mathrm{s}}(x) = p^{\mathrm{T}}V_{\mathrm{s}}(x)$, $x \in \mathcal{D}$, are positive definite, that is, $p^{\mathrm{T}}V_{\mathrm{s}}(x) > 0$, $x \in \mathcal{D}$, $x \neq 0$ [102]. Finally, we assume that $V_{\mathrm{s}}(\cdot)$ is component decoupled, that is, $V_{\mathrm{s}}(x) = [v_{\mathrm{s}1}(x_1), \ldots, v_{\mathrm{s}q}(x_q)]^{\mathrm{T}}$, $x \in \mathcal{D}$, and continuously differentiable. Note that if each *disconnected* subsystem $\mathcal{G}_i$ (i.e., $\mathcal{I}_i(x) \equiv 0$, $i \in \{1, \ldots, q\}$) of $\mathcal{G}$ is lossless with respect to the supply rate $s_i(u_i, y_i)$, then $V_{\mathrm{s}}(\cdot)$ is component decoupled.

Consider the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ given by $y_i = u_{\mathrm{c}i}$ and $u_i = -y_{\mathrm{c}i}$, $i = 1, \ldots, q$. In this case, the closed-loop system $\tilde{\mathcal{G}}$ can be written in terms of the subsystems $\tilde{\mathcal{G}}_i$, $i = 1, \ldots, q$, given by

$$\dot{\tilde{x}}_i(t) = \tilde{f}_{\mathrm{c}i}(\tilde{x}_i(t)) + \tilde{\mathcal{I}}_i(x), \quad \tilde{x}_i(0) = \tilde{x}_{i0}, \quad \tilde{x}_i(t) \notin \tilde{\mathcal{Z}}_i, \quad t \geq 0, \tag{6.27}$$

$$\Delta\tilde{x}_i(t) = \tilde{f}_{\mathrm{d}i}(\tilde{x}_i(t)), \quad \tilde{x}_i(t) \in \tilde{\mathcal{Z}}_i, \tag{6.28}$$

where $t \geq 0$, $\tilde{x}_i(t) \triangleq [x_i^{\mathrm{T}}(t), x_{\mathrm{c}i}^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\tilde{\mathcal{Z}}_i \triangleq \{\tilde{x}_i \in \tilde{\mathcal{D}}_i : (x_{\mathrm{c}i}, h_i(x_i)) \in \mathcal{Z}_{\mathrm{c}i}\}$,

$$\tilde{f}_{\mathrm{c}i}(\tilde{x}_i) \triangleq \left[\begin{array}{c} f_i(x_i) - G_i(x_i)h_{\mathrm{c}i}(x_{\mathrm{c}i}, h_i(x_i)) \\ f_{\mathrm{c}i}(x_{\mathrm{c}i}, h_i(x_i)) \end{array}\right], \quad \tilde{\mathcal{I}}_i(x) \triangleq \left[\begin{array}{c} \mathcal{I}_i(x) \\ 0 \end{array}\right], \tag{6.29}$$

$$\tilde{f}_{\mathrm{d}i}(\tilde{x}_i) \triangleq \left[\begin{array}{c} 0 \\ \eta_i(h_i(x_i)) - x_{\mathrm{c}i} \end{array}\right]. \tag{6.30}$$

Hence, the equations of the motion for the closed-loop system $\tilde{\mathcal{G}}$ have the form

$$\dot{\tilde{x}}(t) = \tilde{f}_{\mathrm{c}}(\tilde{x}(t)), \quad \tilde{x}(t_0) = \tilde{x}_0, \quad \tilde{x}(t) \notin \tilde{\mathcal{Z}}, \quad t \geq t_0, \tag{6.31}$$

$$\Delta\tilde{x}(t) = \tilde{f}_{\mathrm{d}}(\tilde{x}(t)), \quad \tilde{x}(t) \in \tilde{\mathcal{Z}}, \tag{6.32}$$

where $\tilde{x}(t) = [\tilde{x}_1^{\mathrm{T}}(t), \ldots, \tilde{x}_q^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\tilde{f}_{\mathrm{c}}(\tilde{x}) \triangleq [\tilde{f}_{\mathrm{c}1}^{\mathrm{T}}(\tilde{x}_1) + \tilde{\mathcal{I}}_1^{\mathrm{T}}(x), \ldots, \tilde{f}_{\mathrm{c}q}^{\mathrm{T}}(\tilde{x}_q) + \tilde{\mathcal{I}}_q^{\mathrm{T}}(x)]^{\mathrm{T}}$, $\tilde{\mathcal{Z}} \triangleq \cup_{i=1}^{q}\{\tilde{x} \in \tilde{\mathcal{D}} : \tilde{x}_i \in \tilde{\mathcal{Z}}_i\}$, $\tilde{\mathcal{D}} \triangleq \cup_{i=1}^{q}\tilde{\mathcal{D}}_i$, and

$$\tilde{f}_{\mathrm{d}}(\tilde{x}) \triangleq \left[\begin{array}{c} \tilde{f}_{\mathrm{d}1}(\tilde{x}_1)\chi_{\tilde{\mathcal{Z}}_1}(\tilde{x}_1) \\ \vdots \\ \tilde{f}_{\mathrm{d}q}(\tilde{x}_q)\chi_{\tilde{\mathcal{Z}}_q}(\tilde{x}_q) \end{array}\right], \quad \chi_{\tilde{\mathcal{Z}}_i}(\tilde{x}_i) = \left\{\begin{array}{ll} 1, & \tilde{x}_i \in \tilde{\mathcal{Z}}_i \\ 0, & \tilde{x}_i \notin \tilde{\mathcal{Z}}_i \end{array}\right., \quad i = 1, \ldots, q. \tag{6.33}$$

Assume that there exist infinitely differentiable functions $v_{ci} : \mathcal{D}_{ci} \times \mathbb{R}^{l_i} \to \overline{\mathbb{R}}_+$, $i = 1, \ldots, q$, such that $v_{ci}(x_{ci}, y_i) \geq 0$, $x_{ci} \in \mathcal{D}_{ci}$, $y_i \in \mathbb{R}^{l_i}$, and $v_{ci}(x_{ci}, y_i) = 0$ if and only if $x_{ci} = \eta_i(y_i)$ and

$$\dot{v}_{ci}(x_{ci}(t), y_i(t)) = s_{ci}(u_{ci}(t), y_{ci}(t)), \quad (x_{ci}(t), y_i(t)) \notin \tilde{\mathcal{Z}}_i, \quad t \geq 0, \tag{6.34}$$

where $s_{ci} : \mathbb{R}^{l_i} \times \mathbb{R}^{m_i} \to \mathbb{R}$ is such that $s_{ci}(0, 0) = 0$, $i = 1, \ldots, q$.

We associate with the plant a positive-definite, continuously differentiable function $v_{p}(x) \triangleq p^{T} V_{s}(x)$, which we will refer to as the *plant energy* composed of the *subsystem energies* $v_{si}(x_i)$, $i = 1, \ldots, q$. Furthermore, we associate with the controller a nonnegative-definite, infinitely differentiable function $v_{c}(x_{c}, y) \triangleq p^{T} V_{c}(x_{c}, y)$, where $V_{c}(x_{c}, y) \triangleq [v_{c1}(x_{c1}, y_1), \ldots, v_{cq}(x_{cq}, y_q)]^{T}$, called the controller *emulated energy* composed of the *subcontroller emulated energies* $v_{ci}(x_{ci}, y_i)$, $i = 1, \ldots, q$. Finally, we associate with the closed-loop system the function

$$v(\tilde{x}) \triangleq v_{p}(x) + v_{c}(x_{c}, H(x)), \tag{6.35}$$

called the *total energy* composed of the *total subsystem energies* $v_{si}(x_i) + v_{ci}(x_{ci}, y_i)$, $i = 1, \ldots, q$.

Next, we construct the resetting set for each subsystem $\tilde{\mathcal{G}}_i$, $i = 1 \ldots, q$, of the closed-loop system $\tilde{\mathcal{G}}$ in the following form

$$\begin{aligned} \tilde{\mathcal{Z}}_i = & \left\{(x_i, x_{ci}) \in \mathcal{D} \times \mathcal{D}_{ci} : L_{\tilde{f}_{c}} v_{ci}(x_{ci}, h_i(x_i)) = 0 \text{ and } v_{ci}(x_{ci}, h_i(x_i)) > 0\right\} \\ = & \left\{(x_i, x_{ci}) \in \mathcal{D} \times \mathcal{D}_{ci} : s_{ci}(h_i(x_i), h_{ci}(x_{ci}, h_i(x_i))) = 0 \text{ and } v_{ci}(x_{ci}, h_i(x_i)) > 0\right\}, \end{aligned} \tag{6.36}$$

where $i = 1, \ldots, q$. The resetting sets $\tilde{\mathcal{Z}}_i$, $i = 1, \ldots, q$, are thus defined to be the sets of all points in the closed-loop state space that correspond to decreasing subcontroller emulated energy. By resetting the subcontroller states, the subsystem energy can never increase after the first resetting event. Furthermore, if the closed-loop subsystem total energy is conserved between resetting events, then a decrease in subsystem energy is accompanied by a corresponding increase in subsystem emulated energy. Hence, this approach allows the subsystem energy to flow to the subcontroller, where it increases the subcontroller emulated energy but does not allow the subcontroller emulated energy to flow back to the subsystem after the first resetting event. This energy dissipating hybrid decentralized controller effectively enforces a one-way energy transfer between each subsystem and corresponding subcontroller after the first resetting event. For practical implementation, knowledge of x_{ci} and y_i is sufficient to determine whether or not the closed-loop state vector is in the set $\tilde{\mathcal{Z}}_i$, $i = 1, \ldots, q$.

The next theorem gives sufficient conditions for asymptotic stability of the closed-loop system $\tilde{\mathcal{G}}$ using state-dependent hybrid decentralized controllers.

Theorem 6.3. Consider the closed-loop impulsive dynamical system $\tilde{\mathcal{G}}$ given by (6.31) and (6.32). Assume that $\tilde{\mathcal{D}}_{ci} \subset \tilde{\mathcal{D}}$ is a compact positively invariant set with respect to $\tilde{\mathcal{G}}$ such that $0 \in \overset{\circ}{\tilde{\mathcal{D}}}_{ci}$, assume that $\mathcal{G}$ is vector lossless with respect to the vector supply rate $S(u, y) \triangleq [s_1(u_1, y_1), \ldots, s_q(u_q, y_q)]^{\mathrm{T}}$ and with a positive, continuously differentiable vector storage function $V_{\mathrm{s}}(x) = [v_{\mathrm{s}1}(x_1), \ldots, v_{\mathrm{s}q}(x_q)]^{\mathrm{T}}$, $x \in \mathcal{D}$. In addition, assume there exist smooth functions $v_{ci} : \mathcal{D}_{ci} \times \mathbb{R}^{l_i} \to \overline{\mathbb{R}}_+$ such that $v_{ci}(x_{ci}, y_i) \geq 0$, $x_{ci} \in \mathcal{D}_{ci}$, $y_i \in \mathbb{R}^{l_i}$, $v_{ci}(x_{ci}, y_i) = 0$ if and only if $x_{ci} = \eta_i(y_i)$, and (6.34) holds. Finally, assume that every $\tilde{x}_0 \in \overline{\tilde{\mathcal{Z}}}$ is transversal to (6.27) and

$$s_i(u_i, y_i) + s_{ci}(u_{ci}, y_{ci}) = 0, \quad \tilde{x}_i \notin \tilde{\mathcal{Z}}_i, \quad i = 1, \ldots, q, \tag{6.37}$$

where $y_i = u_{ci} = h_i(x_i)$, $u_i = -y_{ci} = -h_{ci}(x_{ci}, h_i(x_i))$, and $\tilde{\mathcal{Z}}_i$, $i = 1, \ldots, q$, is given by (6.36). Then the zero solution $\tilde{x}(t) \equiv 0$ to the closed-loop system $\tilde{\mathcal{G}}$ is asymptotically stable. In addition, the total energy function $v(\tilde{x})$ of $\tilde{\mathcal{G}}$ given by (6.35) is strictly decreasing across resetting events. Finally, if $\mathcal{D} = \mathbb{R}^n$, $\mathcal{D}_{\mathrm{c}} = \mathbb{R}^{n_{\mathrm{c}}}$, and $v(\cdot)$ is radially unbounded, then the zero solution $\tilde{x}(t) \equiv 0$ to $\tilde{\mathcal{G}}$ is globally asymptotically stable.

Proof. First, note that since $v_{ci}(x_{ci}, y_i) \geq 0$, $x_{ci} \in \mathcal{D}_{ci}$, $y_i \in \mathbb{R}^{l_i}$, $i = 1, \ldots, q$, it follows that

$$\overline{\tilde{\mathcal{Z}}}_i = \left\{(x_i, x_{ci}) \in \mathcal{D} \times \mathcal{D}_{ci} : L_{\tilde{f}_{\mathrm{c}}} v_{ci}(x_{ci}, h_i(x_i)) = 0 \text{ and } v_{ci}(x_{ci}, h_i(x_i)) \geq 0\right\}$$

$$= \{(x_i, x_{ci}) \in \mathcal{D} \times \mathcal{D}_{ci} : \mathcal{X}_i(\tilde{x}_i) = 0\}, \tag{6.38}$$

where $\mathcal{X}_i(\tilde{x}_i) = L_{\tilde{f}_c} v_{ci}(x_{ci}, h_i(x_i))$, $i = 1, \ldots, q$. Next, we show that if the transversality condition (6.17) holds, then Assumptions 1–3 hold and, for every $\tilde{x}_0 \in \tilde{\mathcal{D}}_{ci}$ there exists $\tau \geq 0$ such that $\tilde{x}(\tau) \in \tilde{\mathcal{Z}}$. Note that if $\tilde{x}_0 \in \overline{\tilde{\mathcal{Z}}} \backslash \tilde{\mathcal{Z}}$, that is, $v_{ci}(x_{ci}(0), h_i(x_i(0))) = 0$ and $L_{\tilde{f}_c} v_{ci}(x_{ci}(0), h_i(x_i(0))) = 0$, $i \in \{1, \ldots, q\}$, it follows from the transversality condition that there exists $\delta_i > 0$ such that for all $t \in (0, \delta_i]$, $L_{\tilde{f}_c} v_{ci}(x_{ci}(t), h_i(x_i(t))) \neq 0$. Hence, since $v_{ci}(x_{ci}(t), h_i(x_i(t))) = v_{ci}(x_{ci}(0), h_i(x_i(0))) + tL_{\tilde{f}_c} v_{ci}(x_{ci}(\tau), h_i(x_i(\tau)))$ for some $\tau \in (0, t]$ and $v_{ci}(x_{ci}, y_i) \geq 0$, $x_{ci} \in \mathcal{D}_{ci}$, $y_i \in \mathbb{R}^{l_i}$, $i \in \{1, \ldots, q\}$, it follows that $v_{ci}(x_{ci}(t), h_i(x_i(t))) > 0$, $t \in (0, \delta]$, which implies that Assumption 1 is satisfied. Furthermore, if $\tilde{x} \in \tilde{\mathcal{Z}}$ then, since $v_{ci}(x_{ci}, y_i) = 0$ if and only if $x_{ci} = \eta(y_i)$, it follows from (6.34) that $\tilde{x}_i + \tilde{f}_{di}(\tilde{x}_i) \in \overline{\tilde{\mathcal{Z}}}_i \backslash \tilde{\mathcal{Z}}_i$, $i \in \{1, \ldots, q\}$. Hence, Assumption 2 holds.

Next, consider the set $\mathcal{M}_\gamma \triangleq \cup_{i=1}^q \left\{\tilde{x} \in \tilde{\mathcal{D}}_{ci} : v_{ci}(x_{ci}, h_i(x_i)) = \gamma_i\right\}$, where $\gamma_i \geq 0$, $i = 1, \ldots, q$, and $\gamma \triangleq [\gamma_1, \ldots, \gamma_q]^{\mathrm{T}}$. It follows from the transversality condition that for every $\gamma_i \geq 0$, $\mathcal{M}_\gamma$ does not contain any nontrivial trajectory of $\tilde{\mathcal{G}}$, $i = 1, \ldots, q$. To see this, suppose, *ad absurdum*, there exists a nontrivial trajectory $\tilde{x}(t) \in \mathcal{M}_\gamma$, $t \geq 0$, for some $\gamma_i \geq 0$ and for some $i \in \{1, \ldots, q\}$. In this case, it follows that $\frac{\mathrm{d}^k}{\mathrm{d}t^k} v_{ci}(x_{ci}(t), h_i(x_i(t))) = L_{\tilde{f}_c}^k v_{ci}(x_{ci}(t), h_i(x_i(t))) \equiv 0$, $k = 1, 2, \ldots$, $i \in \{1, \ldots, q\}$, which contradicts the transversality condition.

Next, we show that for every $\tilde{x}_0 \notin \tilde{\mathcal{Z}}$, $\tilde{x}_0 \neq 0$, there exists $\tau > 0$ such that $\tilde{x}(\tau) \in \tilde{\mathcal{Z}}$. To see this, suppose, *ad absurdum*, $\tilde{x}_i(t) \notin \tilde{\mathcal{Z}}_i$ for all $i = 1, \ldots, q$, $t \geq 0$, which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} v_{ci}(x_{ci}(t), h_i(x_i(t))) \neq 0, \quad t \geq 0, \quad i = 1, \ldots, q, \tag{6.39}$$

or

$$v_{ci}(x_{ci}(t), h_i(x_i(t))) = 0, \quad t \geq 0, \quad i = 1, \ldots, q. \tag{6.40}$$

If (6.39) holds, then it follows that $v_{ci}(x_{ci}(t), h_i(x_i(t)))$ is a (decreasing or increasing) monotonic function of time. Hence, $v_{ci}(x_{ci}(t), h_i(x_i(t))) \to \gamma_i$ as $t \to \infty$, where $\gamma_i \geq 0$ is a constant for $i = 1, \ldots, q$, which implies that the positive limit set of the closed-loop system is contained in $\mathcal{M}_\gamma$ for some $\gamma_i \geq 0$, $i = 1, \ldots, q$, and hence, is a contradiction. Similarly, if (6.40) holds then $\mathcal{M}_0$ contains a nontrivial trajectory of $\tilde{\mathcal{G}}$ also leading to a contradiction. Hence, for every $\tilde{x}_0 \notin \tilde{\mathcal{Z}}$, there exists $\tau > 0$ such that $\tilde{x}(\tau) \in \tilde{\mathcal{Z}}$. Thus, it follows that for every $\tilde{x}_0 \notin \tilde{\mathcal{Z}}$, $0 < \tau_1(\tilde{x}_0) < \infty$. Now, it follows from Proposition 6.2 that $\tau_1(\cdot)$ is continuous at $\tilde{x}_0 \notin \overline{\tilde{\mathcal{Z}}}$. Furthermore, for all $\tilde{x}_0 \in \overline{\tilde{\mathcal{Z}}} \backslash \tilde{\mathcal{Z}}$ and for every sequence $\{\tilde{x}_{(i)}\}_{i=1}^{\infty} \in \overline{\tilde{\mathcal{Z}}} \backslash \tilde{\mathcal{Z}}$ converging to $\tilde{x}_0 \in \overline{\tilde{\mathcal{Z}}} \backslash \tilde{\mathcal{Z}}$, it follows from the transversality condition and Proposition 6.2 that $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)}) = \tau_1(\tilde{x}_0)$. Next, let $\tilde{x}_0 \in \overline{\tilde{\mathcal{Z}}} \backslash \tilde{\mathcal{Z}}$ and let $\{\tilde{x}_{(i)}\}_{i=1}^{\infty} \in \tilde{\mathcal{D}}_{\mathrm{ci}}$ be such that $\lim_{i\to\infty} \tilde{x}_{(i)} = \tilde{x}_0$ and $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)})$ exists. In this case, it follows from Proposition 6.2 that either $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)}) = 0$ or $\lim_{i\to\infty} \tau_1(\tilde{x}_{(i)}) = \tau_1(\tilde{x}_0)$. Furthermore, since $\tilde{x}_0 \in \overline{\tilde{\mathcal{Z}}} \backslash \tilde{\mathcal{Z}}$ corresponds to the case where $v_{\mathrm{c}i}(x_{\mathrm{c}i0}, h_i(x_{i0})) = 0$, $i \in \{1, \ldots, q\}$, it follows that $x_{\mathrm{c}i0} = \eta_i(h_i(x_{i0}))$, and hence, $\tilde{f}_{\mathrm{d}i}(\tilde{x}_{i0}) = 0$, $i \in \{1, \ldots, q\}$. Now, it follows from Proposition 6.1 that Assumption 3 holds.

To show that the zero solution $\tilde{x}(t) \equiv 0$ to $\tilde{\mathcal{G}}$ is asymptotically stable, consider the Lyapunov function candidate corresponding to the total energy function $v(\tilde{x})$ given by (6.35). Since $\mathcal{G}$ is vector lossless with respect to the vector supply rate $S(u, y)$, and hence, lossless with respect to the supply rate $p^{\mathrm{T}} S(u, y)$, where $p \in \mathbb{R}_+^q$, and (6.34) and (6.37) hold, it follows that

$$\dot{v}(\tilde{x}(t)) = \sum_{i=1}^{q} p_i[s_i(u_i(t), y_i(t)) + s_{\mathrm{c}i}(u_{\mathrm{c}i}(t), y_{\mathrm{c}i}(t))] = 0, \quad \tilde{x}(t) \notin \tilde{\mathcal{Z}}, \tag{6.41}$$

where p_i, $i = 1, \ldots, q$, denotes the ith element of $p \in \mathbb{R}_+^q$. Furthermore, it follows from (6.30) and (6.38) that

$$\begin{aligned}
\Delta v(\tilde{x}(t_k)) &= v_{\mathrm{c}}(x_{\mathrm{c}}(t_k^+), H(x(t_k^+))) - v_{\mathrm{c}}(x_{\mathrm{c}}(t_k), H(x(t_k))) \\
&= -\sum_{i=1}^{q} p_i v_{\mathrm{c}i}(x_{\mathrm{c}i}(t_k), h_i(x_i(t_k))) \chi_{\tilde{\mathcal{Z}}_i}(\tilde{x}_i(t_k)) \\
&< 0, \quad \tilde{x}(t_k) \in \tilde{\mathcal{Z}}, \quad k \in \overline{\mathbb{Z}}_+.
\end{aligned} \tag{6.42}$$

Thus, it follows from Theorem 6.2 that the zero solution $\tilde{x}(t) \equiv 0$ to $\tilde{\mathcal{G}}$ is asymptotically stable. Finally, if $\mathcal{D} = \mathbb{R}^n$, $\mathcal{D}_c = \mathbb{R}^{n_c}$, and $v(\cdot)$ is radially unbounded, then global asymptotic stability is immediate. $\square$

Remark 6.4. If $v_{ci} = v_{ci}(x_{ci}, y_i)$ is only a function of x_{ci} and $v_{ci}(x_{ci})$ is a positive-definite function, $i \in \{1, \ldots, q\}$, then we can choose $\eta_i(y_i) \equiv 0$. In this case, $v_{ci}(x_{ci}) = 0$ if and only if $x_{ci} = 0$.

Remark 6.5. In the proof of Theorem 6.3, we assume that $\tilde{x}_0 \notin \tilde{\mathcal{Z}}$ for $\tilde{x}_0 \neq 0$. This proviso is necessary since it may be possible to reset the states of the closed-loop system to the origin, in which case $\tilde{x}(s) = 0$ for a finite value of s. In this case, for $t > s$, we have $v(\tilde{x}(t)) = v(\tilde{x}(s)) = v(0) = 0$. This situation does not present a problem, however, since reaching the origin in finite time is a stronger condition than reaching the origin as $t \to \infty$.

Remark 6.6. Theorem 6.3 can be generalized to the case where $\mathcal{G}$ is *vector dissipative* with respect to the vector supply rate $S(u, y)$ with the component decoupled vector storage function $V_s(x) = [v_{s1}(x_1), \ldots, v_{sq}(x_q)]^{\mathrm{T}}$, $x \in \mathcal{D}$. Specifically, in this case (6.41) becomes $\dot{v}(\tilde{x}(t)) = \sum_{i=1}^{q} p_i d_i(x_i(t)) \leq 0$, $\tilde{x}(t) \in \tilde{\mathcal{Z}}$, where $d_i : \mathcal{D}_i \to \mathbb{R}$, $i = 1, \ldots, q$, is a continuous, nonnegative-definite dissipation rate function. Now, Theorem 6.3 holds with the additional assumption that the only invariant set contained in $\mathcal{R} \triangleq \cap_{i=1}^{q} \{\tilde{x} \in \tilde{\mathcal{D}}_{ci} : d_i(x_i) = 0\}$ is $\mathcal{M} = \{0\}$.

6.4. Quasi-Thermodynamic Stabilization and Maximum Entropy Control

In this section, we use the recently developed notion of system thermodynamics [104] to develop thermodynamically consistent hybrid decentralized controllers for large-scale systems. Specifically, since our energy-based hybrid controller architecture involves the exchange of energy with conservation laws describing transfer, accumulation, and dissipation of energy between the subcontrollers and the plant subsystems, we construct a modified hybrid controller that guarantees that each subsystem-subcontroller pair $(\mathcal{G}_i, \mathcal{G}_{\mathrm{c}i})$ is consistent with basic thermodynamic principles after the first resetting event. To develop thermodynamically consistent hybrid decentralized controllers consider the closed-loop subsystem-subcontroller pair $(\mathcal{G}_i, \mathcal{G}_{\mathrm{c}i})$ given by (6.27) and (6.28) with $\tilde{\mathcal{Z}}_i$ given by

$$\tilde{\mathcal{Z}}_i \triangleq \left\{ \tilde{x}_i \in \tilde{\mathcal{D}}_i : \phi_i(\tilde{x}_i)(v_{\mathrm{p}i}(\tilde{x}_i) - v_{\mathrm{c}i}(\tilde{x}_i)) = 0 \text{ and } v_{\mathrm{c}i}(x_{\mathrm{c}i}, h_i(x_i)) > 0 \right\}, \quad i = 1, \ldots, q, \tag{6.43}$$

where $\phi_i(\tilde{x}_i) \triangleq s_{\mathrm{c}i}(h_i(x_i), h_{\mathrm{c}i}(x_{\mathrm{c}i}, h_i(x_i)))$, $v_{\mathrm{p}i}(\tilde{x}_i) \triangleq v_{\mathrm{s}i}(x_i)$, and $v_{\mathrm{c}i}(\tilde{x}_i) \triangleq v_{\mathrm{c}i}(x_{\mathrm{c}i}, h_i(x_i))$. We refer to $\phi_i(\cdot)$ as the *net energy flow* function.

We assume that the energy flow function $\phi_i(\tilde{x}_i)$ is infinitely differentiable and the transversality condition (6.17) holds with $\mathcal{X}_i(\tilde{x}_i) = \phi_i(\tilde{x}_i)(v_{\mathrm{p}i}(\tilde{x}_i) - v_{\mathrm{c}i}(\tilde{x}_i))$ for all $i = 1, \ldots, q$. To ensure a thermodynamically consistent energy flow between the subsystem $\mathcal{G}_i$ and subcontroller $\mathcal{G}_{\mathrm{c}i}$ after the first resetting event, each subcontroller resetting logic must be designed in such a way so as to satisfy three key thermodynamic axioms. Namely, between resettings the energy flow function $\phi_i(\cdot)$ must satisfy the following two axioms [101, 104]:

Assumption 4. For the connectivity matrix $\mathcal{C} \in \mathbb{R}^{2\times 2}$ [104, p. 56] associated with the subsystem $\tilde{\mathcal{G}}_l$ defined by

$$\mathcal{C}_{(i,j)} \triangleq \begin{cases} 0, & \text{if } \phi_l(\tilde{x}_l(t)) \equiv 0 \\ 1, & \text{otherwise} \end{cases}, \quad i \neq j, \quad i, j = 1, 2, \quad l = 1, \ldots, q, \quad t \geq t_1^+, \tag{6.44}$$

and

$$\mathcal{C}_{(i,i)} = -\mathcal{C}_{(k,i)}, \quad i \neq k, \quad i, k = 1, 2, \tag{6.45}$$

$\operatorname{rank} \mathcal{C} = 1$, and for $\mathcal{C}_{(i,j)} = 1$, $i \neq j$, $\phi_l(\tilde{x}_l(t)) = 0$ if and only if $v_{\mathrm{p}l}(\tilde{x}_l) = v_{\mathrm{c}l}(\tilde{x}_l)$, $\tilde{x}_l(t) \notin \tilde{\mathcal{Z}}_l$, $l = 1, \ldots, q$, $t \geq t_1^+$.

Assumption 5. $\phi_l(\tilde{x}_i(t))(v_{\mathrm{p}l}(\tilde{x}_i) - v_{\mathrm{c}l}(\tilde{x}_i)) \leq 0$, $\tilde{x}_i(t) \notin \tilde{\mathcal{Z}}_i$, $i = 1, \ldots, q$, $t \geq t_1^+$.

Furthermore, across resettings the energy difference between the subsystem and the subcontroller must satisfy the following axiom (Axiom *iii*) of Section 3.3):

Assumption 6. $[v_{\mathrm{p}i}(\tilde{x}_i + \tilde{f}_{\mathrm{d}i}(\tilde{x}_i)) - v_{\mathrm{c}i}(\tilde{x}_i + \tilde{f}_{\mathrm{d}i}(\tilde{x}_i))][v_{\mathrm{p}i}(\tilde{x}_i) - v_{\mathrm{c}i}(\tilde{x}_i)] \geq 0$, $i = 1, \ldots, q$, $\tilde{x}_i \in \tilde{\mathcal{Z}}_i$.

The fact that $\phi_i(\tilde{x}_i(t)) = 0$ if and only if $v_{\mathrm{p}i}(\tilde{x}_i(t)) = v_{\mathrm{c}i}(\tilde{x}_i(t))$, $\tilde{x}_i(t) \notin \tilde{\mathcal{Z}}_i$, $t \geq t_1^+$, implies that the ith subsystem and the ith subcontroller are *connected*; alternatively, $\phi_i(\tilde{x}_i(t)) \equiv 0$, $t \geq t_1^+$, implies that the ith subsystem and the ith subcontroller are *disconnected*. Assumption 4 implies that if the energies in the ith subsystem and the ith subcontroller are equal, then energy exchange between the ith subsystem $\mathcal{G}_i$ and the ith subcontroller $\mathcal{G}_{\mathrm{c}i}$ is not possible unless a resetting event occurs. This statement is consistent with the *zeroth law of thermodynamics*, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium of an isolated system. Assumption 5 implies that energy flows from a more energetic subsystem to a less energetic subsystem and is consistent with the *second law of thermodynamics*, which states that heat (energy) must flow in the direction of lower temperatures. Finally, Assumption 6 implies that the energy difference between the ith subsystem $\mathcal{G}_i$ and the ith subcontroller $\mathcal{G}_{\mathrm{c}i}$ across resetting instants is monotonic, that is, $[v_{\mathrm{p}i}(\tilde{x}_i(t_k^+)) - v_{\mathrm{c}i}(\tilde{x}_i(t_k^+))][v_{\mathrm{p}i}(\tilde{x}_i(t_k)) - v_{\mathrm{c}i}(\tilde{x}_i(t_k))] \geq 0$ for all $v_{\mathrm{p}i}(\tilde{x}_i) \neq v_{\mathrm{c}i}(\tilde{x}_i)$, $\tilde{x}_i \in \tilde{\mathcal{Z}}_i$, $i = 1, \ldots, q$, $k \in \overline{\mathbb{Z}}_+$.

With the resetting law given by (6.43), it follows that each ith subsystem $\tilde{\mathcal{G}}_i$ of the closed-loop dynamical system $\tilde{\mathcal{G}}$ satisfies Assumptions 4-6 for all $t \geq t_1$. To see this, note that since $\phi_i(\tilde{x}_i) \not\equiv 0$, the connectivity matrix $\mathcal{C}$ is given by

$$\mathcal{C} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{6.46}$$

and hence, $\operatorname{rank} \mathcal{C} = 1$. The second condition in Assumption 4 need not be satisfied since the case where $\phi_i(\tilde{x}_i) = 0$ or $v_{\mathrm{p}i}(\tilde{x}_i) = v_{\mathrm{c}i}(\tilde{x}_i)$, corresponds to a resetting instant. Furthermore, it follows from the definition of the resetting set (6.43) that Assumption 5 is satisfied for each closed-loop subsystem pairs $(\mathcal{G}_i, \mathcal{G}_{\mathrm{c}i})$ for all $t \geq t_1^+$. Finally, since $v_{\mathrm{c}i}(\tilde{x}_i + \tilde{f}_{\mathrm{d}i}(\tilde{x}_i)) = 0$ and $v_{\mathrm{p}i}(\tilde{x}_i + \tilde{f}_{\mathrm{d}i}(\tilde{x}_i)) = v_{\mathrm{p}i}(\tilde{x}_i)$, $\tilde{x}_i \in \tilde{\mathcal{Z}}_i$, it follows from the definition of the resetting set that

$$[v_{\mathrm{p}i}(\tilde{x}_i + \tilde{f}_{\mathrm{d}i}(\tilde{x}_i)) - v_{\mathrm{c}i}(\tilde{x}_i + \tilde{f}_{\mathrm{d}i}(\tilde{x}_i))][v_{\mathrm{p}i}(\tilde{x}_i) - v_{\mathrm{c}i}(\tilde{x}_i)] = v_{\mathrm{p}i}(\tilde{x}_i)[v_{\mathrm{p}i}(\tilde{x}_i) - v_{\mathrm{c}i}(\tilde{x}_i)] \geq 0,$$

$$\tilde{x}_i \in \tilde{\mathcal{Z}}_i, \quad i = 1, \ldots, q, \tag{6.47}$$

and hence, Assumption 6 is satisfied across resettings. Hence, each ith closed-loop subsystem $\tilde{\mathcal{G}}_i$ of the closed-loop system $\tilde{\mathcal{G}}$ is thermodynamically consistent after the first resetting event in the sense of [101, 104] and Chapter 3. Note that this statement is only true for each closed-loop subsystem $\tilde{\mathcal{G}}_i$. For the hybrid closed-loop system $\tilde{\mathcal{G}}$, Assumptions 4-6 may not hold since the interconnection function $\mathcal{I}(x)$ defining $\mathcal{G}$ may not necessarily correspond to a thermodynamically consistent model.

If $\tilde{\mathcal{D}}_{\mathrm{ci}} \subset \tilde{\mathcal{D}}$ is a compact positively invariant set with respect to the closed-loop dynamical system $\tilde{\mathcal{G}}$ given by (6.31) and (6.32) such that $0 \in \overset{\circ}{\tilde{\mathcal{D}}}_{\mathrm{ci}}$, and the transversality condition (6.17) holds with $\mathcal{X}_i(\tilde{x}_i) = \phi_i(\tilde{x}_i)(v_{\mathrm{pi}}(\tilde{x}_i) - v_{\mathrm{ci}}(\tilde{x}_i))$ for all $i = 1, \ldots, q$, then it follows from Theorem 6.3 that the zero solution $\tilde{x}(t) \equiv 0$ to the closed-loop system $\tilde{\mathcal{G}}$, with resetting set $\tilde{\mathcal{Z}}_i$ given by (6.43), is asymptotically stable. Furthermore, in this case, the hybrid decentralized controller (6.22) and (6.23), with resetting set (6.43), is a *quasi-thermodynamically stabilizing* compensator.

Finally, we show that the hybrid decentralized controllers developed in this section and Section 6.3 are maximum entropy controllers. To do this, the following hybrid definition of entropy is needed.

Definition 6.2. For each decentralized subcontroller $\mathcal{G}_{ci}$ given by (6.22)–(6.24), a function $S_{ci} : \overline{\mathbb{R}}_+ \to \mathbb{R}$, $i = 1, \ldots, q$, satisfying

$$S_{\mathrm{ci}}(v_{\mathrm{ci}}(\tilde{x}_i(T))) \geq S_{\mathrm{ci}}(v_{\mathrm{ci}}(\tilde{x}_i(t_1))) - \frac{1}{c_i} \sum_{k \in \mathbb{Z}_{[t_1,T)}} v_{\mathrm{ci}}(\tilde{x}_i(t_k)), \quad T \geq t_1, \quad i = 1, \ldots, q, \tag{6.48}$$

where $k \in \mathbb{Z}_{[t_1,T)} \triangleq \{k : t_1 \leq t_k < T\}$, $c_i > 0$, is called an *entropy* function of $\mathcal{G}_{\mathrm{ci}}$, $i = 1, \ldots, q$.

The next result gives necessary and sufficient conditions for establishing the existence of an entropy function of $\mathcal{G}_{\mathrm{ci}}$, $i = 1, \ldots, q$, over an interval $t \in (t_k, t_{k+1}]$ involving the consecutive resetting times t_k and t_{k+1}, $k \in \mathbb{Z}_+$.

Theorem 6.4. For each decentralized subcontroller $\mathcal{G}_{ci}$ given by (6.22)–(6.24), a function $S_{ci} : \overline{\mathbb{R}}_+ \to \mathbb{R}$, $i = 1, \ldots, q$, is an entropy function of $\mathcal{G}_{ci}$ if and only if

$$S_{ci}(v_{ci}(\tilde{x}_i(\hat{t}))) \geq S_{ci}(v_{ci}(\tilde{x}_i(t))), \quad t_k < t \leq \hat{t} \leq t_{k+1}, \quad i = 1, \ldots, q, \tag{6.49}$$

$$S_{ci}(v_{ci}(\tilde{x}_i(t_k) + \tilde{f}_{di}(\tilde{x}_i(t_k)))) \geq S_{ci}(v_{ci}(\tilde{x}_i(t_k))) - \frac{1}{c_i} v_{ci}(\tilde{x}_i(t_k)), \quad k \in \mathbb{Z}_+, \quad i = 1, \ldots, q. \tag{6.50}$$

Proof. Let $k \in \mathbb{Z}_+$ and suppose $S_{ci}(v_{ci})$ is an entropy function of $\mathcal{G}_{ci}$. Then, (6.48) holds. Now, since for $t_k < t \leq \hat{t} \leq t_{k+1}$, $\mathbb{Z}_{[t,\hat{t})} = \emptyset$, (6.49) is immediate. Next, note that

$$S_{ci}(v_{ci}(\tilde{x}_i(t_k^+))) \geq S_{ci}(v_{ci}(\tilde{x}_i(t_k))) - \frac{1}{c_i} v_{ci}(\tilde{x}_i(t_k)), \quad i = 1, \ldots, q, \tag{6.51}$$

which, since $\mathbb{Z}_{[t_k, t_k^+)} = k$, implies (6.50).

Conversely, suppose (6.49) and (6.50) hold, and let $\hat{t} \geq t \geq t_1$ and $\mathbb{Z}_{[t,\hat{t})} = \{i, i+1, \ldots, j\}$. (Note that if $\mathbb{Z}_{[t,\hat{t})} = \emptyset$ the converse result is a direct consequence of (6.49).) If $\mathbb{Z}_{[t,\hat{t})} \neq \emptyset$, it follows from (6.49) and (6.50) that

$$\begin{aligned} S_{cl}(v_{cl}(\tilde{x}_l(\hat{t}))) - S_{cl}(v_{cl}(\tilde{x}_l(t))) = \; & S_{cl}(v_{cl}(\tilde{x}_l(\hat{t}))) - S_{cl}(v_{cl}(\tilde{x}_l(t_j^+))) \\ & + \sum_{m=0}^{j-i} S_{cl}(v_{cl}(\tilde{x}_l(t_{j-m}) + \tilde{f}_{dl}(\tilde{x}_l(t_{j-m})))) - S_{cl}(v_{cl}(\tilde{x}_l(t_{j-m}))) \\ & + \sum_{m=0}^{j-i-1} S_{cl}(v_{cl}(\tilde{x}_l(t_{j-m}))) - S_{cl}(v_{cl}(\tilde{x}_l(t_{j-m-1}^+))) \\ & + S_{cl}(v_{cl}(\tilde{x}_l(t_i))) - S_{cl}(v_{cl}(\tilde{x}_l(t))) \\ \geq \; & -\frac{1}{c_l} \sum_{m=0}^{j-i} v_{cl}(\tilde{x}_l(t_{j-m})) \\ = \; & -\frac{1}{c_l} \sum_{k \in \mathbb{Z}_{[t,\hat{t})}} v_{cl}(\tilde{x}_l(t_k)), \quad l = 1, \ldots, q, \end{aligned} \tag{6.52}$$

which implies that $S_{ci}(v_{ci})$ is an entropy function of $\mathcal{G}_{ci}$, $i = 1, \ldots, q$. □

The next theorem establishes the existence of an entropy function for $\mathcal{G}_{ci}$, $i = 1, \ldots, q$.

Theorem 6.5. Consider the hybrid decentralized subcontrollers $\mathcal{G}_{\mathrm{c}i}$ given by (6.22)–(6.24), with $\tilde{\mathcal{Z}}_i$ given by (6.36) or (6.43). Then the function $S_{\mathrm{c}i}:\overline{\mathbb{R}}_+\to\mathbb{R}$, $i=1,\ldots,q$, given by

$$S_{\mathrm{c}i}(v_{\mathrm{c}i})=\log_e(c_i+v_{\mathrm{c}i})-\log_e c_i,\quad v_{\mathrm{c}i}\in\overline{\mathbb{R}}_+,\quad i=1,\ldots,q, \tag{6.53}$$

where $c_i>0$, is an entropy function of $\mathcal{G}_{\mathrm{c}i}$, $i=1,\ldots,q$. In addition, for $i=1,\ldots,q$,

$$\dot{S}_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i(t)))>0,\quad \tilde{x}_i(t)\notin\tilde{\mathcal{Z}}_i,\quad t_k<t\le t_{k+1}, \tag{6.54}$$

$$-\frac{1}{c_i}v_{\mathrm{c}i}(\tilde{x}_i(t_k))<\Delta S_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i(t_k)))<-\frac{v_{\mathrm{c}i}(\tilde{x}_i(t_k))}{c_i+v_{\mathrm{c}i}(\tilde{x}_i(t_k))},\quad \tilde{x}_i(t_k)\in\tilde{\mathcal{Z}}_i,\quad k\in\mathbb{Z}_+. \tag{6.55}$$

Proof. Since $\dot{v}_{\mathrm{c}i}(\tilde{x}_i(t))>0$, $\tilde{x}_i(t)\notin\tilde{\mathcal{Z}}_i$, $i=1,\ldots,q$, $t\in(t_k,t_{k+1}]$, $k\in\mathbb{Z}_+$, it follows that

$$\dot{S}_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i(t)))=\frac{\dot{v}_{\mathrm{c}i}(\tilde{x}_i(t))}{c_i+v_{\mathrm{c}i}(\tilde{x}_i(t))}>0,\quad \tilde{x}_i(t)\notin\tilde{\mathcal{Z}}_i,\quad i=1,\ldots,q. \tag{6.56}$$

Furthermore, since $v_{\mathrm{c}i}(\tilde{x}_i(t_k)+\tilde{f}_{\mathrm{d}i}(\tilde{x}_i(t_k)))=0$, $\tilde{x}_i(t_k)\in\tilde{\mathcal{Z}}_i$, $i=1,\ldots,q$, $k\in\mathbb{Z}_+$, it follows that, for $i=1,\ldots,q$,

$$\Delta S_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i(t_k)))=\log_e\left[1-\frac{v_{\mathrm{c}i}(\tilde{x}_i(t_k))}{c_i+v_{\mathrm{c}i}(\tilde{x}_i(t_k))}\right]>-\frac{1}{c_i}v_{\mathrm{c}i}(\tilde{x}_i(t_k)),\quad \tilde{x}_i(t_k)\in\tilde{\mathcal{Z}}_i,\quad k\in\mathbb{Z}_+, \tag{6.57}$$

and

$$\Delta S_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i(t_k)))=\log_e\left[1-\frac{v_{\mathrm{c}i}(\tilde{x}_i(t_k))}{c_i+v_{\mathrm{c}i}(\tilde{x}_i(t_k))}\right]<-\frac{v_{\mathrm{c}i}(\tilde{x}_i(t_k))}{c_i+v_{\mathrm{c}i}(\tilde{x}_i(t_k))},$$
$$\tilde{x}_i(t_k)\in\tilde{\mathcal{Z}}_i,\quad k\in\mathbb{Z}_+, \tag{6.58}$$

where in (6.57) and (6.58) we used the fact that $\frac{x}{1+x}<\log_e(1+x)<x$, $x>-1$, $x\neq 0$. The result is now an immediate consequence of Theorem 6.4. □

Using (6.56), the resetting set $\tilde{\mathcal{Z}}_i$, $i=1,\ldots,q$, given by (6.36) can be rewritten as

$$\tilde{\mathcal{Z}}_i\triangleq\left\{\tilde{x}_i\in\tilde{\mathcal{D}}_i:\frac{\mathrm{d}}{\mathrm{d}t}S_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i))=0\text{ and }v_{\mathrm{c}i}(\tilde{x}_i)>0\right\},\quad i=1,\ldots,q, \tag{6.59}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}S_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i(t))) \triangleq \lim_{\tau\to t^-}\frac{1}{t-\tau}[S_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i(t))) - S_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i(\tau)))]$ whenever limit on the right-hand side exists, and $S_{\mathrm{c}i} = \log_e(c_i+v_{\mathrm{c}i})-\log_e c_i$ denotes the continuously differentiable ith subcontroller entropy. Hence, each decentralized controller $\mathcal{G}_{\mathrm{c}i}$ corresponds to a maximum entropy controller. Alternatively, for $i=1,\ldots,q$, the resetting set $\tilde{\mathcal{Z}}_i$ given by (6.43) can be rewritten as $\{\tilde{x}_i(t_k) : k \in \mathbb{Z}_+\}$, where t_k is the maximum final time such that $S_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i(t))) \leq S_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i(t_1)))$ (or $S_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i(t))) \geq S_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i(t_1))))$ holds under the constraint $v_{\mathrm{p}i}(\tilde{x}_i(t)) \geq v_{\mathrm{c}i}(\tilde{x}_i(t))$ (or $v_{\mathrm{p}i}(\tilde{x}_i(t)) \leq v_{\mathrm{c}i}(\tilde{x}_i(t)))$ for $0 \leq t < t_1$, and $S_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i(t))) \leq S_{\mathrm{c}i}(v_{\mathrm{c}i}(\tilde{x}_i(t_{k+1})))$ holds under the constraint $v_{\mathrm{p}i}(\tilde{x}_i(t)) \geq v_{\mathrm{c}i}(\tilde{x}_i(t))$ for all $t_k \leq t < t_{k+1}$ and $k \in \mathbb{Z}_+$. Hence, each decentralized controller $\mathcal{G}_{\mathrm{c}i}$ corresponds to a constrained maximum entropy controller.

6.5. Hybrid Decentralized Control for Combustion Systems

In this section, we apply our results to the control of *thermoacoustic instabilities* in combustion processes. As noted in Section 5.6, we stress that the combustion model we use can be stabilized by conventional nonlinear control methods. The aim here, however, is to show that hybrid decentralized control provides an extremely efficient mechanism for dissipating energy in the combustion process with far superior performance than any conventional control methodology. In particular, we show that the proposed hybrid decentralized controller provides finite-time stabilization.

To design a decentralized hybrid controller for the combustion system we considered in Section 5.6, recall that this model is given by

$$\dot{x}_1(t) = \alpha_1 x_1(t) + \theta_1 x_2(t) - \beta(x_1(t)x_3(t) + x_2(t)x_4(t)) + u_1(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{6.60}$$

$$\dot{x}_2(t) = -\theta_1 x_1(t) + \alpha_1 x_2(t) + \beta(x_2(t)x_3(t) - x_1(t)x_4(t)) + u_2(t), \quad x_2(0) = x_{20}, \tag{6.61}$$

$$\dot{x}_3(t) = \alpha_2 x_3(t) + \theta_2 x_4(t) + \beta(x_1^2(t) - x_2^2(t)) + u_3(t), \quad x_3(0) = x_{30}, \tag{6.62}$$

$$\dot{x}_4(t) = -\theta_2 x_3(t) + \alpha_2 x_4(t) + 2\beta x_1(t)x_2(t) + u_4(t), \quad x_4(0) = x_{40}, \tag{6.63}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ represent growth/decay constants, $\theta_1, \theta_2 \in \mathbb{R}$ represent frequency shift constants, $\beta = ((\gamma+1)/8\gamma)\omega_1$, where γ denotes the ratio of specific heats, ω_1 is frequency of the fundamental mode, and u_i, $i = 1, \ldots, 4$, are control input signals. For the data parameters $\alpha_1 = 5$, $\alpha_2 = -55$, $\theta_1 = 4$, $\theta_2 = 32$, $\gamma = 1.4$, $\omega_1 = 1$, and $x(0) = [1,\ 1,\ 1,\ 1]^{\mathrm{T}}$, the open-loop ($u_i(t) \equiv 0$, $i = 1,2,3,4$) dynamics (6.60)–(6.63) result in a limit cycle instability.

Next, note that (6.60)–(6.63) can be rewritten in the form of (6.1) and (6.2) with $f_1(x_1, x_2) = [\alpha_1 x_1 + \theta_1 x_2, -\theta_1 x_1 + \alpha_1 x_2]^{\mathrm{T}}$, $f_2(x_3, x_4) = [\alpha_2 x_3 + \theta_2 x_4, -\theta_2 x_3 + \alpha_2 x_4]^{\mathrm{T}}$, $\mathcal{I}_1(x) = [-\beta(x_1 x_3 + x_2 x_4), \beta(x_2 x_3 - x_1 x_4)]^{\mathrm{T}}$, $\mathcal{I}_2(x) = [\beta(x_1^2 - x_2^2), 2\beta x_1 x_2]^{\mathrm{T}}$, $G_1(x_1, x_2) = I_2$, $G_2(x_3, x_4) = I_2$, $y_1 = h_1(x_1, x_2) = [x_1, x_2]^{\mathrm{T}}$, and $y_2 = h_2(x_3, x_4) = [x_3, x_4]^{\mathrm{T}}$. Here, we take $v_{\mathrm{s}1}(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ and $v_{\mathrm{s}2}(x_3, x_4) = \frac{1}{2}(x_3^2 + x_4^2)$ as our subsystem energies. Now, it can be shown that the ith disconnected subsystem of (6.60)–(6.63) is lossless with respect to the supply rate $\hat{u}_i^{\mathrm{T}} y_i$, $i = 1, 2$, where $\hat{u}_1 = [u_1 + \alpha_1 x_1, u_2 + \alpha_1 x_2]^{\mathrm{T}}$ and $\hat{u}_2 = [u_3 + \alpha_2 x_3, u_4 + \alpha_2 x_4]^{\mathrm{T}}$. Furthermore, it can also be shown that (6.60)–(6.63) is lossless with respect to the supply rate $\hat{u}_1^{\mathrm{T}} y_1 + \hat{u}_2^{\mathrm{T}} y_2$.

Next, consider the decentralized dynamic compensator given by (6.22)–(6.24) with $f_{\mathrm{c}1}(x_{\mathrm{c}1}, y_1) = A_{\mathrm{c}1} x_{\mathrm{c}1} + B_{\mathrm{c}1} y_1$, $\eta_1(y_1) = 0$, $h_{\mathrm{c}1}(x_{\mathrm{c}1}, y_1) = B_{\mathrm{c}1}^{\mathrm{T}} x_{\mathrm{c}1}$, $f_{\mathrm{c}2}(x_{\mathrm{c}2}, y_2) \equiv 0$, $\eta_2(y_2) = 0$, and $h_{\mathrm{c}2}(x_{\mathrm{c}2}, y_2) \equiv 0$, where

$$A_{\mathrm{c}1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_{\mathrm{c}1} = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}, \tag{6.64}$$

and subcontroller energy is given by $v_{\mathrm{c}1}(x_{\mathrm{c}1}) = \frac{1}{2} x_{\mathrm{c}1}^{\mathrm{T}} x_{\mathrm{c}1}$. Furthermore, the resetting set (6.36) is given by

$$\mathcal{Z}_1 = \left\{ (x_1, x_2, x_{\mathrm{c}1}) : x_{\mathrm{c}1}^{\mathrm{T}} B_{\mathrm{c}1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,\ x_{\mathrm{c}1} \neq 0 \right\}. \tag{6.65}$$

To illustrate the behavior of the closed-loop impulsive dynamical system, we choose the initial condition $x_{\mathrm{c}1}(0) = [0, 0]^{\mathrm{T}}$. For this system a straightforward, but lengthy, calculation shows that Assumptions 1 and 2 hold. However, the transversality condition is sufficiently complex that we have been unable to show it analytically. This condition was verified

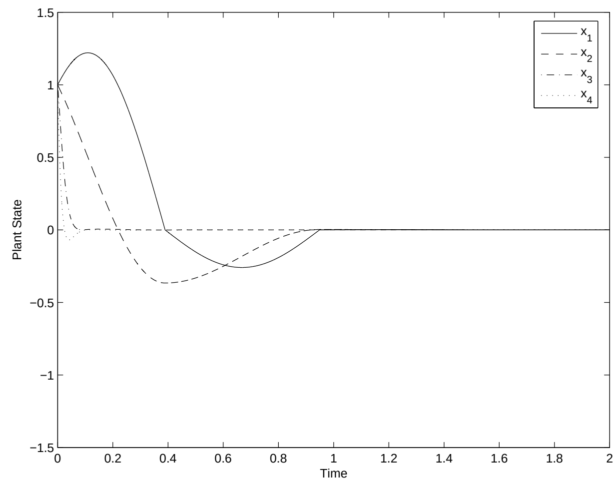

Figure 6.1: Plant state trajectories versus time

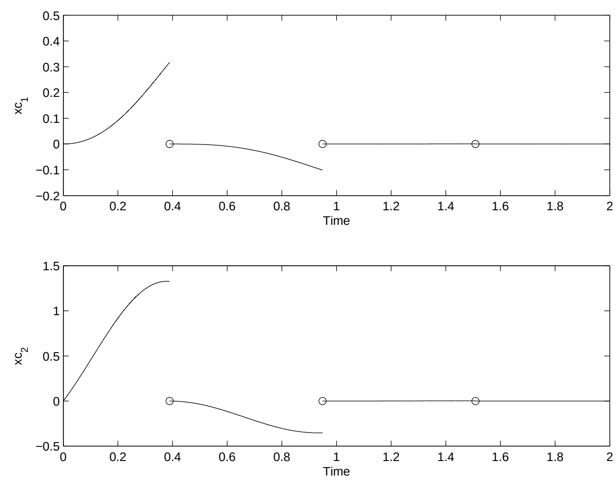

Figure 6.2: Compensator state trajectories versus time

numerically, and hence, Assumption 3 appears to hold. Figure 6.1 shows the state trajectories of the plant versus time, while Figure 6.2 shows the state trajectories of the compensator versus time. Figure 6.3 shows the control inputs u_1 and u_2 versus time. Note that the compensator states are the only states that reset. Furthermore, the control force versus time is partially discontinuous at the resetting times. A comparison of $v_{\rm s1}(x_1, x_2)$, $v_{\rm s2}(x_3, x_4)$, $v_{\rm c1}(x_{\rm c1})$, and $v(x, x_{\rm c1}) \triangleq v_{\rm s1}(x_1, x_2) + v_{\rm s2}(x_3, x_4) + v_{\rm c1}(x_{\rm c1})$ is shown in Figure 6.4. Note that the proposed hybrid decentralized controller achieves finite-time stabilization.

Next, we consider the case where $\alpha_1 = 0$ and $\alpha_2 = 0$. The other parameters remain as

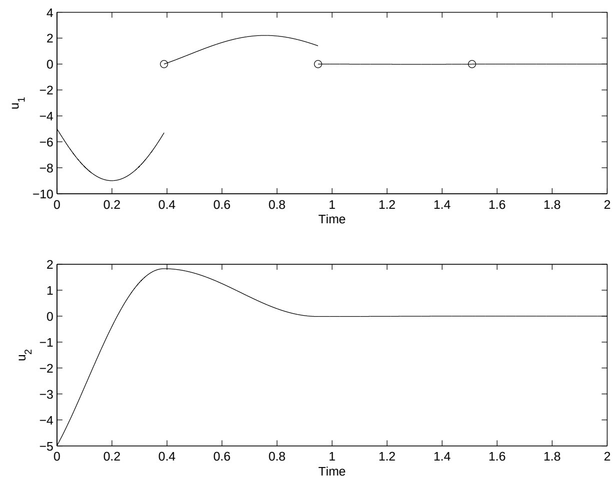

Figure 6.3: u_1 and u_2 versus time

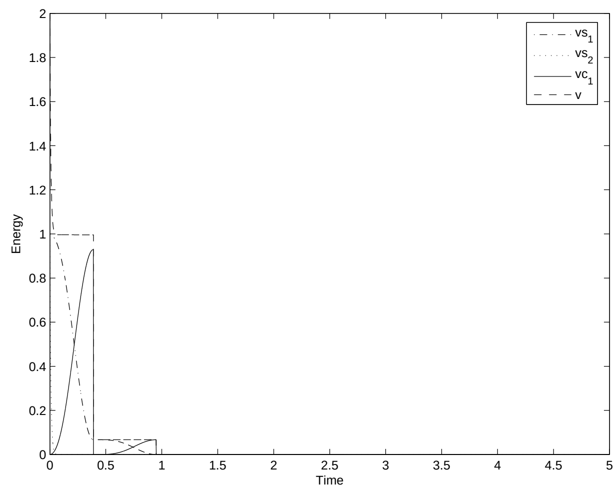

Figure 6.4: $v_{\mathrm{s}1}$, $v_{\mathrm{s}2}$, $v_{\mathrm{c}1}$, and v versus time

before. In this case, the decentralized dynamic compensators are given by (6.22)–(6.24) with $f_{\mathrm{c}i}(x_{\mathrm{c}i}, y_i) = A_{\mathrm{c}i}x_{\mathrm{c}i} + B_{\mathrm{c}i}y_i$, $\eta_i(y_i) = 0$, $h_{\mathrm{c}i}(x_{\mathrm{c}i}, y_i) = B_{\mathrm{c}i}^{\mathrm{T}}x_{\mathrm{c}i}$, $i = 1, 2$, where $A_{\mathrm{c}1}$ and $B_{\mathrm{c}1}$ are given by (6.64),

$$A_{\mathrm{c}2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_{\mathrm{c}2} = \begin{bmatrix} 0 & 0 \\ 16 & 0 \end{bmatrix}, \tag{6.66}$$

and subcontroller energies are given by $v_{\mathrm{c}1}(x_{\mathrm{c}1}) = \frac{1}{2}x_{\mathrm{c}1}^{\mathrm{T}}x_{\mathrm{c}1}$ and $v_{\mathrm{c}2}(x_{\mathrm{c}2}) = \frac{1}{2}x_{\mathrm{c}2}^{\mathrm{T}}x_{\mathrm{c}2}$. Furthermore, the resetting set (6.36) is given by (6.65) and

$$\mathcal{Z}_2 = \left\{(x_3, x_4, x_{\mathrm{c}2}) : x_{\mathrm{c}2}^{\mathrm{T}}B_{\mathrm{c}2}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = 0,\ x_{\mathrm{c}2} \neq 0\right\}. \tag{6.67}$$

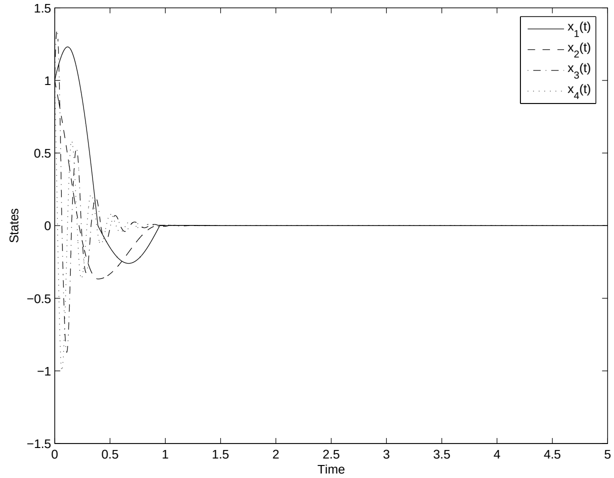

Figure 6.5: Plant state trajectories versus time

Finally, the entropy functions $S_{c1}(v_{c1})$ and $S_{c2}(v_{c2})$ are given by $S_{ci}(v_{ci}) = \log_e[1 + v_{ci}(x_{ci})]$, $i = 1, 2$.

To illustrate the behavior of the closed-loop impulsive dynamical system, we choose initial conditions $x_{c1}(0) = [0, 0]^T$ and $x_{c2}(0) = [0, 0]^T$. For this system a straightforward, but lengthy, calculation shows that Assumptions 1 and 2 hold. However, the transversality condition is sufficiently complex that we have been unable to show it analytically. This condition was verified numerically, and hence, Assumption 3 appears to hold. Figure 6.5 shows the state trajectories of the plant versus time, while Figure 6.6 shows the state trajectories of the compensator versus time. Figure 6.7 shows the control input versus time. Note that the compensator states are the only states that reset. Once again, the proposed hybrid decentralized controller achieves finite-time stabilization. Furthermore, the control force versus time is partially discontinuous at the resetting times. A comparison of $v_{s1}(x_1, x_2)$, $v_{c1}(x_{c1})$, and $v(x, x_{c1}, x_{c2}) \triangleq v_{s1}(x_1, x_2) + v_{s2}(x_3, x_4) + v_{c1}(x_{c1}) + v_{c2}(x_{c2})$ is shown in Figure 6.8, and a comparison of $v_{s2}(x_3, x_4)$, $v_{c2}(x_{c2})$, and $v(x, x_{c1}, x_{c2})$ is shown in Figure 6.9. Finally, Figure 6.10 shows the controller entropy versus time. Note that the entropy of the controller strictly increases between resetting events.

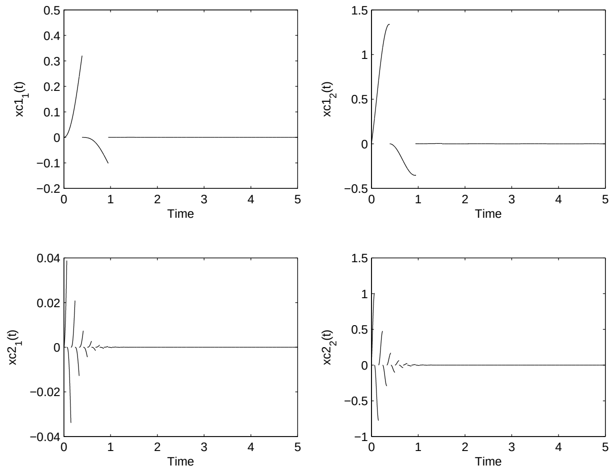

Figure 6.6: Compensator state trajectories versus time

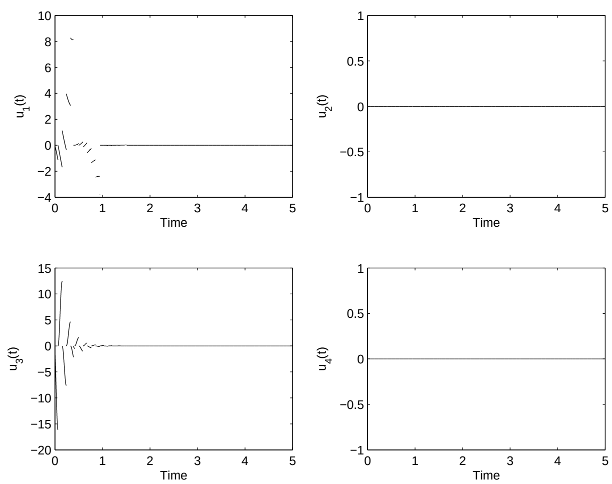

Figure 6.7: Control input versus time

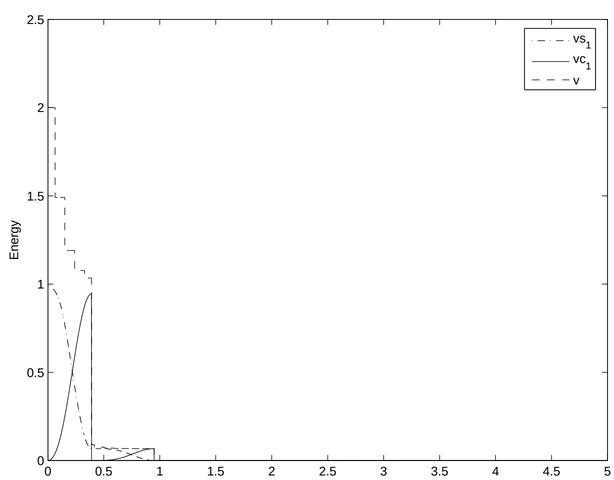

Figure 6.8: v_{s1}, v_{c1}, and v versus time

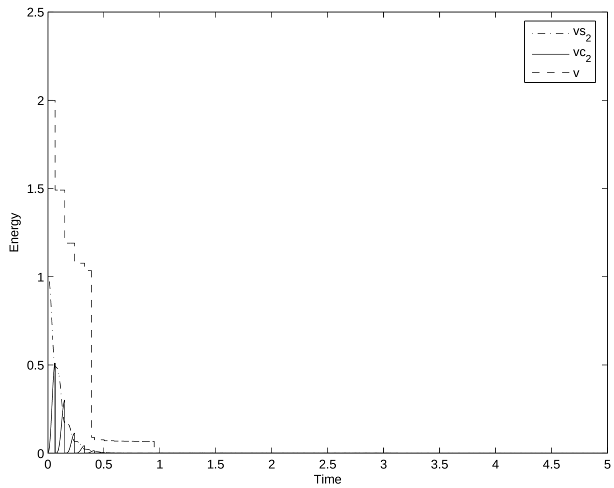

Figure 6.9: v_{s2}, v_{c2}, and v versus time

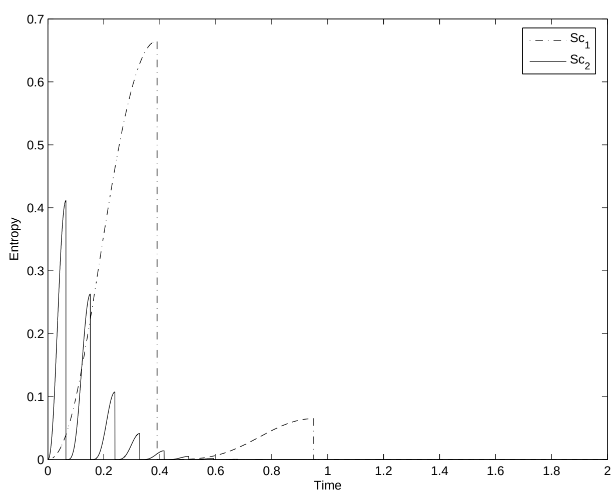

Figure 6.10: Controller entropy versus time

Chapter 7

Finite-Time Stabilization of Nonlinear Dynamical Systems via Control Vector Lyapunov Functions

7.1. Introduction

The notions of asymptotic and exponential stability in dynamical systems theory imply convergence of the system trajectories to an equilibrium state over the infinite horizon. In many applications, however, it is desirable that a dynamical system possesses the property that trajectories that converge to a Lyapunov stable equilibrium state must do so in finite time rather than merely asymptotically. Most of the existing control techniques in the literature ensure that the closed-loop system dynamics of a controlled system are Lipschitz continuous, which implies uniqueness of system solutions in forward and backward times. Hence, convergence to an equilibrium state is achieved over an infinite time interval. In order to achieve convergence in finite time, the closed-loop system dynamics need to be non-Lipschitzian giving rise to non-uniqueness of solutions in backward time. Uniqueness of solutions in forward time, however, can be preserved in the case of finite-time convergence. Sufficient conditions that ensure uniqueness of solutions in forward time in the absence of Lipschitz continuity are given in [1, 76, 140, 243]. In addition, it is shown in [57, Theorem 4.3, p. 59] that uniqueness of solutions in forward time along with continuity of the system dynamics ensure that the system solutions are continuous functions of the system initial conditions even when the dynamics are not Lipschitz continuous.

Finite-time convergence to a Lyapunov stable equilibrium, that is, finite-time stability, was rigorously studied in [30, 33] using Hölder continuous Lyapunov functions. Finite-time stabilization of second-order systems was considered in [28, 112]. More recently, researchers have considered finite-time stabilization of higher-order systems [120] as well as finite-time stabilization using output feedback [121]. Alternatively, discontinuous finite-time stabilizing feedback controllers have also been developed in the literature [78, 211, 212]. However, for practical implementations, discontinuous feedback controllers can lead to chattering due to system uncertainty or measurement noise, and hence, may excite unmodeled high-frequency system dynamics.

In this chapter, we develop a general framework for finite-time stability analysis of nonlinear dynamical systems using vector Lyapunov functions. Specifically, we construct a vector comparison system that is finite-time stable and, using the vector comparison principle [17, 50, 148, 171, 180], relate this finite-time stability property to the stability properties of the nonlinear dynamical system. We show that in the case of a scalar comparison system this result specializes to the result in [30]. Furthermore, we design universal finite-time stabilizing decentralized controllers for large-scale dynamical systems based on the newly proposed notion of a *control vector Lyapunov function* [180]. In addition, we present necessary and sufficient conditions for continuity of such controllers. Moreover, we specialize these results to the case of a scalar Lyapunov function to obtain universal finite-time stabilizers for nonlinear systems that are affine in the control. Finally, we demonstrate the utility of the proposed framework on two numerical examples.

7.2. Mathematical Preliminaries

In this section, we introduce notation and definitions, and present some key results needed for developing the main results. We write $\|\cdot\|$ for an arbitrary spatial vector norm in $\mathbb{R}^n$ and $\mathbf{e} \in \mathbb{R}^q$ for the ones vector of order n, that is, $\mathbf{e} \triangleq [1, \ldots, 1]^{\mathrm{T}}$.

Next, consider the nonlinear dynamical system given by

$$\dot{x}(t) = f(x(t)), \quad x(t_0) = x_0, \quad t \in \mathcal{I}_{x_0}, \tag{7.1}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \in \mathcal{I}_{x_0}$, is the system state vector, $\mathcal{I}_{x_0}$ is the maximal interval of existence of a solution $x(t)$ of (7.1), $\mathcal{D}$ is an open set, $0 \in \mathcal{D}$, $f(0) = 0$, and $f(\cdot)$ is continuous on $\mathcal{D}$. A continuously differentiable function $x : \mathcal{I}_{x_0} \to \mathcal{D}$ is said to be a *solution* of (7.1) on the interval $\mathcal{I}_{x_0} \subset \mathbb{R}$ if $x(\cdot)$ satisfies (7.1) for all $t \in \mathcal{I}_{x_0}$. Recall that every bounded solution to (7.1) can be extended on a semi-infinite time interval $[0, \infty)$ [114]. We assume that (7.1) possesses unique solutions in forward time for all initial conditions except possibly the origin in the following sense. For every $x \in \mathcal{D}\backslash\{0\}$ there exists $\tau_x > 0$ such that, if $y_1 : [0, \tau_1) \to \mathcal{D}$ and $y_2 : [0, \tau_2) \to \mathcal{D}$ are two solutions of (7.1) with $y_1(0) = y_2(0) = x$, then $\tau_x \leq \min\{\tau_1, \tau_2\}$ and $y_1(t) = y_2(t)$ for all $t \in [0, \tau_x)$. Without loss of generality, we assume that for each x, τ_x is chosen to be the largest such number in $\overline{\mathbb{R}}_+$. In this case, we denote the *trajectory* or *solution curve* of (7.1) on $[0, \tau_x)$ satisfying the consistency property $s(0, x) = x$ and the semi-group property $s(t, s(\tau, x)) = s(t + \tau, x)$ for every $x \in \mathcal{D}$ and $t, \tau \in [0, \tau_x)$ by $s(\cdot, x)$ or $s^x(\cdot)$. Sufficient conditions for forward uniqueness in the absence of Lipschitz continuity can be found in [1], [76, Section 10], [140], and [243, Section 1].

The next result presents the vector comparison principle [17,50,148,171,180] for nonlinear dynamical systems.

Theorem 7.1 [180]. Consider the nonlinear dynamical system (7.1). Assume there exists a continuously differentiable vector function $V : \mathcal{D} \to \mathcal{Q} \subseteq \mathbb{R}^q$ such that

$$V'(x)f(x) \leq\leq w(V(x)), \quad x \in \mathcal{D}, \tag{7.2}$$

where $w : \mathcal{Q} \to \mathbb{R}^q$ is a continuous function, $w(\cdot) \in \mathcal{W}$, and

$$\dot{z}(t) = w(z(t)), \quad z(t_0) = z_0, \quad t \in \mathcal{I}_{z_0}, \tag{7.3}$$

has a unique solution $z(t)$, $t \in \mathcal{I}_{z_0}$. If $[t_0,\, t_0 + \tau] \subseteq \mathcal{I}_{x_0} \cap \mathcal{I}_{z_0}$ is a compact interval and $V(x_0) \leq\leq z_0$, $z_0 \in \mathcal{Q}$, then $V(x(t)) \leq\leq z(t)$, $t \in [t_0,\, t_0 + \tau]$.

The next definition introduces the notion of finite-time stability.

Definition 7.1 [30]. Consider the nonlinear dynamical system (7.1). The zero solution $x(t) \equiv 0$ to (7.1) is *finite-time stable* if there exist an open neighborhood $\mathcal{N} \subseteq \mathcal{D}$ of the origin and a function $T : \mathcal{N}\backslash\{0\} \to (0, \infty)$, called the *settling-time function*, such that the following statements hold:

i) *Finite-time convergence.* For every $x \in \mathcal{N}\backslash\{0\}$, $s^x(t)$ is defined on $[0, T(x))$, $s^x(t) \in \mathcal{N}\backslash\{0\}$ for all $t \in [0, T(x))$, and $\lim_{t\to T(x)} s(x, t) = 0$.

ii) *Lyapunov stability.* For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{N}$ and for every $x \in \mathcal{B}_\delta(0)\backslash\{0\}$, $s(t, x) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [0, T(x))$.

The zero solution $x(t) \equiv 0$ of (7.1) is *globally finite-time stable* if it is finite-time stable with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.

Note that if the zero solution $x(t) \equiv 0$ to (7.1) is finite-time stable, then it is asymptotically stable, and hence, finite-time stability is a stronger notion than asymptotic stability. It is shown in [30] that if the zero solution $x(t) \equiv 0$ to (7.1) is finite-time stable, then (7.1) has a unique solution $s(\cdot, \cdot)$ defined on $\mathbb{R}_+ \times \mathcal{N}$ for every initial condition in an open neighborhood of the origin, including the origin, and $s(t, x) = 0$ for all $t \geq T(x)$, $x \in \mathcal{N}$, where $T(0) \triangleq 0$.

7.3. Finite-Time Stability via Vector Lyapunov Functions

We start this section by considering an example of a finite-time stable system with a continuous but non-Lipschitzian vector field.

Example 7.1 [30]. Consider the scalar system

$$\dot{x}(t) = -k\,\mathrm{sign}(x(t))|x(t)|^\alpha, \quad x(0) = x_0, \quad t \geq 0, \tag{7.4}$$

where $x_0 \in \mathbb{R}$, $\mathrm{sign}(x) \triangleq \frac{x}{|x|}$, $x \neq 0$, $\mathrm{sign}(0) \triangleq 0$, $k > 0$, and $\alpha \in (0, 1)$. The right-hand side of (7.4) is continuous everywhere and locally Lipschitz everywhere except the origin. Hence, every initial condition in $\mathbb{R}\backslash\{0\}$ has a unique solution in forward time on a sufficiently small time interval. The solution to (7.4) is obtained by direct integration and is given by

$$s(t,x_0) = \begin{cases} \operatorname{sign}(x_0)\left[|x_0|^{1-\alpha} - k(1-\alpha)t\right]^{\frac{1}{1-\alpha}}, & t < \frac{1}{k(1-\alpha)}|x_0|^{1-\alpha}, \quad x_0 \neq 0, \\ 0, & t \geq \frac{1}{k(1-\alpha)}|x_0|^{1-\alpha}, \quad x_0 \neq 0, \\ 0, & t \geq 0, \quad x_0 = 0. \end{cases} \tag{7.5}$$

It is clear from (7.5) that $i)$ in Definition 7.1 is satisfied with $\mathcal{N} = \mathcal{D} = \mathbb{R}$ and with the settling-time function $T : \mathbb{R} \to \mathbb{R}_+$ given by

$$T(x_0) = \frac{1}{k(1-\alpha)}|x_0|^{1-\alpha}, \quad x_0 \in \mathbb{R}. \tag{7.6}$$

Lyapunov stability follows by considering the Lyapunov function $V(x) = x^2$, $x \in \mathbb{R}$. Thus, the zero solution $x(t) \equiv 0$ to (7.4) is globally finite-time stable. △

Next, we present sufficient conditions for finite-time stability using a vector Lyapunov function involving a vector differential inequality.

Theorem 7.2. Consider the nonlinear dynamical system (7.1). Assume there exist a continuously differentiable vector function $V : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+^q$, where $\mathcal{Q} \subset \mathbb{R}^q$ and $0 \in \mathcal{Q}$, and a positive vector $p \in \mathbb{R}_+^q$ such that $V(0) = 0$, the scalar function $p^{\mathrm{T}}V(x)$, $x \in \mathcal{D}$, is positive definite, and

$$V'(x)f(x) \leq\leq w(V(x)), \quad x \in \mathcal{D}, \tag{7.7}$$

where $w : \mathcal{Q} \to \mathbb{R}^q$ is continuous, $w(\cdot) \in \mathcal{W}$, and $w(0) = 0$. In addition, assume that the vector comparison system

$$\dot{z}(t) = w(z(t)), \quad z(0) = z_0, \quad t \in \mathcal{I}_{z_0}, \tag{7.8}$$

has a unique solution in forward time $z(t)$, $t \in \mathcal{I}_{z_0}$, and there exist a continuously differentiable function $v : \mathcal{Q} \to \mathbb{R}$, real numbers $c > 0$ and $\alpha \in (0,1)$, and a neighborhood $\mathcal{M} \subseteq \mathcal{Q}$ of the origin such that $v(\cdot)$ is positive definite and

$$v'(z)w(z) \leq -c(v(z))^{\alpha}, \quad z \in \mathcal{M}. \tag{7.9}$$

Then the zero solution $x(t) \equiv 0$ to (7.1) is finite-time stable. Moreover, if $\mathcal{N}$ is as in Definition 7.1 and $T : \mathcal{N} \to [0, \infty)$ is the settling-time function, then

$$T(x_0) \leq \frac{1}{c(1-\alpha)} (v(V(x_0)))^{1-\alpha}, \quad x_0 \in \mathcal{N}, \tag{7.10}$$

and $T(\cdot)$ is continuous on $\mathcal{N}$. If, in addition, $\mathcal{D} = \mathbb{R}^n$, $v(\cdot)$ is radially unbounded, and (7.9) holds on $\mathbb{R}^q$, then the zero solution $x(t) \equiv 0$ to (7.1) is globally finite-time stable.

Proof. Assume there exist a continuously differentiable vector function $V : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+^q$ and a positive vector $p \in \mathbb{R}_+^q$ such that $p^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, is positive definite, that is, $p^{\mathrm{T}} V(0) = 0$ and $p^{\mathrm{T}} V(x) > 0$, $x \neq 0$. Note that since $p^{\mathrm{T}} V(x) \leq \max_{i=1,\ldots,q}\{p_i\} \mathbf{e}^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, the function $\mathbf{e}^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, is also positive definite.

Let $\mathcal{V} \subseteq \mathcal{M}$ be a bounded open set such that $0 \in \mathcal{V}$ and $\overline{\mathcal{V}} \subset \mathcal{Q}$. Then $\partial\mathcal{V}$ is compact and $0 \notin \partial\mathcal{V}$. Now, it follows from Weierstrass' theorem that the continuous function $v(\cdot)$ attains a minimum on $\partial\mathcal{V}$ and since $v(\cdot)$ is positive definite, $\min_{z \in \partial\mathcal{V}} v(z) > 0$. Let $0 < \beta < \min_{z \in \partial\mathcal{V}} v(z)$ and $\mathcal{D}_\beta \triangleq \{z \in \mathcal{V} : v(z) \leq \beta\}$. It follows from (7.9) that $\mathcal{D}_\beta \subset \mathcal{M}$ is invariant with respect to (7.8). Furthermore, it follows from (7.9), the positive definiteness of $v(\cdot)$, and standard Lyapunov arguments that for every $\hat{\varepsilon} > 0$ there exists $\hat{\delta} > 0$ such that $\mathcal{B}_{\hat{\delta}}(0) \subset \mathcal{D}_\beta \subset \mathcal{M}$ and

$$\|z(t)\|_1 \leq \hat{\varepsilon}, \quad \|z_0\|_1 < \hat{\delta}, \tag{7.11}$$

where $\|\cdot\|_1$ denotes the absolute sum norm, $\mathcal{B}_{\hat{\delta}}(0)$ is defined in terms of the absolute sum norm $\|\cdot\|_1$, and $t \in \mathcal{I}_{z_0}$. Moreover, since the solution $z(t)$ to (7.8) is bounded for all $t \in \mathcal{I}_{z_0}$, it can be extended on the semi-infinite interval $[0, \infty)$ [114], and hence, $z(t)$ is defined for all $t \geq 0$. Furthermore, it follows from Theorem 7.1 with $q = 1$, $w(y) = -cy^\alpha$, and $z(t) = s(t, v(z_0))$, where $\alpha \in (0, 1)$, that

$$v(z(t)) \leq s(t, v(z_0)), \quad z_0 \in \mathcal{B}_{\hat{\delta}}(0), \quad t \in [0, \infty), \tag{7.12}$$

where $s(\cdot,\cdot)$ is given by (7.5) with $k = c$. Now, it follows from (7.5), (7.12), and the positive definiteness of $v(\cdot)$ that

$$z(t) = 0, \quad t \geq \frac{1}{c(1-\alpha)}(v(z_0))^{1-\alpha}, \quad z_0 \in \mathcal{B}_{\hat{\delta}}(0), \tag{7.13}$$

which implies finite-time convergence of the trajectories of (7.8) for all $z_0 \in \mathcal{B}_{\hat{\delta}}(0)$. This along with (7.11) implies finite-time stability of the zero solution $z(t) \equiv 0$ to (7.8).

Next, it follows from the continuity of $V(\cdot)$ that there exists $\delta_1 > 0$ such that $\|V(x_0)\|_1 < \hat{\delta}$ for all $\|x_0\| < \delta_1$, where $\|\cdot\|$ is the Euclidian norm on $\mathbb{R}^n$. Now, choose $z_0 = V(x_0) \in \mathcal{B}_{\hat{\delta}}(0)$ for all $\|x_0\| < \delta_1$. In this case, it follows from (7.7) and Theorem 7.1 that $V(x(t)) \leq\leq z(t)$ on a compact interval $[0, \tau_{x_0}]$, where $[0, \tau_{x_0})$ is the maximal interval of existence of the solution $x(t)$ to (7.1). Since $z(t)$, $t \geq 0$, is bounded and $\mathbf{e}^{\mathrm{T}}V(\cdot)$ is positive definite it follows that $x(t)$, $t \in [0, \tau_{x_0}]$, is bounded, and hence, $x(t)$ can be extended to the semi-infinite interval $[0, \infty)$. Using (7.13) it follows that

$$\mathbf{e}^{\mathrm{T}}V(x(t)) = \mathbf{e}^{\mathrm{T}}z(t) = 0, \quad t \geq \frac{1}{c(1-\alpha)}(v(z_0))^{1-\alpha}, \quad z_0 = V(x_0) \in \mathcal{B}_{\hat{\delta}}(0). \tag{7.14}$$

Since $\mathbf{e}^{\mathrm{T}}V(\cdot)$ is positive definite, it follows that

$$x(t) = 0, \quad t \geq \frac{1}{c(1-\alpha)}(v(V(x_0)))^{1-\alpha}, \quad \|x_0\| < \delta_1, \tag{7.15}$$

which implies finite-time convergence of the trajectories of (7.1) for all $\|x_0\| < \delta_1$. Furthermore, it follows from (7.15) that the settling-time function satisfies

$$T(x_0) \leq \frac{1}{c(1-\alpha)}(v(V(x_0)))^{1-\alpha}, \quad \|x_0\| < \delta_1. \tag{7.16}$$

Next, note that since $\mathbf{e}^{\mathrm{T}}V(x)$, $x \in \mathcal{D}$, is positive definite, there exist $r > 0$ and class $\mathcal{K}$ functions [111] α, $\beta : [0, r] \to \overline{\mathbb{R}}_+$ such that $\mathcal{B}_r(0) \subset \mathcal{D}$, where $\mathcal{B}_r(0)$ is defined in terms of the Euclidean norm $\|\cdot\|$, and

$$\alpha(\|x\|) \leq \mathbf{e}^{\mathrm{T}}V(x) \leq \beta(\|x\|), \quad x \in \mathcal{B}_r(0). \tag{7.17}$$

Let $\varepsilon > 0$ and choose $0 < \hat{\varepsilon} < \min\{\varepsilon, r\}$. In this case, it follows from the Lyapunov stability of the nonlinear vector comparison system (7.8) that there exists $\mu = \mu(\hat{\varepsilon}) = \mu(\varepsilon) > 0$ such that if $\|z_0\|_1 < \mu$, then $\|z(t)\|_1 < \alpha(\hat{\varepsilon})$, $t \geq 0$. Now, choose $z_0 = V(x_0) \geq\geq 0$, $x_0 \in \mathcal{D}$. Since $V(x)$, $x \in \mathcal{D}$, is continuous, $\mathbf{e}^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, is also continuous. Hence, for $\mu = \mu(\hat{\varepsilon}) > 0$ there exists $\delta = \delta(\mu(\hat{\varepsilon})) = \delta(\varepsilon) > 0$ such that $\delta < \min\{\delta_1, \hat{\varepsilon}\}$, and if $\|x_0\| < \delta$, then $\mathbf{e}^{\mathrm{T}} V(x_0) = \mathbf{e}^{\mathrm{T}} z_0 = \|z_0\|_1 < \mu$, which implies that $\|z(t)\|_1 < \alpha(\hat{\varepsilon})$, $t \geq 0$. Now, with $z_0 = V(x_0) \geq\geq 0$, $x_0 \in \mathcal{D}$, and the assumption that $w(\cdot) \in \mathcal{W}$ it follows from (7.7) and Theorem 7.1 that $0 \leq\leq V(x(t)) \leq\leq z(t)$ on any compact interval $[0, \tau]$, and hence, $\mathbf{e}^{\mathrm{T}} z(t) = \|z(t)\|_1$, $t \in [0, \tau]$.

Let $\tau > 0$ be such that $x(t) \in \mathcal{B}_r(0)$, $t \in [0, \tau]$, for all $x_0 \in \mathcal{B}_\delta(0)$. Thus, using (7.17), if $\|x_0\| < \delta$, then

$$\alpha(\|x(t)\|) \leq \mathbf{e}^{\mathrm{T}} V(x(t)) \leq \mathbf{e}^{\mathrm{T}} z(t) < \alpha(\hat{\varepsilon}), \quad t \in [0, \tau], \tag{7.18}$$

which implies $\|x(t)\| < \hat{\varepsilon} < \varepsilon$, $t \in [0, \tau]$. Now, suppose, *ad absurdum*, that for some $x_0 \in \mathcal{B}_\delta(0)$ there exists $\hat{t} > \tau$ such that $\|x(\hat{t})\| = \hat{\varepsilon}$. Then, for $z_0 = V(x_0)$ and the compact interval $[0, \hat{t}]$ it follows from (7.7) and Theorem 7.1 that $V(x(\hat{t})) \leq\leq z(\hat{t})$, which implies that $\alpha(\hat{\varepsilon}) = \alpha(\|x(\hat{t})\|) \leq \mathbf{e}^{\mathrm{T}} V(x(\hat{t})) \leq \mathbf{e}^{\mathrm{T}} z(\hat{t}) < \alpha(\hat{\varepsilon})$, leading to a contradiction. Hence, for a given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x_0 \in \mathcal{B}_\delta(0)$, $\|x(t)\| < \varepsilon$, $t \geq t_0$, which implies Lyapunov stability of the zero solution $x(t) \equiv 0$ to (7.1). This, along with (7.15), implies finite-time stability of the zero solution $x(t) \equiv 0$ to (7.1) with $\mathcal{N} \triangleq \mathcal{B}_\delta(0)$. Equation (7.10) implies that $T(\cdot)$ is continuous at the origin, and hence, by Proposition 2.4 of [30], continuous on $\mathcal{N}$.

Finally, if $\mathcal{D} = \mathbb{R}^n$ and $v(\cdot)$ is radially unbounded, then global finite-time stability follows using standard arguments. $\square$

Assume the conditions of Theorem 7.2 are satisfied with $q = 1$. In this case, there exists a continuously differentiable, positive definite function $V : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}_+$ such that (7.7) holds, and there exists a continuously differentiable, positive definite function $v : \mathcal{Q} \to \overline{\mathbb{R}}_+$ such that (7.9) holds. Since $q = 1$ and $\mathcal{M}$ is a neighborhood of the origin, it follows that there exists $\gamma > 0$ such that $[0, \gamma] \subset \mathcal{M}$. Furthermore, since $v(\cdot)$ is positive definite, there exists $\beta > 0$ such that $v'(z) \geq 0$ for all $z \in [0, \beta]$. Next, consider the function $\tilde{v}(x) \triangleq v(V(x))$, $x \in \mathcal{D}$, and note that $\tilde{v}(\cdot)$ is positive definite. Define $\mathcal{V} \triangleq \{x \in \mathcal{D} : V(x) \leq \min\{\beta, \gamma\}\}$. Then it follows from (7.7) and (7.9) that

$$\begin{aligned} \dot{\tilde{v}}(x) &= v'(V(x))V'(x)f(x) \\ &\leq v'(V(x))w(V(x)) \\ &\leq -c(v(V(x)))^{\alpha} \\ &= -c(\tilde{v}(x))^{\alpha}, \quad x \in \mathcal{V}, \end{aligned} \tag{7.19}$$

which implies condition (4.7) in Theorem 4.2 of [30]. Thus, in the case where $q = 1$, Theorem 7.2 specializes to Theorem 4.2 of [30].

The next result is a specialization of Theorem 7.2 to the case where the structure of the comparison dynamics directly guarantees finite-time stability of the comparison system. That is, there is *no* need to require the existence of a scalar function $v(\cdot)$ such that (7.9) holds in order to guarantee finite-time stability of the nonlinear dynamical system (7.1).

Corollary 7.1. Consider the nonlinear dynamical system (7.1). Assume there exist a continuously differentiable vector function $V : \mathcal{D} \to \mathcal{Q} \cap \overline{\mathbb{R}}^q_+$, where $\mathcal{Q} \subset \mathbb{R}^q$ and $0 \in \mathcal{Q}$, and a positive vector $p \in \mathbb{R}^q_+$ such that $V(0) = 0$, the scalar function $p^{\mathrm{T}}V(x)$, $x \in \mathcal{D}$, is positive definite, and

$$V'(x)f(x) \leq\leq W(V(x))^{[\alpha]}, \quad x \in \mathcal{D}, \tag{7.20}$$

where $\alpha \in (0, 1)$, $W \in \mathbb{R}^{q \times q}$ is essentially nonnegative and Hurwitz, and $(V(x))^{[\alpha]} \triangleq [(V_1(x))^{\alpha}, \ldots, (V_q(x))^{\alpha}]^{\mathrm{T}}$. Then the zero solution $x(t) \equiv 0$ to (7.1) is finite-time stable. If, in addition, $\mathcal{D} = \mathbb{R}^n$, then the zero solution $x(t) \equiv 0$ to (7.1) is globally finite-time stable.

Proof. Consider the comparison system given by

$$\dot{z}(t) = W(z(t))^{[\alpha]}, \quad z(0) = z_0, \quad t \geq 0, \tag{7.21}$$

where $z_0 \in \overline{\mathbb{R}}_+^q$. Note that the right-hand side in (7.21) is of class $\mathcal{W}$ and is essentially nonnegative and, hence, the solutions to (7.21) are nonnegative for all nonnegative initial conditions [94]. Since $W \in \mathbb{R}^{q\times q}$ is essentially nonnegative and Hurwitz, it follows from Theorem 3.2 of [94] that there exist positive vectors $\hat{p} \in \mathbb{R}_+^q$ and $r \in \mathbb{R}_+^q$ such that

$$0 = W^{\mathrm{T}}\hat{p} + r. \tag{7.22}$$

Now, consider the Lyapunov function $v(z) = \hat{p}^{\mathrm{T}} z$, $z \in \overline{\mathbb{R}}_+^q$. Note that $v(0) = 0$, $v(z) > 0$, $z \in \overline{\mathbb{R}}_+^q$, $z \neq 0$, and $v(\cdot)$ is radially unbounded. Let $\beta \triangleq \min_{i=1,\ldots,q} r_i$, $\gamma \triangleq \max_{i=1,\ldots,q} \hat{p}_i^{\alpha}$, where r_i and $\hat{p}_i$ are the ith components of $r \in \mathbb{R}_+^q$ and $\hat{p} \in \mathbb{R}_+^q$, respectively. Then

$$\begin{aligned}
\dot{v}(z) &= \hat{p}^{\mathrm{T}} W z^{[\alpha]} \\
&= -r^{\mathrm{T}} z^{[\alpha]} \\
&\leq -\frac{\beta}{\gamma}\gamma\left(\sum_{i=1}^{q} z_i^{\alpha}\right) \\
&\leq -\frac{\beta}{\gamma}\left(\sum_{i=1}^{q} \hat{p}_i^{\alpha} z_i^{\alpha}\right) \\
&\leq -\frac{\beta}{\gamma}\left(\sum_{i=1}^{q} \hat{p}_i z_i\right)^{\alpha} \\
&\leq -\frac{\beta}{\gamma}(v(z))^{\alpha} \\
&= -c(v(z))^{\alpha}, \quad z \in \overline{\mathbb{R}}_+^q,
\end{aligned} \tag{7.23}$$

where $c \triangleq \frac{\beta}{\gamma}$. Thus, it follows from Theorem 4.2 of [30] that the comparison system (7.21) is finite-time stable with the settling-time function $T(z_0) \leq \frac{1}{c(1-\alpha)}(v(z_0))^{1-\alpha}$, $z_0 \in \overline{\mathbb{R}}_+^q$. Next, it follows from Corollary 4.1 of [180] that the nonlinear dynamical system (7.1) is asymptotically stable with the domain of attraction $\mathcal{N} \subset \mathcal{D}$. Now, the result is a direct consequence of Theorem 7.2. □

Remark 7.1. If the conditions of Corollary 7.1 hold, then the nonlinear dynamical system (7.1) has a settling-time function $T(x_0) \leq \frac{1}{c(1-\alpha)}(v(V(x_0)))^{1-\alpha}$, $x_0 \in \mathcal{N}$, where $v(z) = \hat{p}^{\mathrm{T}} z$, $z \in \overline{\mathbb{R}}_+^q$.

7.4. Finite-Time Stabilization of Large-Scale Dynamical Systems

In the recent paper [180], the notion of a *control vector Lyapunov function* was introduced as a generalization of the classical notion of a control Lyapunov function. Furthermore, a universal stabilizing feedback control law was constructed based on a control vector Lyapunov function [180]. In this section, we show that this control law can be used to stabilize large-scale dynamical systems in finite time provided that the comparison system possesses non-Lipschitzian dynamics.

Specifically, consider the large-scale dynamical system composed of q interconnected subsystems given by

$$\dot{x}_i(t) = f_i(x(t)) + G_i(x(t))u_i(t), \quad t \geq t_0, \quad i = 1, \ldots, q, \tag{7.24}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ satisfying $f_i(0) = 0$ and $G_i : \mathbb{R}^n \to \mathbb{R}^{n_i \times m_i}$ are continuous functions for all $i = 1, \ldots, q$, and $u_i(\cdot)$, $i = 1, \ldots, q$, satisfy sufficient regularity conditions such that the nonlinear dynamical system (7.24) has a unique solution forward in time. Let $V = [V_1, \ldots, V_q]^{\mathrm{T}} : \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$ be a component decoupled continuously differentiable vector function, that is, $V(x) = [V_1(x_1), \ldots, V_q(x_q)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$, $p \in \mathbb{R}_+^q$ be a positive vector, and $w : \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ be a continuous function such that $V(0) = 0$, $p^{\mathrm{T}} V(x)$, $x \in \mathbb{R}^n$, is positive definite, and $w(\cdot) \in \mathcal{W}$ with $w(0) = 0$. Define $\alpha_i(x) \triangleq V_i'(x_i) f_i(x)$, $x \in \mathbb{R}^n$, and $\beta_i(x) \triangleq G_i^{\mathrm{T}}(x) V_i'^{\mathrm{T}}(x_i)$, $x \in \mathbb{R}^n$, and assume that

$$V_i'(x_i) f_i(x) < w_i(V(x)), \quad x \in \mathcal{R}_i, \quad i = 1, \ldots, q, \tag{7.25}$$

where $\mathcal{R}_i \triangleq \{x \in \mathbb{R}^n, x \neq 0 : \beta_i(x) = 0\}$, $i = 1, \ldots, q$. Construct the feedback control law

$\phi(x) = [\phi_1^{\mathrm{T}}(x), \ldots, \phi_q^{\mathrm{T}}(x)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$, given by

$$\phi_i(x) = \begin{cases} -\left(c_{0i} + \frac{(\alpha_i(x) - w_i(V(x))) + \sqrt{(\alpha_i(x) - w_i(V(x)))^2 + (\beta_i^{\mathrm{T}}(x)\beta_i(x))^2}}{\beta_i^{\mathrm{T}}(x)\beta_i(x)}\right)\beta_i(x), & \beta_i(x) \neq 0, \\ 0, & \beta_i(x) = 0, \end{cases} \tag{7.26}$$

where $c_{0i} > 0$, $i = 1, \ldots, q$.

The vector Lyapunov derivative components $\dot{V}_i(\cdot)$, $i = 1, \ldots, q$, along the trajectories of the closed-loop dynamical system (7.24), with $u = \phi(x)$, $x \in \mathbb{R}^n$, given by (7.26), is given by

$$\begin{aligned} \dot{V}_i(x_i) &= V_i'(x_i)[f_i(x) + G_i(x)\phi_i(x)] \\ &= \alpha_i(x) + \beta_i^{\mathrm{T}}(x)\phi_i(x) \\ &= \begin{cases} -c_{0i}\beta_i^{\mathrm{T}}(x)\beta_i(x) - \sqrt{(\alpha_i(x) - w_i(V(x)))^2 + (\beta_i^{\mathrm{T}}(x)\beta_i(x))^2} \\ +w_i(V(x)), & \beta_i(x) \neq 0, \\ \qquad \alpha_i(x), & \beta_i(x) = 0, \end{cases} \\ &< w_i(V(x)), \quad x \in \mathbb{R}^n. \end{aligned} \tag{7.27}$$

It follows from Theorem 7.2 that if there exist $v : \overline{\mathbb{R}}_+^q \to \mathbb{R}$, $c > 0$, and $\alpha \in (0, 1)$ such that $v(\cdot)$ is positive definite and

$$v'(z)w(z) \leq -c(v(z))^{\alpha}, \quad z \in \mathcal{M}, \tag{7.28}$$

where $\mathcal{M}$ is a neighborhood of $\overline{\mathbb{R}}_+^q$ containing the origin, then the zero solution $x(t) \equiv 0$ to (7.24) is finite-time stable with the settling time $T(x_0) \leq \frac{1}{c(1-\alpha)}(v(V(x_0)))^{1-\alpha}$, $x_0 \in \mathbb{R}^n$. In this case, it follows from Theorem 5.1 of [180] that $V(x)$, $x \in \mathbb{R}^n$, is a control vector Lyapunov function.

Remark 7.2. If $\mathcal{R}_i = \emptyset$, $i = 1, \ldots, q$, then the function $w(\cdot)$ in (7.26) can be chosen to be

$$w(z) = Wz^{[\alpha]}, \quad z \in \overline{\mathbb{R}}_+^q, \tag{7.29}$$

where $W \in \mathbb{R}^{q \times q}$ is essentially nonnegative and asymptotically stable, $\alpha \in (0, 1)$, and $z^{[\alpha]} \triangleq [z_1^{\alpha}, \ldots, z_q^{\alpha}]^{\mathrm{T}}$. In this case, condition (7.28) need *not* be verified and it follows from Corollary

7.1 that the close-loop system (7.24) and (7.26) with $w(\cdot)$ given by (7.29) is finite-time stable and, hence, the controller (7.26) is finite-time stabilizing controller for (7.24).

Since $f_i(\cdot)$ and $G_i(\cdot)$ are continuous and $V_i(\cdot)$ is continuously differentiable for all $i = 1, \ldots, q$, it follows that $\alpha_i(x)$ and $\beta_i(x)$, $x \in \mathbb{R}^n$, $i = 1, \ldots, q$, are continuous functions, and hence, $\phi_i(x)$ given by (7.26) is continuous for all $x \in \mathbb{R}^n$ if either $\beta_i(x) \neq 0$ or $\alpha_i(x) - w_i(V(x)) < 0$ for all $i = 1, \ldots, q$. Hence, the feedback control law given by (7.26) is continuous everywhere except for the origin. The following result provides necessary and sufficient conditions under which the feedback control law given by (7.26) is guaranteed to be continuous at the origin in addition to being continuous everywhere else.

Proposition 7.1 [180]. The feedback control law $\phi(x)$ given by (7.26) is continuous on $\mathbb{R}^n$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 < \|x\| < \delta$ there exists $u_i \in \mathbb{R}^{m_i}$ such that $\|u_i\| < \varepsilon$ and $\alpha_i(x) + \beta_i^{\mathrm{T}}(x)u_i < w_i(V(x))$, $i = 1, \ldots, q$.

The following corollary addressing the case where $q = 1$ is immediate from the above arguments. In this case, the nonlinear dynamical system (7.24) specializes to

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{7.30}$$

where $x_0 \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $f(0) = 0$ and $G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are continuous functions.

Corollary 7.2. Consider the nonlinear dynamical system (7.30). Assume there exists a continuously differentiable function $V : \mathcal{D} \to \overline{\mathbb{R}}_+$ such that $V(\cdot)$ is positive definite, $w(V(x)) \triangleq -c(V(x))^\alpha$, $x \in \mathbb{R}^n$, and

$$V'(x)f(x) \leq w(V(x)) = -c(V(x))^\alpha, \quad x \in \mathcal{R}, \tag{7.31}$$

where $c > 0$, $\alpha \in (0,1)$, $\mathcal{R} \triangleq \{x \in \mathbb{R}^n, x \neq 0 : V'(x)G(x) = 0\}$. Then the nonlinear dynamical system (7.30) with the feedback controller $u = \phi(x)$, $x \in \mathbb{R}^n$, given by

$$\phi(x) = \begin{cases} -\left(c_0 + \frac{(\alpha(x)-w(V(x)))+\sqrt{(\alpha(x)-w(V(x)))^2+(\beta^{\mathrm{T}}(x)\beta(x))^2}}{\beta^{\mathrm{T}}(x)\beta(x)}\right)\beta(x), & \beta(x) \neq 0, \\ 0, & \beta(x) = 0, \end{cases} \tag{7.32}$$

where $c_0 > 0$, $\alpha(x) \triangleq V'(x)f(x)$, $x \in \mathbb{R}^n$, and $\beta(x) \triangleq G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)$, $x \in \mathbb{R}^n$, is finite-time stable with the settling time $T(x_0) \leq \frac{1}{c(1-\alpha)}(V(x_0))^{1-\alpha}$, $x_0 \in \mathbb{R}^n$. Furthermore, $V(\cdot)$ is a control Lyapunov function.

Next, we show that the control law (7.32) ensures finite-time stability for a perturbed version of (7.30) with bounded perturbations. Specifically, consider the more accurate description of the system (7.30) given by the perturbed model

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t) + g(t, x(t)), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{7.33}$$

where $g : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function that captures disturbances, uncertainties, parameter variations, or modeling errors. Assume that there exists a continuously differentiable function $V : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ such that the conditions of Corollary 7.2 are satisfied. Then it follows from Theorem 5.2 of [30] that there exist $\delta_0 > 0$, $\ell > 0$, $\tau > 0$, and an open neighborhood $\mathcal{V}$ of the origin such that for every continuous function $g(\cdot, \cdot)$ with

$$\delta = \sup_{[t_0,\infty)\times\mathbb{R}^n} \|g(t, x)\| < \delta_0, \tag{7.34}$$

the solutions $x(t)$, $t \geq t_0$, to the closed-loop system (7.33) with $u(t)$ given by (7.32) and $x_0 \in \mathcal{V}$ are such that $x(t) \in \mathcal{V}$, $t \geq t_0$, and

$$\|x(t)\| \leq \ell\delta^{\gamma}, \quad t \geq \tau, \tag{7.35}$$

where $\gamma = \frac{1-\alpha}{\alpha}$. Note that, if in Corollary 7.2, $\alpha \in (0, \frac{1}{2})$, then $\gamma > 1$ which makes the bound in (7.35) smaller for sufficiently small δ compared to the case when $0 < \gamma < 1$. In addition, if $g(\cdot, \cdot)$ is such that

$$\|g(t, x)\| \leq L\|x\|, \quad (t, x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{7.36}$$

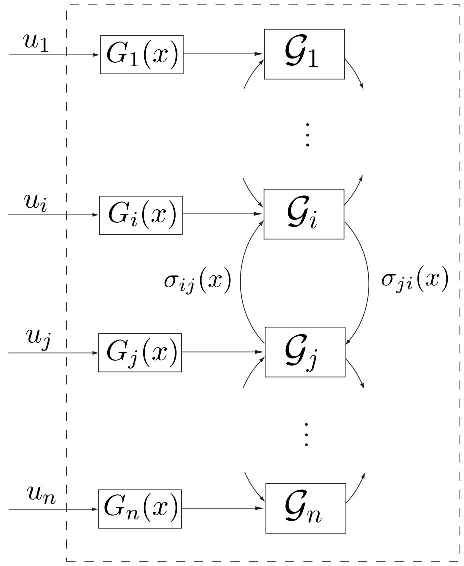

Figure 7.1: Large-scale dynamical system $\mathcal{G}$

where $L \geq 0$, then it follows from Theorem 5.3 of [30] that $x(t) = 0$, $t \geq \tau$, for all $x_0 \in \mathcal{V}$. Finally, if $g : \mathbb{R}^n \to \mathbb{R}^n$ is only a function of the dynamical system state and

$$\|g(x)\| \leq L\|x\|, \quad x \in \mathbb{R}^n, \tag{7.37}$$

where $L \geq 0$, then it follows from Theorem 5.4 of [30] that the zero solution $x(t) \equiv 0$ to the closed-loop system (7.33) with $u(t)$ given by (7.32) is finite-time stable.

Next, consider the large-scale dynamical system $\mathcal{G}$ shown in Figure 7.1 involving energy exchange between n interconnected subsystems. Let $x_i : [0, \infty) \to \overline{\mathbb{R}}_+$ denote the energy (and hence a nonnegative quantity) of the ith subsystem, let $u_i : [0, \infty) \to \mathbb{R}$ denote the control input to the ith subsystem, and let $\sigma_{ij} : \overline{\mathbb{R}}^n_+ \to \overline{\mathbb{R}}_+$, $i \neq j$, $i, j = 1, \ldots, n$, denote the instantaneous rate of energy flow from the jth subsystem to the ith subsystem.

An energy balance yields the large-scale dynamical system [104]

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{7.38}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^{\mathrm{T}}$, $t \geq t_0$, $f_i(x) = \sum_{j=1, j\neq i}^{n} \phi_{ij}(x)$, where $\phi_{ij}(x) \triangleq \sigma_{ij}(x) - \sigma_{ji}(x)$, $x \in \overline{\mathbb{R}}^n_+$, $i \neq j$, $i, j = 1, \ldots, q$, denotes the net energy flow from the jth subsystem to the ith subsystem, $G(x) = \mathrm{diag}[G_1(x_1), \ldots, G_n(x_n)]$, $x \in \overline{\mathbb{R}}^n_+$, $G_i(x_i) = 0$ if and only if

$x_i = 0$ for all $i = 1, \ldots, n$, and $u(t) \in \mathbb{R}^n$, $t \geq t_0$. Here, we assume that $\sigma_{ij}(x) = 0$, $x \in \overline{\mathbb{R}}^n_+$, whenever $x_j = 0$, $i \neq j$, $i, j = 1, \ldots, n$. In this case, $f(\cdot)$ is *essentially nonnegative* [94, 104] (i.e., $f_i(x) \geq 0$ for all $x \in \overline{\mathbb{R}}^n_+$ such that $x_i = 0$, $i = 1, \ldots, n$). The above constraint implies that if the energy of the jth subsystem of $\mathcal{G}$ is zero, then this subsystem cannot supply any energy to its surroundings. In addition, we assume that $\phi_{ij}(x') \leq \phi_{ij}(x'')$, $i \neq j$, $i, j = 1, \ldots, n$, for all $x', x'' \in \mathbb{R}^n$ such that $x'_i = x''_i$ and $x'_k \leq x''_k$, $k \neq i$, where x_i is the ith component of x. The above assumption implies that the more energy the surroundings of the ith subsystem possess, the more energy is gained by the ith subsystem from the energy exchange due to subsystem interconnections. Finally, in order to ensure that the trajectories of the closed-loop system remain in the nonnegative orthant of the state space for all nonnegative initial conditions, we seek a feedback control law $u(\cdot)$ that guarantees the closed-loop system dynamics are essentially nonnegative [94].

For the dynamical system $\mathcal{G}$, consider the control vector Lyapunov function candidate $V(x) = [V_1(x_1), \ldots, V_n(x_n)]^{\mathrm{T}}$, $x \in \overline{\mathbb{R}}^n_+$, given by

$$V(x) = [x_1, \ldots, x_n]^{\mathrm{T}}, \quad x \in \overline{\mathbb{R}}^n_+. \tag{7.39}$$

Note that $V(0) = 0$ and $\mathbf{e}^{\mathrm{T}} V(x)$, $x \in \overline{\mathbb{R}}^n_+$, is positive definite and radially unbounded. Furthermore, consider the function

$$w(V(x)) = [-V_1^{1/2}(x_1) + \sum_{j=1, j\neq 1}^{n} \phi_{1j}(V(x)), \ldots, -V_n^{1/2}(x_n) + \sum_{j=1, j\neq n}^{n} \phi_{nj}(V(x))]^{\mathrm{T}}, \quad x \in \overline{\mathbb{R}}^n_+, \tag{7.40}$$

and note that it follows from the above constraints that $w(\cdot) \in \mathcal{W}$ and $w(0) = 0$. Furthermore, note that $\mathcal{R}_i \triangleq \{x \in \overline{\mathbb{R}}^n_+, x_i \neq 0 : V'_i(x_i) G_i(x_i) = 0\} = \{x \in \overline{\mathbb{R}}^n_+, x_i \neq 0 : x_i = 0\} = \emptyset$, and hence, condition (7.25) is satisfied for $V(\cdot)$ and $w(\cdot)$ given by (7.39) and (7.40), respectively.

Next, consider the vector comparison system

$$\dot{z}(t) = w(z(t)), \quad z(t_0) = z_0, \quad t \geq t_0, \tag{7.41}$$

where $z_0 \in \overline{\mathbb{R}}_+^n$ and the ith component of $w(z)$ is given by $w_i(z) = -z_i^{1/2} + \sum_{j=1, j\neq i}^n \phi_{ij}(z)$, $z \in \overline{\mathbb{R}}_+^n$. In addition, consider the Lyapunov function candidate $v(z) = \mathbf{e}^{\mathrm{T}} z$, $z \in \overline{\mathbb{R}}_+^n$, and note that $v(\cdot)$ is radially unbounded, $v(0) = 0$, $v(z) > 0$, $z \in \overline{\mathbb{R}}_+^n$, $z \neq 0$, and

$$\begin{aligned} v'(z)w(z) &= -\sum_{i=1}^n z_i^{1/2} + \sum_{i=1}^n \sum_{j=1, j\neq i}^n \phi_{ij}(z) \\ &= -\sum_{i=1}^n z_i^{1/2} \\ &\leq -\left(\sum_{i=1}^n z_i\right)^{1/2} \\ &= -(v(z))^{1/2}, \quad z \in \overline{\mathbb{R}}_+^n. \end{aligned} \tag{7.42}$$

Thus, it follows from Theorem 7.2 with $c = 1$, $\alpha = \frac{1}{2}$, and $\mathcal{M} = \overline{\mathbb{R}}_+^n$ that the large-scale dynamical system (7.38) is finite-time stable with a settling time $T(x_0) \leq 2(\mathbf{e}^{\mathrm{T}} x_0)^{1/2}$, $x_0 \in \overline{\mathbb{R}}_+^n$, and $V(x)$, $x \in \overline{\mathbb{R}}_+^n$, given by (7.39) is a control vector Lyapunov function for (7.38).

Finally, the feedback control law $\phi(x) = [\phi_1^{\mathrm{T}}(x), \ldots, \phi_n^{\mathrm{T}}(x)]^{\mathrm{T}}$, where $\phi_i(x)$, $i = 1, \ldots, n$, is given by (7.26) with $\alpha_i(x) = V_i'(x_i) f_i(x) = \sum_{j=1, j\neq i}^n \phi_{ij}(x)$, $\beta_i(x) = G_i(x_i)$, $x \in \overline{\mathbb{R}}_+^n$, and $c_{0i} > 0$, $i = 1, \ldots, n$, is a finite-time globally stabilizing decentralized feedback controller for (7.38). It can be seen from the structure of the feedback control law that the closed-loop system dynamics are essentially nonnegative. Furthermore, since $\alpha_i(x) - w_i(V(x)) = (V_i(x_i))^{1/2}$, $x \in \overline{\mathbb{R}}_+^n$, $i = 1, \ldots, n$, this feedback controller is fully independent from $f(x)$ which represents the internal interconnections of the large-scale system dynamics, and hence, is robust against full modeling uncertainty in $f(x)$.

7.5. Finite-Time Stabilization for Large-Scale Homogeneous Systems

In this section, we use geometric homogeneity developed in [13, 33] to construct finite-time controllers for large-scale homogeneous systems. First, we introduce the concept of homogeneity in relation to a scaling operation or dilation.

Definition 7.2 [13,33]. Let $x \triangleq [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$. A *dilation* $\Delta_\lambda(x) : (\lambda, x_1, \ldots, x_n) \mapsto (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n)$ is a mapping that assigns to every $\lambda > 0$ a diffeomorphism $\Delta_\lambda(x) = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n)$, where $(x_1, \ldots, x_n)$ is a suitable coordinate on $\mathbb{R}^n$ and $r_i > 0$, $i = 1, \ldots, n$, are constants. A function $V : \mathbb{R}^n \to \mathbb{R}$ is *homogeneous of degree* $l \in \mathbb{R}$ *with respect to the dilation* $\Delta_\lambda(x)$ if $V(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) = \lambda^l V(x_1, \ldots, x_n)$. Finally, a vector field $f(x) \triangleq [f_1(x), \ldots, f_n(x)]^{\mathrm{T}} : \mathbb{R}^n \to \mathbb{R}$ is *homogeneous of degree* $k \in \mathbb{R}$ *with respect to the dilation* $\Delta_\lambda(x)$ if $f_i(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) = \lambda^{k+r_i} f_i(x_1, \ldots, x_n)$, $\lambda > 0$, $i = 1, \ldots, n$.

Proposition 7.2 [33]. Consider the nonlinear dynamical system (7.1). Assume $f(\cdot)$ is homogeneous of degree $k \in \mathbb{R}$ with respect to the dilation $\Delta_\lambda(x)$. Furthermore, assume $f(\cdot)$ is continuous on $\mathcal{D}$ and $x = 0$ is an asymptotically stable equilibrium point of (7.1). If $k < 0$, then $x = 0$ is a finite-time stable equilibrium point of (7.1). Alternatively, suppose $f(x) = g_1(x) + \cdots + g_q(x)$, $x \in \mathcal{D}$, where for each $i = 1, \ldots, q$, the vector field $g_i(\cdot)$ is continuous on $\mathcal{D}$, homogeneous of degree $k_i \in \mathbb{R}$ with respect to the dilation $\Delta_\lambda(x)$, and $k_1 < \cdots < k_q$. If $x = 0$ is a finite-time-stable equilibrium point of $g_1(\cdot)$, then $x = 0$ is a finite-time-stable equilibrium point of $f(\cdot)$.

Remark 7.3. If in Theorem 7.2 the comparison function $w(\cdot)$ is homogeneous of degree $k < 0$ with respect to the dilation $\Delta_\lambda(z)$ and $z = 0$ is an asymptotically stable equilibrium point of (7.8), then the zero solution $x(t) \equiv 0$ to (7.1) is finite-time stable. In this case, there is *no* need to construct a scalar positive-definite function $v(\cdot)$ such that (7.9) holds.

Now, consider the large-scale dynamical system $\mathcal{G}$ involving energy exchange between n interconnected subsystems given by (7.38). Furthermore, assume that there exists a constant $k \in \mathbb{R}$ such that

$$\phi_{ij}(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) = \lambda^{r_i+k} \phi_{ij}(x_1, \ldots, x_n), \quad i, j = 1, \ldots, q, \quad i \neq j, \tag{7.43}$$

for every $\lambda > 0$ and for given $r_i > 0$, $i = 1, \ldots, n$. Next, consider the decentralized controller given by

$$u_i = \psi_i(x_i), \quad i = 1, \ldots, n, \tag{7.44}$$

with $\psi_i(x_i)$ satisfying

$$G_i(\lambda^{r_i} x_i)\psi_i(\lambda^{r_i} x_i) = \lambda^{r_i+l} G_i(x_i)\psi_i(x_i), \quad i = 1, \ldots, n, \quad x \in \mathbb{R}^n, \tag{7.45}$$

and

$$\sum_{i=1}^{n} G_i(x_i)\psi_i(x_i) < 0, \quad x \in \mathbb{R}^n, \tag{7.46}$$

for every $\lambda > 0$ and for given $r_i > 0$, $i = 1, \ldots, n$, where $l \in \mathbb{R}$, $G(x) = \mathrm{diag}[G_1(x_1), \ldots, G_n(x_n)]$, and $G_i(x_i) = 0$ if and only if $x_i = 0$, $i = 1, \ldots, n$. If $l = k < 0$, then it follows from Proposition 7.2 that the closed-loop system (7.38) with $u(t) = [\psi_1(x_1), \ldots, \psi_n(x_n)]^{\mathrm{T}}$ is globally finite-time stable. Alternatively, if $l < k$ and $l < 0$, then it follows from Proposition 7.2 that the closed-loop system (7.38) with $u(t) = [\psi_1(x_1), \ldots, \psi_n(x_n)]^{\mathrm{T}}$ is finite-time stable.

Note that if $l < k$ and $l < 0$, then stability is only local [33]. In order to obtain a global result in this case, we need to examine the control vector Lyapunov function of the large-scale homogeneous system. Specifically, for the dynamical system $\mathcal{G}$ given by (7.38), consider the control vector Lyapunov function candidate $V(\cdot)$ given by (7.39). Furthermore, consider the function

$$w(V(x)) = [-\sigma_1(V_1(x_1)) + \sum_{j=1, j\neq 1}^{n} \phi_{1j}(V(x)), \ldots, -\sigma_n(V_n(x_n)) + \sum_{j=1, j\neq n}^{n} \phi_{nj}(V(x))]^{\mathrm{T}},$$
$$x \in \overline{\mathbb{R}}_+^n, \tag{7.47}$$

where $\sigma_i(\cdot)$ satisfies $\sigma_i(\lambda^{r_i} x_i) = \lambda^{r_i+l}\sigma_i(x_i)$ for each $\lambda > 0$ and given $r_i > 0$, $i = 1, \ldots, n$, $l < 0$, $x_i \in \overline{\mathbb{R}}_+$, $\sigma_i(0) = 0$, $\sigma_i(z) > 0$ for $z \neq 0$, $z \in \mathbb{R}$, and $\phi_{ij}(\cdot)$ satisfies (7.43) with $k > l$ and $i, j = 1, \ldots, n$, $i \neq j$.

Next, consider the comparison system given by (7.41) where the ith component of $w(z)$ is given by $w_i(z) = -\sigma_i(z_i) + \sum_{j=1, j\neq i}^{n} \phi_{ij}(z)$, $z \in \overline{\mathbb{R}}_+^n$. Then it follows from Proposition 7.2

that (7.41) is finite-time stable. Furthermore, consider the Lyapunov function candidate $v(z) = \mathbf{e}^{\mathrm{T}} z$, $z \in \overline{\mathbb{R}}_+^n$, and note that $v(\cdot)$ is radially unbounded, $v(0) = 0$, $v(z) > 0$, $z \in \overline{\mathbb{R}}_+^n$, $z \neq 0$, and

$$
\begin{aligned}
v'(z)w(z) &= -\sum_{i=1}^{n} \sigma_i(z_i) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \phi_{ij}(z) \\
&= -\sum_{i=1}^{n} \sigma_i(z_i) \\
&< 0, \quad z \neq 0, \quad z \in \overline{\mathbb{R}}_+^n,
\end{aligned} \tag{7.48}
$$

which implies that (7.41) is globally asymptotically stable. Hence, (7.41) is globally asymptotically stable, and thus, the large-scale homogeneous system (7.38) with $u_i = \psi_i(x_i)$, $i = 1, \ldots, n$, is globally finite-time stable and $V(\cdot)$ given by (7.39) is a control vector Lyapunov function for (7.38). Finally, (7.26) with $\alpha_i(x) = V_i'(x_i) f_i(x) = \sum_{j=1, j\neq i}^{n} \phi_{ij}(x)$, $\beta_i(x) = G_i(x_i)$, $x \in \overline{\mathbb{R}}_+^n$, and $c_{0i} > 0$, $i = 1, \ldots, n$, is a finite-time globally stabilizing decentralized feedback controller for (7.38). It can be seen from the structure of the feedback control law that the closed-loop system dynamics are essentially nonnegative. Furthermore, since $\alpha_i(x) - w_i(V(x)) = \sigma_i(V_i(x_i))$, $x \in \overline{\mathbb{R}}_+^n$, $i = 1, \ldots, n$, this feedback controller is fully independent from $f(x)$ which represents the internal interconnections of the large-scale system dynamics, and hence, is robust against full modeling uncertainty in $f(x)$.

7.6. Illustrative Numerical Examples

In our first example we consider the large-scale dynamical system shown in Figure 7.1 with the power balance equation (7.38) where $\sigma_{ij}(x) = \sigma_{ij} x_j^2$, $\sigma_{ij} \geq 0$, $i \neq j$, $i, j = 1, \ldots, n$, and $G_i(x_i) = x_i^{1/4}$, $i = 1, \ldots, n$. Note that in this case $\phi_{ij}(x') \leq \phi_{ij}(x'')$, $i \neq j$, $i, j = 1, \ldots, n$, for all $x', x'' \in \overline{\mathbb{R}}_+^n$ such that $x_i' = x_i''$ and $x_k' \leq x_k''$, $k \neq i$. Furthermore, with $u_i = -2x_i^{1/4}$, $i = 1, \ldots, n$, the conditions of Proposition 7.1 are satisfied, and hence, the feedback control law (7.26) is continuous on $\overline{\mathbb{R}}_+^n$. For our simulation we set $n = 3$, $\sigma_{12} = 2$, $\sigma_{13} = 3$, $\sigma_{21} = 1.5$, $\sigma_{23} = 0.3$, $\sigma_{31} = 4.4$, $\sigma_{32} = 0.6$, $c_{01} = 1$, $c_{02} = 1$, and $c_{03} = 0.25$, with initial condition

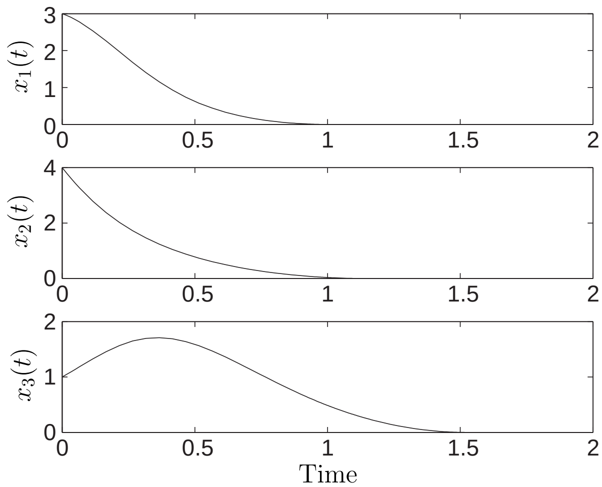

Figure 7.2: Controlled system states versus time

$x_0 = [3, 4, 1]^{\mathrm{T}}$. Figure 7.2 shows the states of the closed-loop system versus time and Figure 7.3 shows control signal for each decentralized control channel as a function of time.

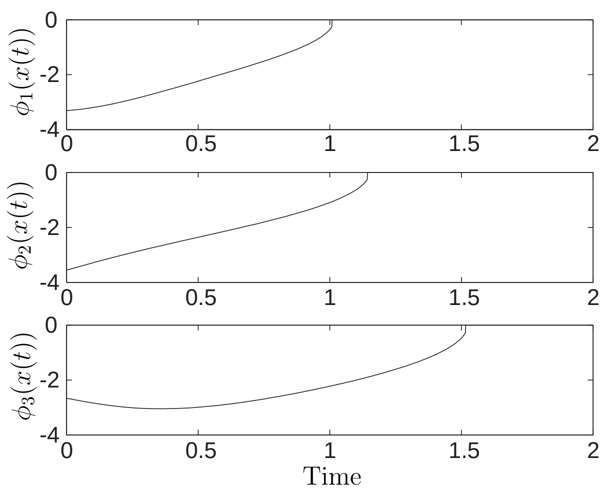

Figure 7.3: Control signals in each decentralized control channel versus time

For the next example we consider control of thermoacoustic instabilities in combustion processes. Engineering applications involving steam and gas turbines and jet and ramjet engines for power generation and propulsion technology involve combustion processes. Due to the inherent coupling between several intricate physical phenomena in these processes involving acoustics, thermodynamics, fluid mechanics, and chemical kinetics, the dynamic behavior of combustion systems is characterized by highly complex nonlinear models [10, 11, 61, 136].

The unstable dynamic coupling between heat release in combustion processes generated by reacting mixtures releasing chemical energy and unsteady motions in the combustor develop acoustic pressure and velocity oscillations which can severely impact operating conditions and system performance. These pressure oscillations, known as *thermoacoustic instabilities*, often lead to high vibration levels causing mechanical failures, high levels of acoustic noise, high burn rates, and even component melting. Hence, the need for active control to mitigate combustion-induced pressure instabilities is critical.

Next, we design a finite-time stabilizing controller for the combustion system we considered in Section 5.6. Recall that this model is given by

$$\dot{x}_1(t) = \alpha_1 x_1(t) + \theta_1 x_2(t) - \beta(x_1(t)x_3(t) + x_2(t)x_4(t)) + u_1(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{7.49}$$

$$\dot{x}_2(t) = -\theta_1 x_1(t) + \alpha_1 x_2(t) + \beta(x_2(t)x_3(t) - x_1(t)x_4(t)) + u_2(t), \quad x_2(0) = x_{20}, \tag{7.50}$$

$$\dot{x}_3(t) = \alpha_2 x_3(t) + \theta_2 x_4(t) + \beta(x_1^2(t) - x_2^2(t)) + u_3(t), \quad x_3(0) = x_{30}, \tag{7.51}$$

$$\dot{x}_4(t) = -\theta_2 x_3(t) + \alpha_2 x_4(t) + 2\beta x_1(t)x_2(t) + u_4(t), \quad x_4(0) = x_{40}, \tag{7.52}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ represent growth/decay constants, $\theta_1, \theta_2 \in \mathbb{R}$ represent frequency shift constants, $\beta = ((\gamma+1)/8\gamma)\omega_1$, where γ denotes the ratio of specific heats, ω_1 is the frequency of the fundamental mode, and u_i, $i = 1, \ldots, 4$, are control input signals. For the data parameters $\alpha_1 = 5$, $\alpha_2 = -55$, $\theta_1 = 4$, $\theta_2 = 32$, $\gamma = 1.4$, $\omega_1 = 1$, and $x_0 = [2, 3, 1, 1]^{\mathrm{T}}$, the open-loop (i.e., $u_i(t) \equiv 0, i = 1, \ldots, 4$) dynamics (7.49)–(7.52) result in a limit cycle instability.

To stabilize this system in finite time we design a feedback control law given by (7.32), where $V(x) = \frac{1}{2}x^{\mathrm{T}}x$, $x \in \mathbb{R}^4$, $c = 1$, $c_0 = 1$, $\alpha = \frac{3}{4}$. In this case, $V'(x) = x^{\mathrm{T}}$, $G(x) = I_4$, and hence, $\mathcal{R} = \{x \in \mathbb{R}^4, x \neq 0 : x^{\mathrm{T}} = 0\} = \emptyset$. Thus, condition (7.31) is trivially satisfied and it follows from Corollary 7.2 that the closed-loop system (7.49)–(7.52) with the feedback control law (7.32) is finite-time stable. Furthermore, the hypothesis of Proposition 7.1 are satisfied for the case where $q = 1$, and hence, the control law (7.32) is continuous in $\mathbb{R}^4$.

Specifically, with $u = -f(x) - 2^{-3/4}g(x)$, where

$$f(x) = \begin{bmatrix} \alpha_1 x_1 + \theta_1 x_2 - \beta(x_1 x_3 + x_2 x_4) \\ -\theta_1 x_1 + \alpha_1 x_2 + \beta(x_2 x_3 - x_1 x_4) \\ \alpha_2 x_3 + \theta_2 x_4 + \beta(x_1^2 - x_2^2) \\ -\theta_2 x_3 + \alpha_2 x_4 + 2\beta x_1 x_2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} x_1^{1/3} \\ x_2^{1/3} \\ x_3^{1/3} \\ x_4^{1/3} \end{bmatrix}, \tag{7.53}$$

the inequality

$$\alpha(x) + \beta^{\mathrm{T}}(x)u \leq w(V(x)), \quad 0 < \|x\| < \delta, \tag{7.54}$$

is satisfied, where $\alpha(x) \triangleq V'(x)f(x)$, $\beta(x) \triangleq G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)$, $w(V(x)) = -(V(x))^{3/4}$, $x \in \mathbb{R}^4$, and $0 < \delta < 1$. To see this, note that

$$\begin{aligned} \alpha(x) + \beta^{\mathrm{T}}(x)u &= -2^{-3/4}x^{\mathrm{T}}g(x) \\ &= -2^{-3/4}\sum_{i=1}^{4} x_i^{4/3} \\ &\leq -2^{-3/4}\left(\sum_{i=1}^{4} x_i^2\right)^{3/4} \\ &= -(V(x))^{3/4} \\ &= w(V(x)), \quad 0 < \|x\| < \delta < 1. \end{aligned} \tag{7.55}$$

In addition, since $f(\cdot)$ and $g(\cdot)$ are continuous and $f(0) = g(0) = 0$, it follows from (7.55) that for every $\varepsilon > 0$, there exists $0 < \delta < 1$ such that for all $0 < \|x\| < \delta$ there exists $u \in \mathbb{R}^4$ such that $\|u\| < \varepsilon$ and inequality (7.54) holds. Thus, the feedback control law (7.32) is continuous in $\mathbb{R}^4$. Figure 7.4 shows the states of the closed-loop system versus time and Figure 7.5 shows the control signals versus time.

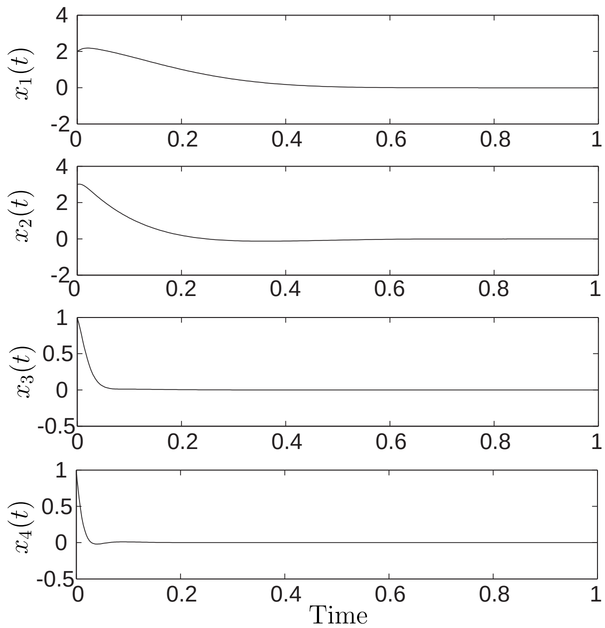

Figure 7.4: Controlled system states versus time

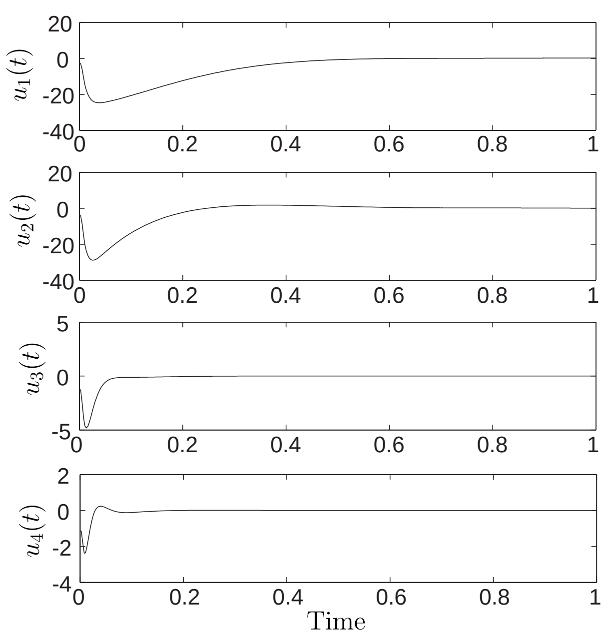

Figure 7.5: Control signals in each control channel versus time

Chapter 8

Finite-Time Semistability and Consensus for Nonlinear Dynamical Networks

8.1. Introduction

In a recent series of papers the authors in [31, 32] developed a unified stability analysis framework for systems having a continuum of equilibria. Since every neighborhood of a nonisolated equilibrium contains another equilibrium, a nonisolated equilibrium cannot be asymptotically stable. Hence, asymptotic stability is not the appropriate notion of stability for systems having a continuum of equilibria. Two notions that are of particular relevance to such systems are convergence and semistability. Convergence is the property whereby every system solution converges to a limit point that may depend on the system initial condition. Semistability is the additional requirement that all solutions converge to limit points that are Lyapunov stable. Semistability for an equilibrium thus implies Lyapunov stability, and is implied by asymptotic stability. It is important to note that semistability is not merely equivalent to asymptotic stability of the set of equilibria. Indeed, it is possible for a trajectory to converge to the set of equilibria without converging to any one equilibrium point as examples in [32] show.

The dependence of the limiting state on the initial state is seen in numerous dynamical systems including compartmental systems [134] which arise in chemical kinetics, biomedical, environmental, economic, power, and thermodynamic systems [104]. For these systems, every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium, and hence, these systems are semistable. Semistability is especially pertinent to networks of dynamic agents which exhibit convergence to a state of consensus in which the agents agree on certain quantities of interest. Semistability was first introduced in [47] for linear systems, and applied to matrix second-order systems in [23]. References [32] and [31] consider semistability of nonlinear systems, and give several stability results for systems having a continuum of equilibria based on nontangency and arc length of trajectories, respectively.

In addition to semistability, it is desirable that a dynamical system that exhibits semistability also possesses the property that trajectories that converge to a Lyapunov stable system state must do so in finite time rather than merely asymptotically. Finite-time convergence to an isolated Lyapunov stable equilibrium, that is, finite-time stability, was rigorously studied in [30], although finite-time stabilization of second-order systems was considered earlier in [28, 112]. More recently, researchers have considered finite-time stabilization of higher-order systems [120] as well as finite-time stabilization using output feedback [121]. Alternatively, discontinuous finite-time stabilizing feedback controllers have been developed in the literature [78, 211, 212]. However, in practical implementation, discontinuous feedback controllers can lead to chattering behavior due to system uncertainty or measurement noise, and hence, may excite unmodeled high-frequency system dynamics.

In this chapter, we merge the theories of semistability and finite-time stability developed in [30–32] to develop a rigorous framework for finite-time semistability. In Section 8.3, we extend the theory of semistability given in [31, 32] by presenting new Lyapunov theorems as well as the first converse Lyapunov theorem for semistability, which holds with a smooth (i.e., infinitely differentiable) Lyapunov function. Next, in Section 8.4, we establish finite-time semistability theory. We present the notions of finite-time convergence and finite-time semistability for nonlinear dynamical systems, and develop several sufficient Lyapunov stability theorems for finite-time semistability. Following [33], we exploit homogeneity as a means for verifying finite-time convergence in Section 8.5. Our main result in this direction asserts that a homogeneous system is finite-time semistable if and only if it is semistable and has a negative degree of homogeneity. This main result depends on a converse Lyapunov result for homogeneous semistable systems, which we develop. While our converse result resembles a related result for asymptotically stable systems given in [33, 209], the proof of our result is rendered more difficult by the fact that our result does not hold under the notions of homogeneity considered in [33, 209]. More specifically, while previous treatments of homogeneity involved Euler vector fields representing asymptotically stable dynamics, our results involve homogeneity with respect to a semi-Euler vector field representing a semistable system having the same equilibria as the dynamics of interest. Consequently, our theory precludes the use of dilations commonly used in the literature on homogeneous systems (such as [209]), and requires us to adopt a more geometric description of homogeneity (see [33] and references therein).

Next, in Section 8.6, we use the main results of this chapter to develop a general, thermodynamically motivated framework for designing semistable protocols in dynamical networks for achieving coordination tasks in finite time. Distributed decision-making for coordination of networks of dynamic agents involving information flow can be naturally captured by graph-theoretic notions. These dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles (UAV's), autonomous underwater vehicles (AUV's), distributed sensor networks, air and ground transportation systems, swarms of air and space vehicle formations [72], and congestion control in communication networks, to cite but a few examples. Hence, it is not surprising that a considerable research effort has been devoted to control of networks and control over networks in recent years [72, 135, 166, 185, 187]. However, with the notable exception of [58], finite-time coordination has not been addressed in the literature.

In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In such applications, it is important to develop information consensus protocols for networks of dynamic agents. An essential feature of the closed-loop dynamics under any control algorithm that achieves consensus in a dynamical network is the existence of a continuum of equilibria representing a state of consensus. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state. From a practical viewpoint, it is not sufficient to only guarantee that a network converges to a state of consensus since steady state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequently, semistable. Hence, in Section 8.7, we use the results from Sections 8.4–8.6 to develop a unified distributed control framework based on finite-time semistability for addressing the consensus problem in networks of agents.

We begin by establishing notation and definitions in Section 8.2.

8.2. Notation and Definitions

The notation used in this chapter is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\overline{\mathbb{R}}_+$ denotes the set of nonnegative real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $(\cdot)^{\mathrm{T}}$ denotes transpose, and "$\circ$" denotes the composition operator. For $A \in \mathbb{R}^{n\times m}$ we write $\operatorname{rank} A$ to denote the rank of A. Furthermore, $\partial\mathcal{S}$ and $\overline{\mathcal{S}}$ denote the boundary and the closure of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. We write $\|\cdot\|$ for the Euclidean vector norm, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball *centered* at α with *radius* ε, $\operatorname{dist}(p, \mathcal{M})$ for the distance from a point p to the set $\mathcal{M}$, that is, $\operatorname{dist}(p, \mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \|p-x\|$, $x(t) \to \mathcal{M}$ as $t \to \infty$ to denote that $x(t)$ approaches the set $\mathcal{M}$, that is, for each $\varepsilon > 0$ there exists $T > 0$ such that $\operatorname{dist}(x(t), \mathcal{M}) < \varepsilon$ for all $t > T$, and $V'(x)$ for the Fréchet derivative of V at x.

In this chapter, we consider nonlinear dynamical systems of the form

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \in \mathcal{I}_{x_0}, \tag{8.1}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \in \mathcal{I}_{x_0}$, is the system state vector, $\mathcal{D}$ is an open set, $f : \mathcal{D} \to \mathbb{R}^n$ is continuous on $\mathcal{D}$, $f^{-1}(0) \triangleq \{x \in \mathcal{D} : f(x) = 0\}$ is nonempty, and $\mathcal{I}_{x_0} = [0, \tau_{x_0})$, $0 \leq \tau_{x_0} \leq \infty$, is the maximal interval of existence for the solution $x(\cdot)$ of (8.1). A continuously differentiable function $x : \mathcal{I}_{x_0} \to \mathcal{D}$ is said to be a *solution* of (8.1) on the interval $\mathcal{I}_{x_0} \subset \mathbb{R}$ if x satisfies (8.1) for all $t \in \mathcal{I}_{x_0}$. The continuity of f implies that, for every $x_0 \in \mathcal{D}$, there exist $\tau_0 < 0 < \tau_1$ and a solution $x(\cdot)$ of (8.1) defined on (τ_0, τ_1) such that $x(0) = x_0$. A solution x is said to be *right maximally defined* if x cannot be extended on the right (either uniquely or nonuniquely) to a solution of (8.1). Here, we assume that for every initial condition $x_0 \in \mathcal{D}$, (8.1) has a unique right maximally defined solution, and this unique solution is defined on $[0, \infty)$. Under these assumptions, the solutions of (8.1) define a continuous *global semiflow* on $\mathcal{D}$, that is, $s : [0, \infty) \times \mathcal{D} \to \mathcal{D}$ is a jointly continuous function satisfying the *consistency property* $s(0, x) = x$ and the *semi-group property* $s(t, s(\tau, x)) = s(t + \tau, x)$ for every $x \in \mathcal{D}$ and $t, \tau \in [0, \infty)$. Furthermore, we assume that for every initial condition $x_0 \in \mathcal{D} \backslash f^{-1}(0)$, (8.1) has a local unique solution for negative time. Given $t \in [0, \infty)$ we denote the *flow* $s(t, \cdot) : \mathcal{D} \to \mathcal{D}$ of (8.1) by $s_t(x_0)$ or s_t. Likewise, given $x \in \mathcal{D}$ we denote the *solution curve* or *trajectory* $s(\cdot, x) : [0, \infty) \to \mathcal{D}$ of (8.1) by $s^x(t)$ or s^x. Finally, the image of $\mathcal{U} \subset \mathcal{D}$ under the flow s_t is defined as $s_t(\mathcal{U}) \triangleq \{y : y = s_t(x_0) \text{ for all } x_0 \in \mathcal{U}\}$.

A set $\mathcal{M} \subseteq \mathbb{R}^n$ is *positively invariant* if $s_t(\mathcal{M}) \subseteq \mathcal{M}$ for all $t \geq 0$. The set $\mathcal{M}$ is *negatively invariant* if, for every $z \in \mathcal{M}$ and every $t \geq 0$, there exists $x \in \mathcal{M}$ such that $s(t, x) = z$ and $s(\tau, x) \in \mathcal{M}$ for all $\tau \in [0, t]$. The set $\mathcal{M}$ is *invariant* if $s_t(\mathcal{M}) = \mathcal{M}$, $t \geq 0$. Note that a set is invariant if and only if it is positively and negatively invariant. Finally, a set $\mathcal{E} \subseteq \mathbb{R}^n$ is *connected* if and only if every pair of open sets $\mathcal{U}_i \subseteq \mathbb{R}^n$, $i = 1, 2$, satisfying $\mathcal{E} \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$ and $\mathcal{U}_i \cap \mathcal{E} \neq \emptyset$, $i = 1, 2$, has a nonempty intersection. A *connected component* of the set $\mathcal{E} \subseteq \mathbb{R}^n$ is a connected subset of $\mathcal{E}$ that is not properly contained in any connected subset of $\mathcal{E}$.

8.3. Lyapunov and Converse Lyapunov Theory for Semistability

In this section, we develop necessary and sufficient conditions for semistability. In order to develop necessary and sufficient conditions for finite-time semistability, we first need to establish a converse Lyapunov theorem for semistability. This extends some of the results in [13,145,170,209,238]. Converse Lyapunov theorems were extensively studied in [145,170]. In particular, Massera [170] proved a converse Lyapunov theorem under the assumption that the vector field f is locally Lipschitz continuous. For locally Lipschitz continuous vector fields, it has been shown that asymptotic stability implies the existence of a smooth (i.e., infinitely differentiable) Lyapunov function. Kurzweil [145] proved the existence of smooth Lyapunov functions for asymptotic stability under the assumption of f only being continuous. Unlike asymptotic stability, Lyapunov stability for autonomous dynamical systems does not imply the existence of a continuous Lyapunov function. However, semistability does imply the existence of a smooth Lyapunov function. Before stating this result, we first present several definitions and a key proposition.

Definition 8.1 [32]. An equilibrium point $x \in \mathcal{D}$ of (8.1) is *Lyapunov stable* if for every open subset $\mathcal{N}_\varepsilon$ of $\mathcal{D}$ containing x, there exists an open subset $\mathcal{N}_\delta$ of $\mathcal{D}$ containing x such that $s_t(\mathcal{N}_\delta) \subset \mathcal{N}_\varepsilon$ for all $t \geq 0$. An equilibrium point $x \in \mathcal{D}$ of (8.1) is *semistable* if it is Lyapunov stable and there exists an open subset $\mathcal{U}$ of $\mathcal{D}$ containing x such that for all initial conditions in $\mathcal{U}$, the trajectory of (8.1) converges to a Lyapunov stable equilibrium point, that is, $\lim_{t\to\infty} s(t,x) = y$, where $y \in \mathcal{D}$ is a Lyapunov stable equilibrium point of (8.1) and $x \in \mathcal{U}$. If, in addition, $\mathcal{U} = \mathcal{D} = \mathbb{R}^n$, then the equilibrium point $x \in \mathcal{D}$ of (8.1) is a *globally semistable equilibrium*. The system (8.1) is said to be *Lyapunov stable* if every equilibrium point of (8.1) is Lyapunov stable. The system (8.1) is said to be *semistable* if every equilibrium point of (8.1) is semistable. Finally, (8.1) is said to be *globally semistable* if every equilibrium of (8.1) is globally semistable.

Definition 8.2. The *domain of semistability* is the set of points $x_0 \in \mathcal{D}$ such that if $x(t)$ is a solution to (8.1) with $x(0) = x_0$, $t \geq 0$, then $x(t)$ converges to a Lyapunov stable equilibrium point in $\mathcal{D}$.

Note that if (8.1) is semistable, then its domain of semistability contains the set of equilibria in its interior. The following proposition gives a sufficient condition for a trajectory of (8.1) to converge to a limit. For this result, $\mathcal{D}_{\rm c} \subseteq \mathcal{D}$ denotes a positively invariant set with respect to (8.1) so that the orbit $\mathcal{O}_x$ of (8.1) is contained in $\mathcal{D}_{\rm c}$ for all $x \in \mathcal{D}_{\rm c}$.

Proposition 8.1. Consider the nonlinear dynamical system (8.1) and let $x \in \mathcal{D}_{\rm c}$. If the positive limit set $\omega(x)$ of (8.1) contains a Lyapunov stable equilibrium point y, then $y = \lim_{t\to\infty} s(t,x)$, that is, $\omega(x) = \{y\}$.

Proof. The proof of the result appears in [32]. For completeness of exposition, we provide a proof here. Suppose $y \in \omega(x)$ is Lyapunov stable and let $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_{\rm c}$ be an open neighborhood of y. Since y is Lyapunov stable, there exists an open neighborhood $\mathcal{N}_\delta \subset \mathcal{D}_{\rm c}$ of y such that $s_t(\mathcal{N}_\delta) \subseteq \mathcal{N}_\varepsilon$ for every $t \geq 0$. Now, since $y \in \omega(x)$, it follows that there exists $\tau \geq 0$ such that $s(\tau, x) \in \mathcal{N}_\delta$. Hence, $s(t+\tau, x) = s_t(s(\tau, x)) \in s_t(\mathcal{N}_\delta) \subseteq \mathcal{N}_\varepsilon$ for every $t > 0$. Since $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_{\rm c}$ is arbitrary, it follows that $y = \lim_{t\to\infty} s(t,x)$. Thus, $\lim_{n\to\infty} s(t_n, x) = y$ for every sequence $\{t_n\}_{n=1}^{\infty}$, and hence, $\omega(x) = \{y\}$. $\square$

Next, we present alternative equivalent characterizations of semistability of (8.1).

Proposition 8.2. Consider the nonlinear dynamical system (8.1). Then the following statements are equivalent:

i) The system (8.1) is semistable.

ii) For each $x_{\rm e} \in f^{-1}(0)$, there exist class $\mathcal{K}$ and $\mathcal{L}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, respectively, and $\delta = \delta(x_{\rm e}) > 0$, such that if $\|x_0 - x_{\rm e}\| < \delta$, then $\|x(t) - x_{\rm e}\| \leq \alpha(\|x_0 - x_{\rm e}\|)$, $t \geq 0$, and $\operatorname{dist}(x(t), f^{-1}(0)) \leq \beta(t)$, $t \geq 0$.

iii) For each $x_{\rm e} \in f^{-1}(0)$, there exist class $\mathcal{K}$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, a class $\mathcal{L}$ function $\beta(\cdot)$, and $\delta = \delta(x_{\rm e}) > 0$, such that if $\|x_0 - x_{\rm e}\| < \delta$, then $\operatorname{dist}(x(t), f^{-1}(0)) \leq \alpha_1(\|x(t) - x_{\rm e}\|)\beta(t) \leq \alpha_2(\|x_0 - x_{\rm e}\|)\beta(t)$, $t \geq 0$.

Proof. $(i) \Rightarrow ii)$). Suppose (8.1) is semistable and let $x_{\rm e} \in f^{-1}(0)$. It follows from Lemma 4.5 of [141] that there exists $\delta = \delta(x_{\rm e}) > 0$ and a class $\mathcal{K}$ function $\alpha(\cdot)$ such that if $\|x_0 - x_{\rm e}\| \leq \delta$, then $\|x(t) - x_{\rm e}\| \leq \alpha(\|x_0 - x_{\rm e}\|)$, $t \geq 0$. Without loss of generality, we may assume that δ is such that $\overline{\mathcal{B}_\delta(x_{\rm e})}$ is contained in the domain of semistability of (8.1). Hence, for every $x_0 \in \overline{\mathcal{B}_\delta(x_{\rm e})}$, $\lim_{t\to\infty} x(t) = x^* \in f^{-1}(0)$ and, consequently, $\lim_{t\to\infty} \mathrm{dist}(x(t), f^{-1}(0)) = 0$.

For each $\varepsilon > 0$ and $x_0 \in \overline{\mathcal{B}_\delta(x_{\rm e})}$, define $T_{x_0}(\varepsilon)$ to be the infimum of T with the property that $\mathrm{dist}(x(t), f^{-1}(0)) < \varepsilon$ for all $t \geq T$, that is, $T_{x_0}(\varepsilon) \triangleq \inf\{T : \mathrm{dist}(x(t), f^{-1}(0)) < \varepsilon, t \geq T\}$. For each $x_0 \in \overline{\mathcal{B}_\delta(x_{\rm e})}$, the function $T_{x_0}(\varepsilon)$ is nonnegative and nonincreasing in ε, and $T_{x_0}(\varepsilon) = 0$ for sufficiently large ε.

Next, let $T(\varepsilon) \triangleq \sup\{T_{x_0}(\varepsilon) : x_0 \in \overline{\mathcal{B}_\delta(x_{\rm e})}\}$. We claim that T is well defined. To show this, consider $\varepsilon > 0$ and $x_0 \in \overline{\mathcal{B}_\delta(x_{\rm e})}$. Since $\mathrm{dist}(s(t, x_0), f^{-1}(0)) < \varepsilon$ for every $t > T_{x_0}(\varepsilon)$, it follows from the continuity of s that, for every $\eta > 0$, there exists an open neighborhood $\mathcal{U}$ of x_0 such that $\mathrm{dist}(s(t, z), f^{-1}(0)) < \varepsilon$ for every $z \in \mathcal{U}$. Hence, $\limsup_{z\to x_0} T_z(\varepsilon) \leq T_{x_0}(\varepsilon)$ implying that the function $x_0 \mapsto T_{x_0}(\varepsilon)$ is upper semicontinuous at the arbitrarily chosen point x_0, and hence on $\overline{\mathcal{B}_\delta(x_{\rm e})}$. Since an upper semicontinuous function defined on a compact set achieves its supremum, it follows that $T(\varepsilon)$ is well defined. The function $T(\cdot)$ is the pointwise supremum of a collection of nonegative and nonincreasing functions, and is hence nonegative and nonincreasing. Moreover, $T(\varepsilon) = 0$ for every $\varepsilon > \max\{\alpha(\|x_0 - x_{\rm e}\|) : x_0 \in \overline{\mathcal{B}_\delta(x_{\rm e})}\}$.

Let $\psi(\varepsilon) \triangleq \frac{2}{\varepsilon}\int_{\varepsilon/2}^{\varepsilon} T(\sigma)\mathrm{d}\sigma + \frac{1}{\varepsilon} \geq T(\varepsilon) + \frac{1}{\varepsilon}$. The function $\psi(\varepsilon)$ is positive, continuous, strictly decreasing, and $\psi(\varepsilon) \to 0$ as $\varepsilon \to \infty$. Choose $\beta(\cdot) = \psi^{-1}(\cdot)$. Then $\beta(\cdot)$ is positive, continuous, strictly decreasing, and $\beta(\sigma) \to 0$ as $\sigma \to \infty$. Furthermore, $T(\beta(\sigma)) < \psi(\beta(\sigma)) = \sigma$. Hence, $\mathrm{dist}(x(t), f^{-1}(0)) \leq \beta(t)$, $t \geq 0$.

$(ii) \Rightarrow iii)$). Suppose $ii)$ holds and let $x_{\rm e} \in f^{-1}(0)$. Then it follows from Lemma 4.5 of [141] that $x_{\rm e}$ is Lyapunov stable. Choosing x_0 sufficiently close to $x_{\rm e}$, it follows from the inequality $\|x(t) - x_{\rm e}\| \leq \alpha(\|x_0 - x_{\rm e}\|)$, $t \geq 0$, that trajectories of (8.1) starting sufficiently close to $x_{\rm e}$ are bounded, and hence, the positive limit set of (8.1) is nonempty.

Since $\lim_{t\to\infty} \operatorname{dist}(x(t), f^{-1}(0)) = 0$, it follows that the positive limit set is contained in $f^{-1}(0)$. Now, since every point in $f^{-1}(0)$ is Lyapunov stable, it follows from Proposition 5.4 of [32] that $\lim_{t\to\infty} x(t) = x^*$, where $x^* \in f^{-1}(0)$ is Lyapunov stable. If $x^* = x_{\mathrm{e}}$, then it follows using similar arguments as above that there exists a class $\mathcal{L}$ function $\hat{\beta}(\cdot)$ such that $\operatorname{dist}(x(t), f^{-1}(0)) \le \|x(t)-x_{\mathrm{e}}\| \le \hat{\beta}(t)$ for every x_0 satisfying $\|x_0-x_{\mathrm{e}}\| < \delta$ and $t \ge 0$. Hence, $\operatorname{dist}(x(t), f^{-1}(0)) \le \sqrt{\|x(t) - x_{\mathrm{e}}\|}\sqrt{\hat{\beta}(t)}$, $t \ge 0$. Next, consider the case where $x^* \ne x_{\mathrm{e}}$ and let $\alpha_1(\cdot)$ be a class $\mathcal{K}$ function. In this case, note that $\lim_{t\to\infty} \operatorname{dist}(x(t), f^{-1}(0))/\alpha_1(\|x(t) - x_{\mathrm{e}}\|) = 0$, and hence, it follows using similar arguments as above that there exists a class $\mathcal{L}$ function $\beta(\cdot)$ such that $\operatorname{dist}(x(t), f^{-1}(0)) \le \alpha_1(\|x(t) - x_{\mathrm{e}}\|)\beta(t)$, $t \ge 0$. Finally, note that $\alpha_1 \circ \alpha$ is of class $\mathcal{K}$ (by Lemma 4.2 of [141]), and hence, *iii*) follows immediately.

(*iii*) $\Rightarrow$ *i*)). Suppose *iii*) holds and let $x_{\mathrm{e}} \in f^{-1}(0)$. Then it follows that $\alpha_1(\|x(t)-x_{\mathrm{e}}\|) \le \alpha_2(\|x(0) - x_{\mathrm{e}}\|)$, $t \ge 0$, that is, $\|x(t) - x_{\mathrm{e}}\| \le \alpha(\|x(0) - x_{\mathrm{e}}\|)$, where $t \ge 0$ and $\alpha = \alpha_1^{-1} \circ \alpha_2$ is of class $\mathcal{K}$ (by Lemma 4.2 of [141]). It now follows from Lemma 4.5 of [141] that x_{e} is Lyapunov stable. Since x_{e} was chosen arbitrarily, it follows that every equilibrium point is Lyapunov stable. Furthermore, $\lim_{t\to\infty} \operatorname{dist}(x(t), f^{-1}(0)) = 0$. Choosing x_0 sufficiently close to x_{e}, it follows from the inequality $\|x(t) - x_{\mathrm{e}}\| \le \alpha(\|x_0 - x_{\mathrm{e}}\|)$, $t \ge 0$, that trajectories of (8.1) starting sufficiently close to x_{e} are bounded, and hence, the positive limit set of (8.1) is nonempty. Since every point in $f^{-1}(0)$ is Lyapunov stable, it follows from Proposition 5.4 of [32] that $\lim_{t\to\infty} x(t) = x^*$, where $x^* \in f^{-1}(0)$ is Lyapunov stable. Hence, by definition, (8.1) is semistable. □

Given a continuous function $V : \mathcal{D} \to \mathbb{R}$, the *upper right Dini derivative* of V along the solution of (8.1) is defined by $\dot{V}(s(t,x)) \triangleq \limsup_{h\to 0^+} \frac{1}{h}[V(s(t+h,x)) - V(s(t,x))]$. It is easy to see that $\dot{V}(x_{\mathrm{e}}) = 0$ for every $x_{\mathrm{e}} \in f^{-1}(0)$. Finally, if $V(\cdot)$ is continuously differentiable, then $\dot{V}(x) = V'(x)f(x)$.

Next, we present a sufficient condition for semistability.

Theorem 8.1. Consider the system (8.1). Let $\mathcal{U}$ be an open neighborhood of $f^{-1}(0)$ and assume there exists a continuously differentiable function $V : \mathcal{U} \to \mathbb{R}$ such that $V'(x)f(x) < 0$, $x \in \mathcal{U} \backslash f^{-1}(0)$. If (8.1) is Lyapunov stable, then (8.1) is semistable.

Proof. Since (8.1) is Lyapunov stable by assumption, for every $z \in f^{-1}(0)$, there exists an open neighborhood $\mathcal{V}_z$ of z such that $s([0,\infty) \times \mathcal{V}_z)$ is bounded and contained in $\mathcal{U}$. The set $\mathcal{V} \triangleq \bigcup_{z \in f^{-1}(0)} \mathcal{V}_z$ is an open neighborhood of $f^{-1}(0)$ contained in $\mathcal{U}$. Consider $x \in \mathcal{V}$ so that there exists $z \in f^{-1}(0)$ such that $x \in \mathcal{V}_z$ and $s(t,x) \in \mathcal{U}$, $t \geq 0$. Since $s([0,\infty) \times \mathcal{V}_z)$ is bounded it follows that the positive limit set of x is nonempty and invariant. Furthermore, it follows from the assumption that $\dot{V}(s(t,x)) \leq 0$, $t \geq 0$, and hence, it follows from the Krasovskii-LaSalle invariant set theorem [141, p. 128] that $s(t,x) \to \mathcal{M}$ as $t \to \infty$, where $\mathcal{M}$ is the largest invariant set contained in the set $\mathcal{R} = \{y \in \mathcal{U} : V'(y)f(y) = 0\}$. Note that $\mathcal{R} = f^{-1}(0)$ is invariant, and hence, $\mathcal{M} = \mathcal{R}$, which implies that $\lim_{t\to\infty} \operatorname{dist}(s(t,x), f^{-1}(0)) = 0$. Finally, since every point in $f^{-1}(0)$ is Lyapunov stable, it follows from Proposition 8.1 that $\lim_{t\to\infty} s(t,x) = x^*$, where $x^* \in f^{-1}(0)$ is Lyapunov stable. Hence, by definition, (8.1) is semistable. □

Next, we present a slightly more general theorem for semistability wherein we do not assume that all points in $\dot{V}^{-1}(0)$ are Lyapunov stable but rather we assume that all points in the largest invariant subset of $\dot{V}^{-1}(0)$ are Lyapunov stable.

Theorem 8.2. Consider the nonlinear dynamical system (8.1) and let $\mathcal{Q}$ be an open neighborhood of $f^{-1}(0)$. Suppose the orbit $\mathcal{O}_x$ of (8.1) is bounded for all $x \in \mathcal{Q}$ and assume that there exists a continuously differentiable function $V : \mathcal{Q} \to \mathbb{R}$ such that

$$V'(x)f(x) \leq 0, \quad x \in \mathcal{Q}. \tag{8.2}$$

If every point in the largest invariant subset $\mathcal{M}$ of $\{x \in \mathcal{Q} : V'(x)f(x) = 0\}$ is Lyapunov stable, then (8.1) is semistable.

Proof. Since every solution of (8.1) is bounded, it follows from the hypotheses on $V(\cdot)$ that, for every $x \in \mathcal{Q}$, the positive limit set $\omega(x)$ of (8.1) is nonempty and contained in the largest invariant subset $\mathcal{M}$ of $\{x \in \mathcal{Q} : V'(x)f(x) = 0\}$. Since every point in $\mathcal{M}$ is a Lyapunov stable equilibrium, it follows from Proposition 8.1 that $\omega(x)$ contains a single point for every $x \in \mathcal{Q}$ and $\lim_{t\to\infty} s(t,x)$ exists for every $x \in \mathcal{Q}$. Now, since $\lim_{t\to\infty} s(t,x) \in \mathcal{M}$ is Lyapunov stable for every $x \in \mathcal{Q}$, semistability is immediate. $\square$

Example 8.1. Consider the nonlinear dynamical system given by

$$\dot{x}_1(t) = \sigma_{12}(x_2(t)) - \sigma_{21}(x_1(t)), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{8.3}$$

$$\dot{x}_2(t) = \sigma_{21}(x_1(t)) - \sigma_{12}(x_2(t)), \ x_2(0) = x_{20}, \tag{8.4}$$

where $x_1, x_2 \in \mathbb{R}$, $\sigma_{ij}(\cdot)$, $i, j = 1, 2$, $i \neq j$, are Lipschitz continuous, $\sigma_{12}(x_2) - \sigma_{21}(x_1) = 0$ if and only if $x_1 = x_2$, and $(x_1 - x_2)(\sigma_{12}(x_2) - \sigma_{21}(x_1)) \leq 0$, $x_1, x_2 \in \mathbb{R}$. Note that $f^{-1}(0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha \in \mathbb{R}\}$. To show that (8.3) and (8.4) is semistable, consider the Lyapunov function candidate $V(x_1, x_2) = \frac{1}{2}(x_1 - \alpha)^2 + \frac{1}{2}(x_2 - \alpha)^2$, where $\alpha \in \mathbb{R}$. Now, it follows that

$$\begin{aligned}
\dot{V}(x_1, x_2) = {} & (x_1 - \alpha)[\sigma_{12}(x_2) - \sigma_{21}(x_1)] \\
& +(x_2 - \alpha)[\sigma_{21}(x_1) - \sigma_{12}(x_2)] \\
= {} & x_1[\sigma_{12}(x_2) - \sigma_{21}(x_1)] \\
& +x_2[\sigma_{21}(x_1) - \sigma_{12}(x_2)] \\
= {} & (x_1 - x_2)[\sigma_{12}(x_2) - \sigma_{21}(x_1)] \\
\leq {} & 0, \quad (x_1, x_2) \in \mathbb{R} \times \mathbb{R},
\end{aligned} \tag{8.5}$$

which implies that $x_1 = x_2 = \alpha$ is Lyapunov stable.

Next, let $\mathcal{R} \triangleq \{(x_1, x_2) \in \mathbb{R}^2 : \dot{V}(x_1, x_2) = 0\} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha \in \mathbb{R}\}$. Since $\mathcal{R}$ consists of equilibrium points, it follows that $\mathcal{M} = \mathcal{R}$. Hence, for any $x_1(0), x_2(0) \in$

$\mathbb{R}$, $(x_1(t), x_2(t)) \to \mathcal{M}$ as $t \to \infty$. Hence, it follows from Theorem 8.2 that $x_1 = x_2 = \alpha$ is semistable for all $\alpha \in \mathbb{R}$. $\triangle$

Next, we provide a converse Lyapunov theorem for semistability.

Theorem 8.3. Consider the system (8.1). Suppose (8.1) is semistable with the domain of semistability $\mathcal{D}_0$. Then there exist a smooth nonnegative function $V : \mathcal{D}_0 \to \overline{\mathbb{R}}_+$ and a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that *i)* $V(x) = 0$, $x \in f^{-1}(0)$, *ii)* $V(x) \geq \alpha(\mathrm{dist}(x, f^{-1}(0)))$, $x \in \mathcal{D}_0$, and *iii)* $V'(x)f(x) < 0$, $x \in \mathcal{D}_0 \backslash f^{-1}(0)$.

Proof. For any given solution $x(t)$ of (8.1), the change of time variable from t to $\tau = \int_0^t (1 + \|f(x(s))\|)\mathrm{d}s$ results in the dynamical system

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau} = \frac{f(\bar{x}(\tau))}{1 + \|f(\bar{x}(\tau))\|}, \quad \bar{x}(0) = x_0, \quad \tau \geq 0, \tag{8.6}$$

where $\bar{x}(\tau) = x(t)$. With a slight abuse of notation, let $\bar{s}(t, x)$, $t \geq 0$, denote the solution of (8.6) starting from $x \in \mathcal{D}_0$. Note that (8.6) implies that $\|\bar{s}(t, x) - \bar{s}(\tau, x)\| \leq |t - \tau|$, $x \in \mathcal{D}_0$, $t, \tau \geq 0$.

Next, define the function $U : \mathcal{D}_0 \to \overline{\mathbb{R}}_+$ by

$$U(x) \triangleq \sup_{t \geq 0} \left\{ \frac{1 + 2t}{1 + t} \mathrm{dist}(\bar{s}(t, x), f^{-1}(0)) \right\}, \quad x \in \mathcal{D}_0. \tag{8.7}$$

Note that $U(\cdot)$ is well defined since (8.6) is semistable. Clearly, *i)* holds with $V(\cdot)$ replaced by $U(\cdot)$. Furthermore, since $U(x) \geq \mathrm{dist}(x, f^{-1}(0))$, $x \in \mathcal{D}_0$, it follows that *ii)* holds with $V(\cdot)$ replaced by $U(\cdot)$.

To show that $U(\cdot)$ is continuous on $\mathcal{D}_0 \backslash f^{-1}(0)$, define $T : \mathcal{D}_0 \backslash f^{-1}(0) \to [0, \infty)$ by $T(z) \triangleq \inf\{h : \mathrm{dist}(\bar{s}(t, z), f^{-1}(0)) < \mathrm{dist}(z, f^{-1}(0))/2 \text{ for all } t \geq h > 0\}$, and denote $\mathcal{W}_\varepsilon \triangleq \{x \in \mathcal{D}_0 : \mathrm{dist}(x, f^{-1}(0)) < \varepsilon\}$. Note that $\mathcal{W}_\varepsilon \supset f^{-1}(0)$ is open. Consider $z \in \mathcal{D}_0 \backslash f^{-1}(0)$ and define $\lambda \triangleq \mathrm{dist}(z, f^{-1}(0)) > 0$ and let $x_\mathrm{e} \triangleq \lim_{t \to \infty} \bar{s}(t, z)$. Since x_e is Lyapunov stable, it follows that there exists an open neighborhood $\mathcal{V}$ of x_e such that all solutions of (8.6) in $\mathcal{V}$

remain in $\mathcal{W}_{\lambda/2}$. Since $x_{\rm e}$ is semistable, it follows that there exists $h > 0$ such that $\bar{s}(h, z) \in \mathcal{V}$. Consequently, $\bar{s}(h+t, z) \in \mathcal{W}_{\lambda/2}$ for all $t \geq 0$, and hence, it follows that $T(z)$ is well defined. Next, by continuity of solutions of (8.6) on compact time intervals, it follows that there exists a neighborhood $\mathcal{U}$ of z such that $\mathcal{U} \cap f^{-1}(0) = \emptyset$ and $\bar{s}(T(z), y) \in \mathcal{V}$ for all $y \in \mathcal{U}$. Now, it follows from the choice of $\mathcal{V}$ that $\bar{s}(T(z)+t, y) \in \mathcal{W}_{\lambda/2}$ for all $t \geq 0$ and $y \in \mathcal{U}$. Then, for every $t > T(z)$ and $y \in \mathcal{U}$, $[(1+2t)/(1+t)]\mathrm{dist}(\bar{s}(t, y), f^{-1}(0)) \leq 2\mathrm{dist}(\bar{s}(t, y), f^{-1}(0)) \leq \lambda$. Therefore, for each $y \in \mathcal{U}$,

$$\begin{aligned} U(z) - U(y) &= \sup_{t\geq 0}\left\{\frac{1+2t}{1+t}\mathrm{dist}(\bar{s}(t,z), f^{-1}(0))\right\} - \sup_{t\geq 0}\left\{\frac{1+2t}{1+t}\mathrm{dist}(\bar{s}(t,y), f^{-1}(0))\right\} \\ &= \sup_{0\leq t\leq T(z)}\left\{\frac{1+2t}{1+t}\mathrm{dist}(\bar{s}(t,z), f^{-1}(0))\right\} \\ &\quad - \sup_{0\leq t\leq T(z)}\left\{\frac{1+2t}{1+t}\mathrm{dist}(\bar{s}(t,y), f^{-1}(0))\right\}. \end{aligned} \tag{8.8}$$

Hence,

$$\begin{aligned} |U(z) - U(y)| &\leq \sup_{0\leq t\leq T(z)}\left|\frac{1+2t}{1+t}\left(\mathrm{dist}(\bar{s}(t,z), f^{-1}(0)) - \mathrm{dist}(\bar{s}(t,y), f^{-1}(0))\right)\right| \\ &\leq 2\sup_{0\leq t\leq T(z)}\left|\mathrm{dist}(\bar{s}(t,z), f^{-1}(0)) - \mathrm{dist}(\bar{s}(t,y), f^{-1}(0))\right| \\ &\leq 2\sup_{0\leq t\leq T(z)}\mathrm{dist}(\bar{s}(t,z), \bar{s}(t,y)), \quad z \in \mathcal{D}_0\backslash f^{-1}(0), \quad y \in \mathcal{U}. \end{aligned} \tag{8.9}$$

Now, it follows from continuous dependence of solutions $\bar{s}(\cdot,\cdot)$ on system initial conditions (Theorem 3.4 of Chapter I of [114]) and (8.9) that $U(\cdot)$ is continuous at z. Furthermore, it follows from (8.9) that, for every sufficiently small $h > 0$,

$$\begin{aligned} |U(\bar{s}(h,z)) - U(z)| &\leq 2\sup_{0\leq t\leq T(z)}\|\bar{s}(t, \bar{s}(h,z)) - \bar{s}(t,z)\| \\ &= 2\sup_{0\leq t\leq T(z)}\|\bar{s}(t+h,z) - \bar{s}(t,z)\| \leq 2h, \end{aligned}$$

which implies that $|\dot{U}(z)| \leq 2$. Since $z \in \mathcal{D}_0\backslash f^{-1}(0)$ was chosen arbitrarily, it follows that $U(\cdot)$ is continuous, $|\dot{U}(\cdot)| \leq 2$, and $T(\cdot)$ is well defined on $\mathcal{D}_0\backslash f^{-1}(0)$.

To show that $U(\cdot)$ is continuous on $f^{-1}(0)$, consider $x_{\rm e} \in f^{-1}(0)$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{D}_0\backslash f^{-1}(0)$ that converges to $x_{\rm e}$. Since $x_{\rm e}$ is Lyapunov stable, it follows from

Lemma 4.5 of [141] that $x(t) \equiv x_{\mathrm{e}}$ is the unique solution to (8.6) with $x_0 = x_{\mathrm{e}}$. By continuous dependence of solutions $\bar{s}(\cdot,\cdot)$ on system initial conditions (Theorem 3.4 of Chapter I of [114]), $\bar{s}(t, x_n) \to \bar{s}(t, x_{\mathrm{e}}) = x_{\mathrm{e}}$ as $n \to \infty$, $t \geq 0$.

Let $\varepsilon > 0$ and note that it follows from *ii*) of Proposition 3.1 that there exists $\delta = \delta(x_{\mathrm{e}}) > 0$ such that, for every solution of (8.6) in $\mathcal{B}_\delta(x_{\mathrm{e}})$, there exists $\hat{T} = \hat{T}(x_{\mathrm{e}}, \varepsilon) > 0$ such that $\bar{s}_t(\mathcal{B}_\delta(x_{\mathrm{e}})) \subset \mathcal{W}_\varepsilon$ for all $t \geq \hat{T}$. Next, note that there exists a positive integer N_1 such that $x_n \in \mathcal{B}_\delta(x_{\mathrm{e}})$ for all $n \geq N_1$. Now, it follows from (8.7) that

$$U(x_n) \leq 2 \sup_{0 \leq t \leq \hat{T}} \operatorname{dist}(\bar{s}(t, x_n), f^{-1}(0)) + 2\varepsilon, \quad n \geq N_1. \tag{8.10}$$

Next, it follows from Lemma 3.1 of Chapter I of [114] that $\bar{s}(\cdot, x_n)$ converges to $\bar{s}(\cdot, x_{\mathrm{e}})$ uniformly on $[0, \hat{T}]$. Hence,

$$\begin{aligned}
\lim_{n\to\infty} \sup_{0\leq t\leq \hat{T}} \operatorname{dist}(\bar{s}(t, x_n), f^{-1}(0)) &= \sup_{0\leq t\leq \hat{T}} \operatorname{dist}(\lim_{n\to\infty} \bar{s}(t, x_n), f^{-1}(0)) \\
&= \sup_{0\leq t\leq \hat{T}} \operatorname{dist}(x_{\mathrm{e}}, f^{-1}(0)) \\
&= 0,
\end{aligned}$$

which implies that there exists a positive integer $N_2 = N_2(x_{\mathrm{e}}, \varepsilon) \geq N_1$ such that $\sup_{0\leq t\leq \hat{T}} \operatorname{dist}$ $(\bar{s}(t, x_n), f^{-1}(0)) < \varepsilon$ for all $n \geq N_2$. Combining (8.10) with the above result yields $U(x_n) < 4\varepsilon$ for all $n \geq N_2$, which implies that $\lim_{n\to\infty} U(x_n) = 0 = U(x_{\mathrm{e}})$.

Next, we show that $U(\bar{x}(\tau))$ is strictly decreasing along the solution of (8.6) on $\mathcal{D} \backslash f^{-1}(0)$. Note that for every $x \in \mathcal{D}_0 \backslash f^{-1}(0)$ and $0 < h \leq 1/2$ such that $\bar{s}(h, x) \in \mathcal{D}_0 \backslash f^{-1}(0)$, it follows from the arguments preceding (8.8) that, for sufficiently small h, the supremum in the definition of $U(\bar{s}(h, x))$ is reached at some time $\hat{t}$ such that $0 \leq \hat{t} \leq T(x)$. Hence,

$$\begin{aligned}
U(\bar{s}(h, x)) &= \operatorname{dist}(\bar{s}(\hat{t} + h, x), f^{-1}(0)) \frac{1 + 2\hat{t}}{1 + \hat{t}} \\
&= \operatorname{dist}(\bar{s}(\hat{t} + h, x), f^{-1}(0)) \frac{1 + 2\hat{t} + 2h}{1 + \hat{t} + h} \left[1 - \frac{h}{(1 + 2\hat{t} + 2h)(1 + \hat{t})}\right] \\
&\leq U(x) \left[1 - \frac{h}{2(1 + T(x))^2}\right],
\end{aligned} \tag{8.11}$$

which implies that $\dot{U}(x) \leq -\frac{1}{2}U(x)(1+T(x))^{-2} < 0$, $x \in \mathcal{D}_0\backslash f^{-1}(0)$, and hence, *iii)* holds with $V(\cdot)$ replaced by $U(\cdot)$. The function $U(\cdot)$ now satisfies all of the conditions of the theorem except for smoothness.

To obtain smoothness, note that since $|\dot{U}(x)| \leq 2$ for every $x \in \mathcal{D}_0$, it follows that $\dot{U}(x)$ satisfies a boundedness condition in the sense of Wilson [238]. By Theorem 2.5 of [238], there exists a smooth function $W : \mathcal{D}_0\backslash f^{-1}(0) \to \mathbb{R}$ satisfying $|W(x) - U(x)| < \frac{1}{4}U(x)(1+T(x))^{-2} < \frac{1}{2}U(x)$ and $\dot{W}(x) \leq -\frac{1}{4}U(x)(1+T(x))^{-2} < 0$ for $x \in \mathcal{D}_0\backslash f^{-1}(0)$. Next, we extend $W(\cdot)$ to all of $\mathcal{D}_0$ by taking $W(z) = 0$ for $z \in f^{-1}(0)$. Now, $W(\cdot)$ is a continuous Lyapunov function which is smooth on $\mathcal{D}_0\backslash f^{-1}(0)$. Taking $V(x) = W(x)e^{-(W(x))^{-2}}$, and noting that $W(x) > \frac{1}{2}U(x) > \frac{1}{2}\mathrm{dist}(x, f^{-1}(0))$, $x \in \mathcal{D}_0\backslash f^{-1}(0)$, so that $V(\cdot)$ satisfies *ii)* with $\alpha(r) \triangleq (r/2)e^{-4/r^2}$, we obtain the desired smooth Lyapunov function. □

8.4. Finite-Time Semistability of Nonlinear Dynamical Systems

In this section, we establish the notion of finite-time semistability and develop sufficient Lyapunov stability theorems for finite-time semistability.

Definition 8.3. An equilibrium point $x_e \in f^{-1}(0)$ of (8.1) is said to be *finite-time-semistable* if there exist an open neighborhood $\mathcal{U} \subseteq \mathcal{D}$ of x_e and a function $T : \mathcal{U}\backslash f^{-1}(0) \to (0, \infty)$, called the *settling-time function*, such that the following statements hold:

i) For every $x \in \mathcal{U}\backslash f^{-1}(0)$, $s(t, x) \in \mathcal{U}\backslash f^{-1}(0)$ for all $t \in [0, T(x))$, and $\lim_{t\to T(x)} s(t, x)$ exists and is contained in $\mathcal{U} \cap f^{-1}(0)$.

ii) x_e is semistable.

An equilibrium point $x_e \in f^{-1}(0)$ of (8.1) is said to be *globally finite-time-semistable* if it is finite-time-semistable with $\mathcal{D} = \mathcal{U} = \mathbb{R}^n$. The system (8.1) is said to be *finite-time-semistable* if every equilibrium point in $f^{-1}(0)$ is finite-time-semistable. Finally, (8.1) is said to be *globally finite-time-semistable* if every equilibrium point in $f^{-1}(0)$ is globally finite-time-semistable.

It is easy to see from Definition 8.3 that, for all $x \in \mathcal{U}$, $T(x) = \inf\{t \in \overline{\mathbb{R}}_+ : f(s(t,x)) = 0\}$, where $T(\mathcal{U} \cap f^{-1}(0)) = \{0\}$.

Lemma 8.1. Suppose (8.1) is finite-time-semistable. Let $x_e \in f^{-1}(0)$ be an equilibrium point of (8.1) and let $\mathcal{U} \subseteq \mathcal{D}$ be as in Definition 8.3. Furthermore, let $T : \mathcal{U} \to \overline{\mathbb{R}}_+$ be the settling-time function. Then T is continuous on $\mathcal{U}$ if and only if T is continuous at each $z_e \in \mathcal{U} \cap f^{-1}(0)$.

Proof. The proof is similar to the proof of Proposition 2.4 given in [30] and, hence, is omitted. □

Next, we introduce a new definition which is weaker than finite-time semistability and is needed for the next result.

Definition 8.4. The system (8.1) is said to be *finite-time convergent to* $\mathcal{M} \subseteq f^{-1}(0)$ for $\mathcal{D}_0 \subseteq \mathcal{D}$ if for every $x_0 \in \mathcal{D}_0$, there exists a finite-time $T = T(x_0) > 0$ such that $x(t) \in \mathcal{M}$ for all $t \geq T$.

The next result gives a sufficient condition for characterizing finite-time convergence.

Proposition 8.3. Let $\mathcal{D}_0 \subseteq \mathcal{D}$ be positively invariant and $\mathcal{M} \subseteq f^{-1}(0)$. Assume that there exists a continuous function $V : \mathcal{D}_0 \to \mathbb{R}$ such that $\dot{V}(\cdot)$ is defined everywhere on $\mathcal{D}_0$, $V(x) = 0$ if and only if $x \in \mathcal{M} \subset \mathcal{D}_0$, and

$$-c_1|V(x)|^\alpha \leq \dot{V}(x) \leq -c_2|V(x)|^\alpha, \quad x \in \mathcal{D}_0 \backslash \mathcal{M}, \tag{8.12}$$

where $c_1 \geq c_2 > 0$ and $0 < \alpha < 1$. Then (8.1) is finite-time convergent to $\mathcal{M}$ for $\{x \in \mathcal{D}_0 : V(x) \geq 0\}$. Alternatively, if V is nonnegative and

$$\dot{V}(x) \leq -c_3(V(x))^{\alpha}, \quad x \in \mathcal{D}_0 \backslash \mathcal{M}, \tag{8.13}$$

where $c_3 > 0$, then (8.1) is finite-time convergent to $\mathcal{M}$ for $\mathcal{D}_0$.

Proof. Note that (8.12) is also true for $x \in \mathcal{M}$. Applying the comparison lemma (Theorems 4.1 and 4.2 of [243]) to (8.12) yields $\mu(t, V(x), c_1) \leq V(s(t, x)) \leq \mu(t, V(x), c_2)$, $x \in \{z \in \mathcal{D}_0 : V(z) \geq 0\}$, where μ is given by

$$\mu(t, z, c) \triangleq \begin{cases} (|z|^{1-\alpha} - c(1-\alpha)t)^{\frac{1}{1-\alpha}}, & 0 \leq t < \frac{|z|^{1-\alpha}}{c(1-\alpha)}, \quad \alpha < 1, \\ 0, & t \geq \frac{|z|^{1-\alpha}}{c(1-\alpha)}, \quad \alpha < 1. \end{cases} \tag{8.14}$$

Hence, $V(s(t, x)) = 0$ for $t \geq \frac{|V(x)|^{1-\alpha}}{c_2(1-\alpha)}$, which implies that $s(t, x) \in \mathcal{M}$ for $t \geq \frac{|V(x)|^{1-\alpha}}{c_2(1-\alpha)}$. The conclusion follows. The second part of the conclusion can be proved similarly. □

The next result establishes a relationship between finite-time convergence and finite-time semistability.

Theorem 8.4. Assume that there exists a continuous nonnegative function $V : \mathcal{D} \to \overline{\mathbb{R}}_+$ such that $\dot{V}(\cdot)$ is defined everywhere on $\mathcal{D}$, $V^{-1}(0) = f^{-1}(0)$, and there exists an open neighborhood $\mathcal{U} \subseteq \mathcal{D}$ such that $\mathcal{U} \cap f^{-1}(0)$ is nonempty and

$$\dot{V}(x) \leq w(V(x)), \quad x \in \mathcal{U} \backslash f^{-1}(0), \tag{8.15}$$

where $w : \mathbb{R} \to \mathbb{R}$ is continuous, $w(0) = 0$, and

$$\dot{z}(t) = w(z(t)), \quad z(0) = z_0 \in \mathbb{R}, \quad t \geq 0, \tag{8.16}$$

has a unique solution in forward time. If (8.16) is finite-time convergent to the origin for $\overline{\mathbb{R}}_+$ and every point in $\mathcal{U} \cap f^{-1}(0)$ is a Lyapunov stable equilibrium point of (8.1), then every point in $\mathcal{U} \cap f^{-1}(0)$ is finite-time-semistable. Moreover, the settling-time function of (8.1) is continuous on an open neighborhood of $\mathcal{U} \cap f^{-1}(0)$. Finally, if $\mathcal{U} = \mathcal{D}$, then (8.1) is finite-time-semistable.

Proof. Consider $x_{\mathrm{e}} \in \mathcal{U} \cap f^{-1}(0)$. Since $x(t) \equiv x_{\mathrm{e}}$ is Lyapunov stable, it follows that there exists an open positively invariant set $\mathcal{S} \subseteq \mathcal{U}$ containing x_{e}. Next, it follows from (8.15) that

$$\dot{V}(s(t,x)) \leq w(V(s(t,x))), \quad x \in \mathcal{S}, \quad t \geq 0. \tag{8.17}$$

Now, applying the comparison lemma (Theorem 4.1 of [243]) to the inequality (8.17) with the comparison system (8.16) yields

$$V(s(t,x)) \leq \psi(t, V(x)), \quad t \geq 0, \quad x \in \mathcal{S}, \tag{8.18}$$

where $\psi : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is the global semiflow of (8.16). Since (8.16) is finite-time convergent to the origin for $\overline{\mathbb{R}}_+$, it follows from (8.18) and the nonnegativity of $V(\cdot)$ that

$$V(s(t,x)) = 0, \quad t \geq \hat{T}(V(x)), \quad x \in \mathcal{S}, \tag{8.19}$$

where $\hat{T}(\cdot)$ denotes the settling-time function of (8.16).

Next, since $s(0,x) = x$, $s(\cdot,\cdot)$ is jointly continuous, and $V(s(t,x)) = 0$ is equivalent to $f(s(t,x)) = 0$ on $\mathcal{S}$, it follows that $\inf\{t \in \overline{\mathbb{R}}_+ : f(s(t,x)) = 0\} > 0$ for $x \in \mathcal{S}\backslash f^{-1}(0)$. Furthermore, it follows from (8.19) that $\inf\{t \in \overline{\mathbb{R}}_+ : f(s(t,x)) = 0\} < \infty$ for $x \in \mathcal{S}$. Define $T : \mathcal{S}\backslash f^{-1}(0) \to \overline{\mathbb{R}}_+$ by $T(x) = \inf\{t \in \overline{\mathbb{R}}_+ : f(s(t,x)) = 0\}$. Then it follows that every point in $\mathcal{S} \cap f^{-1}(0)$ is finite-time-semistable and T is the settling-time function on $\mathcal{S}$. Furthermore, it follows from (8.19) that $T(x) \leq \hat{T}(V(x))$, $x \in \mathcal{S}$. Since the settling time function of a one-dimensional finite-time stable system is continuous at the equilibrium, it follows that T is continuous at each point in $\mathcal{S} \cap f^{-1}(0)$. Since $x_{\mathrm{e}} \in \mathcal{U} \cap f^{-1}(0)$ was chosen arbitrarily, it follows that every point in $\mathcal{U} \cap f^{-1}(0)$ is finite-time-semistable, while Lemma 8.1 implies that T is continuous on an open neighborhood of $\mathcal{U} \cap f^{-1}(0)$.

The last statement follows by noting that, if $\mathcal{U} = \mathcal{D}$, then $\mathcal{U}$ is positively invariant by our assumptions on (8.1), and hence, the preceding arguments hold with $\mathcal{S} = \mathcal{U}$. □

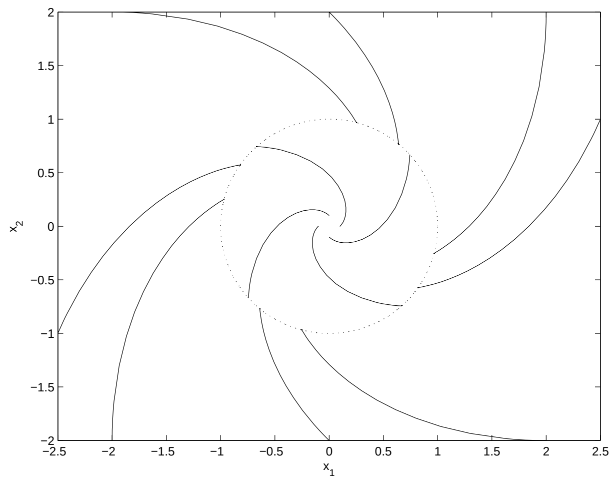

Figure 8.1: Phase portrait for Example 8.2

Example 8.2. Consider the nonlinear dynamical system given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} (1-x_1^2(t)-x_2^2(t))^{\frac{1}{3}}(x_1(t)-x_2(t)) \\ (1-x_1^2(t)-x_2^2(t))^{\frac{1}{3}}(x_1(t)+x_2(t)) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad t \geq 0, \tag{8.20}$$

where $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$. For this system, we show that all the points in $\mathcal{S}^1 \triangleq \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2+x_2^2 = 1\}$ are finite-time-semistable. To see this, consider $V(x) = \frac{1}{4}(x_1^2+x_2^2-1)^2$. Let $0 < c < 1$ and $\mathcal{U} = \{(x_1, x_2) \in \mathbb{R}^n : x_1^2 + x_2^2 > c\}$. Then $\dot{V}(x) = -(x_1^2 + x_2^2)|x_1^2 + x_2^2 - 1|^{\frac{4}{3}} \leq -2^{\frac{4}{3}}c(V(x))^{\frac{2}{3}}$ for all $(x_1, x_2) \in \mathcal{U}$. Next, we show that every point in $\mathcal{S}^1$ is Lyapunov stable. This can be shown by using the nontangency-based Lyapunov tests developed in [32]. In particular, it follows from Example 4.2 of [32] that for every $x \in \mathcal{S}^1$, f is nontangent to $\mathcal{S}^1$. Now, it follows from Corollary 7.2 of [32] that every point in $\mathcal{S}^1$ is Lyapunov stable. Hence, with $c_3 = c2^{\frac{4}{3}}$, $\alpha = \frac{2}{3}$, and $w(x) = -c_3\mathrm{sign}(x)|x|^{\alpha}$, it follows from the second conclusion of Proposition 8.3 and Theorem 8.4 that every point in $\mathcal{S}^1$ is finite-time-semistable. Figure 8.1 shows the phase portrait of (8.20). $\triangle$

Theorem 8.5. Assume that there exists a continuous nonnegative function $V : \mathcal{D} \to \overline{\mathbb{R}}_+$ such that $\dot{V}(\cdot)$ is defined everywhere on $\mathcal{D}$, $V^{-1}(0) = f^{-1}(0)$, and there exists an open neighborhood $\mathcal{U} \subseteq \mathcal{D}$ such that $\mathcal{U} \cap f^{-1}(0)$ is nonempty and (8.13) holds for all $x \in \mathcal{U} \backslash f^{-1}(0)$.

Furthermore, assume that there exists a continuous nonnegative function $W : \mathcal{U} \to \overline{\mathbb{R}}_+$ such that $\dot{W}(\cdot)$ is defined everywhere on $\mathcal{U}$, $W^{-1}(0) = \mathcal{U} \cap f^{-1}(0)$, and

$$\|f(x)\| \leq -c_0 \dot{W}(x), \quad x \in \mathcal{U} \backslash f^{-1}(0), \tag{8.21}$$

where $c_0 > 0$. Then every point in $\mathcal{U} \cap f^{-1}(0)$ is finite-time-semistable.

Proof. For any $x_e \in \mathcal{U} \cap f^{-1}(0)$, since $W(x) \geq 0 = W(x_e)$ for all $x \in \mathcal{U}$, it follows from i) of Theorem 5.2 of [31] that x_e is a Lyapunov stable equilibrium and, hence, every point in $\mathcal{U} \cap f^{-1}(0)$ is Lyapunov stable. Now, it follows from the second conclusion of Proposition 8.3 and Theorem 8.4, with $w(x) = -c_3 \mathrm{sign}(x)|x|^\alpha$, that every point in $\mathcal{U} \cap f^{-1}(0)$ is finite-time-semistable. $\square$

Example 8.3. Consider the dynamical system given by (8.20). Let $V(x) = \frac{1}{4}(x_1^2 + x_2^2 - 1)^2$ and $\hat{V}(x) = \frac{1}{2}(\sqrt{x_1^2 + x_2^2} - 1)^2$. It follows from Example 8.2 that $\dot{V}(x) \leq -2^{\frac{4}{3}} c_1 (V(x))^{\frac{2}{3}}$ for all $x \in \mathcal{U}$, where $\mathcal{U}$ is as in Example 8.2. Since $\|f(x)\| = |x_1^2 + x_2^2 - 1|^{\frac{1}{3}} \sqrt{x_1^2 + x_2^2}$ and $\dot{\hat{V}}(x) = (\sqrt{x_1^2 + x_2^2} - 1)\sqrt{x_1^2 + x_2^2}(1 - x_1^2 - x_2^2)^{\frac{1}{3}}$ for all $x \in \mathcal{U}$, it follows that $\|f(x)\| = -(2\hat{V}(x))^{-\frac{1}{2}} \dot{\hat{V}}(x)$ for all $x \in \mathcal{U} \backslash \mathcal{S}^1$. Now, taking $W(x) = (2\hat{V}(x))^{\frac{1}{2}}$ yields $\|f(x)\| = -\dot{W}(x)$ for all $x \in \mathcal{U} \backslash \mathcal{S}^1$. Hence, it follows from Theorem 8.5 that every point in $\mathcal{S}^1$ is finite-time-semistable. $\triangle$

8.5. Homogeneity and Finite-Time Semistability

In this section, we develop necessary and sufficient conditions for finite-time semistability of homogeneous dynamical systems. In the sequel, we will need to consider a complete vector field ν on $\mathbb{R}^n$ such that the solutions of the differential equation $\dot{y}(t) = \nu(y(t))$ define a continuous *global flow* $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$, where $\nu^{-1}(0) = f^{-1}(0)$. For each $\tau \in \mathbb{R}$, the map $\psi_\tau(\cdot) = \psi(\tau, \cdot)$ is a homeomorphism and $\psi_\tau^{-1} = \psi_{-\tau}$. We define a function $V : \mathbb{R}^n \to \mathbb{R}$ to be *homogeneous of degree* $l \in \mathbb{R}$ *with respect to* ν if and only if $(V \circ \psi_\tau)(x) = e^{l\tau} V(x)$,

$\tau \in \mathbb{R}$, $x \in \mathbb{R}^n$. Our assumptions imply that every connected component of $\mathbb{R}^n \backslash f^{-1}(0)$ is invariant under ν.

The *Lie derivative* of a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ with respect to ν is given by $L_\nu V(x) \triangleq \lim_{t \to 0^+} \frac{1}{t}[V(\psi(t,x)) - V(x)]$, whenever the limit on the right-hand side exists. If V is a continuous homogeneous function of degree $l > 0$, then $L_\nu V$ is defined everywhere and satisfies $L_\nu V = lV$. We assume that the vector field ν is a *semi-Euler vector field*, that is, the dynamical system

$$\dot{y}(t) = -\nu(y(t)), \quad y(0) = y_0, \quad t \geq 0, \tag{8.22}$$

is globally semistable. Thus, for each $x \in \mathbb{R}^n$, $\lim_{\tau \to \infty} \psi(-\tau, x) = x^* \in \nu^{-1}(0)$, and for each $x_{\mathrm{e}} \in \nu^{-1}(0)$, there exists $z \in \mathbb{R}^n$ such that $x_{\mathrm{e}} = \lim_{\tau \to \infty} \psi(-\tau, z)$. Finally, we say that the vector field f is *homogeneous of degree* $k \in \mathbb{R}$ *with respect to* ν if and only if $\nu^{-1}(0) = f^{-1}(0)$ and, for every $t \in \overline{\mathbb{R}}_+$ and $\tau \in \mathbb{R}$,

$$s_t \circ \psi_\tau = \psi_\tau \circ s_{e^{k\tau}t}. \tag{8.23}$$

Note that if $V : \mathbb{R}^n \to \mathbb{R}$ is a homogeneous function of degree l such that $L_f V(x)$ is defined everywhere, then $L_f V(x)$ is a homogeneous function of degree $l + k$. Finally, note that if ν and f are continuously differentiable in a neighborhood of $x \in \mathbb{R}^n$, then (8.23) holds at x for sufficiently small t and τ if and only if $[\nu, f](x) = kf(x)$ in a neighborhood of $x \in \mathbb{R}^n$, where the Lie bracket $[\nu, f]$ of ν and f can be computed by using $[\nu, f] = \frac{\partial f}{\partial x}\nu - \frac{\partial \nu}{\partial x} f$.

The following lemmas are needed for the main results of this section.

Lemma 8.2. Consider the dynamical system (8.22). Let $\mathcal{D}_{\mathrm{c}} \subset \mathbb{R}^n$ be a compact set satisfying $\mathcal{D}_{\mathrm{c}} \cap \nu^{-1}(0) = \emptyset$. Then for every open set $\mathcal{U}$ satisfying $\mathcal{U} \supset \nu^{-1}(0)$, there exist $\tau_1, \tau_2 > 0$ such that $\psi_{-t}(\mathcal{D}_{\mathrm{c}}) \subset \mathcal{U}$ for all $t > \tau_1$ and $\psi_\tau(\mathcal{D}_{\mathrm{c}}) \cap \mathcal{U} = \emptyset$ for all $\tau > \tau_2$.

Proof. Let $\mathcal{U}$ be an open neighborhood of $\nu^{-1}(0)$. Since every $z \in \nu^{-1}(0)$ is Lyapunov stable under ν, it follows that there exists an open neighborhood $\mathcal{V}_z$ containing z such that

$\psi_{-t}(\mathcal{V}_z) \subseteq \mathcal{U}$ for all $t \geq 0$. Hence, $\mathcal{V} \triangleq \bigcup_{z\in\nu^{-1}(0)} \mathcal{V}_z$ is open and $\psi_{-t}(\mathcal{V}) \subseteq \mathcal{U}$ for all $t \geq 0$. Next, consider the collection of nested sets $\{\mathcal{D}_t\}_{t>0}$, where $\mathcal{D}_t = \{x \in \mathcal{D}_{\mathrm{c}} : \psi_h(x) \notin \mathcal{V}, h \in [-t, 0]\} = \mathcal{D}_{\mathrm{c}} \cap (\mathbb{R}^n \backslash (\bigcup_{h\in[-t,0]} \psi_h^{-1}(\mathcal{V})))$, $t > 0$. For each $t > 0$, $\mathcal{D}_t$ is a compact set. Therefore, if $\mathcal{D}_t$ is nonempty for each $t > 0$, then there exists $x \in \bigcap_{t>0} \mathcal{D}_t$, that is, there exists $x \in \mathcal{D}_{\mathrm{c}}$ such that $\psi_{-t}(x) \notin \mathcal{V}$ for all $t > 0$, which contradicts the fact that the domain of semistability of (8.22) is $\mathbb{R}^n$. Hence, there exists $\tau > 0$ such that $\mathcal{D}_\tau = \emptyset$, that is, $\mathcal{D}_{\mathrm{c}} \subset \bigcup_{h\in[-\tau,0]} \psi_h^{-1}(\mathcal{V})$. Therefore, for every $t > \tau$, $\psi_{-t}(\mathcal{D}_{\mathrm{c}}) \subset \bigcup_{h\in[-\tau,0]} \psi_{-t}(\psi_h^{-1}(\mathcal{V})) = \bigcup_{h\in[-\tau,0]} \psi_{-t-h}(\mathcal{V}) \subseteq \mathcal{U}$. The second conclusion follows using similar arguments as above. □

Lemma 8.3. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree $k \in \mathbb{R}$ with respect to ν and (8.1) is (locally) semistable. Then the domain of semistability of (8.1) is $\mathbb{R}^n$.

Proof. Let $\mathcal{A} \subseteq \mathbb{R}^n$ be the domain of semistability and $x \in \mathbb{R}^n$. Note that $\mathcal{A}$ is an open neighborhood of $\nu^{-1}(0)$. Since every point in $\nu^{-1}(0)$ is a globally semistable equilibrium under $-\nu$, there exists $\tau > 0$ such that $z = \psi_{-\tau}(x) \in \mathcal{A}$. Then it follows from (8.23) that $s(t, x) = s(t, \psi_\tau(z)) = \psi_\tau(s(e^{k\tau}t, z))$. Since $\lim_{t\to\infty} s(t, z) = x^* \in f^{-1}(0)$, it follows that $\lim_{t\to\infty} s(t, x) = \lim_{t\to\infty} \psi_\tau(s(e^{k\tau}t, z)) = \psi_\tau(\lim_{t\to\infty} s(e^{k\tau}t, z)) = \psi_\tau(x^*) = x^*$, which implies that $x \in \mathcal{A}$. Since $x \in \mathbb{R}^n$ is arbitrary, $\mathcal{A} = \mathbb{R}^n$. □

Theorem 8.6. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree $k \in \mathbb{R}$ with respect to ν and (8.1) is semistable. Then for every $l > \max\{-k, 0\}$, there exists a continuous nonnegative function $V : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ that is homogeneous of degree l with respect to ν, continuously differentiable on $\mathbb{R}^n \backslash f^{-1}(0)$, and satisfies $V^{-1}(0) = f^{-1}(0)$, $V'(x)f(x) < 0$, $x \in \mathbb{R}^n \backslash f^{-1}(0)$, and for each $x_{\mathrm{e}} \in f^{-1}(0)$ and each bounded open neighborhood $\mathcal{D}_0$ containing x_{e}, there exist $c_1 = c_1(\mathcal{D}_0) \geq c_2 = c_2(\mathcal{D}_0) > 0$ such that

$$-c_1[V(x)]^{\frac{l+k}{l}} \leq V'(x)f(x) \leq -c_2[V(x)]^{\frac{l+k}{l}}, \quad x \in \mathcal{D}_0. \tag{8.24}$$

Proof. Choose $l > \max\{-k, 0\}$. First, we prove that there exists a continuous Lyapunov function V on $\mathbb{R}^n$ that is homogeneous of degree l with respect to ν, continuously differentiable on $\mathbb{R}^n \backslash f^{-1}(0)$, and $V'(x)f(x) < 0$ for $x \in \mathbb{R}^n \backslash f^{-1}(0)$. Choose any nondecreasing smooth function $g : \overline{\mathbb{R}}_+ \to [0, 1]$ such that $g(s) = 0$ for $s \leq a$, $g(s) = 1$ for $s \geq b$, and $g'(s) > 0$ on (a, b), where $0 < a < b$ are constants. It follows from Theorem 8.3 and Lemma 8.3 that there exists a continuously differentiable Lyapunov function $U(\cdot)$ on $\mathbb{R}^n$ satisfying all of the properties in Theorem 8.3.

Next, define

$$V(x) \triangleq \int_{-\infty}^{+\infty} e^{-l\tau} g(U(\psi(\tau, x)))\mathrm{d}\tau, \quad x \in \mathbb{R}^n. \tag{8.25}$$

Let $\mathcal{U}$ be a bounded open set satisfying $\overline{\mathcal{U}} \cap f^{-1}(0) = \emptyset$. Since every point in $\nu^{-1}(0)$ is a globally semistable equilibrium point under $-\nu$, it follows that for each $x \in \overline{\mathcal{U}}$, $\lim_{\tau \to +\infty} U(\psi(\tau, x)) = +\infty$ and $\lim_{\tau \to +\infty} U(\psi(-\tau, x)) = 0$. Now, it follows from Lemma 8.2 that there exists time instants $\tau_1 < \tau_2$ such that for each $x \in \overline{\mathcal{U}}$, $U(\psi(\tau, x)) \leq a$ for all $\tau \leq \tau_1$ and $U(\psi(\tau, x)) \geq b$ for all $\tau \geq \tau_2$. Hence,

$$V(x) = \int_{\tau_1}^{\tau_2} e^{-l\tau} g(U(\psi(\tau, x)))\mathrm{d}\tau + \frac{e^{-l\tau_2}}{l}, \quad x \in \mathcal{U}, \tag{8.26}$$

which implies that V is well defined, positive, and continuously differentiable on $\mathcal{U}$.

Next, since $U(\cdot)$ satisfies *i*) and *ii*) of Theorem 8.3 it follows from (8.25) and (8.26) that $V^{-1}(0) = f^{-1}(0)$. Since for any $\sigma \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$V(\psi(\sigma, x)) = \int_{-\infty}^{+\infty} e^{-l\tau} g(U(\psi(\tau + \sigma, x)))\mathrm{d}\tau = e^{l\sigma} V(x), \tag{8.27}$$

by definition, V is homogeneous of degree l. In addition, it follows from (8.23) and (8.26) that

$$\begin{aligned} V'(x)f(x) &= \int_{\tau_1}^{\tau_2} e^{-l\tau} g'(U(\psi(\tau, x))) \frac{\mathrm{d}}{\mathrm{d}t} U(s(e^{-k\tau}t, \psi(\tau, x)))\Big|_{t=0} \mathrm{d}\tau \\ &= \int_{\tau_1}^{\tau_2} e^{-(l+k)\tau} g'(U(\psi(\tau, x))) U'(\psi(\tau, x)) f(\psi(\tau, x))\mathrm{d}\tau < 0, \quad x \in \mathcal{U}, \end{aligned} \tag{8.28}$$

which implies that $V'f$ is negative and continuous on $\mathcal{U}$. Now, since $\mathcal{U}$ is arbitrary, it follows that V is well defined and continuously differentiable, and $V'f$ is negative and continuous on $\mathbb{R}^n \backslash f^{-1}(0)$.

Next, to show continuity at points in $f^{-1}(0)$, we define $T : \mathbb{R}^n \backslash f^{-1}(0) \to \mathbb{R}$ by $T(x) = \sup\{t \in \mathbb{R} : U(\psi(\tau, x)) \leq a \text{ for all } \tau \leq t\}$, and note that the continuity of U implies that $U(\psi(T(x), x)) = a$ for all $x \in \mathbb{R}^n \backslash f^{-1}(0)$. Let $x_{\mathrm{e}} \in f^{-1}(0)$, and consider a sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n \backslash f^{-1}(0)$ converging to x_{e}. We claim that the sequence $\{T(x_k)\}_{k=1}^{\infty}$ has no bounded subsequence so that $\lim_{k\to\infty} T(x_k) = \infty$. To prove our claim by contradiction, suppose $\{T(x_{k_i})\}_{i=1}^{\infty}$ is a bounded subsequence. Without loss of generality, we may assume that the sequence $\{T(x_{k_i})\}_{i=1}^{\infty}$ converges to $h \in \mathbb{R}$. Then, by joint continuity of ψ, $\lim_{i\to\infty} \psi(T(x_{k_i}), x_{k_i}) = \psi(h, x_{\mathrm{e}}) = x_{\mathrm{e}}$, so that $\lim_{i\to\infty} U(\psi(T(x_{k_i}), x_{k_i})) = U(x_{\mathrm{e}}) = 0$. However, this contradicts our observation above that $U(\psi(T(x), x)) = a$ for all $x \in \mathbb{R}^n \backslash f^{-1}(0)$. The contradiction leads us to conclude that $\lim_{k\to\infty} T(x_k) = \infty$. Now, for each $k = 1, 2, \ldots,$ it follows that

$$V(x_k) = \int_{T(x_k)}^{\infty} e^{-l\tau} g(U(\psi(\tau, x_k)))\mathrm{d}\tau \leq \int_{T(x_k)}^{\infty} e^{-l\tau}\mathrm{d}\tau = l^{-1}e^{-lT(x_k)},$$

so that $\lim_{k\to\infty} V(x_k) = 0 = V(x_{\mathrm{e}})$. Since x_{e} was chosen arbitrarily, it follows that V is continuous at every $x_{\mathrm{e}} \in f^{-1}(0)$.

To show that V possesses the last property, let $x_{\mathrm{e}} \in f^{-1}(0)$, and choose a bounded open neighborhood $\mathcal{D}_0$ of x_{e}. Let $\mathcal{Q} = \psi(\mathbb{R}_+ \times \mathcal{D}_0)$. For every $\varepsilon > 0$, denote $\mathcal{Q}_\varepsilon = \mathcal{Q} \cap V^{-1}(\varepsilon)$. For every $\varepsilon > 0$, define the continuous map $\tau_\varepsilon : \mathbb{R}^n \backslash f^{-1}(0) \to \mathbb{R}$ by $\tau_\varepsilon(x) \triangleq l^{-1}\ln(\varepsilon/V(x))$, and note that, for every $x \in \mathbb{R}^n \backslash f^{-1}(0)$, $\psi(t, x) \in V^{-1}(\varepsilon)$ if and only if $t = \tau_\varepsilon(x)$. Next, define $\beta_\varepsilon : \mathbb{R}^n \backslash f^{-1}(0) \to \mathbb{R}^n$ by $\beta_\varepsilon \triangleq \psi(\tau_\varepsilon(x), x)$. Note that, for every $\varepsilon > 0$, β_ε is continuous, and $\beta_\varepsilon(x) \in V^{-1}(\varepsilon)$ for every $x \in \mathbb{R}^n \backslash f^{-1}(0)$.

Consider $\varepsilon > 0$. $\mathcal{Q}_\varepsilon$ is the union of the images of connected components of $\mathcal{D}_0 \backslash f^{-1}(0)$ under the continuous map β_ε. Since every connected component of $\mathbb{R}^n \backslash f^{-1}(0)$ is invariant under ν, it follows that the image of each connected component $\mathcal{U}$ of $\mathbb{R}^n \backslash f^{-1}(0)$ under β_ε

is contained in $\mathcal{U}$ itself. In particular, the images of connected components of $\mathcal{D}_0\backslash f^{-1}(0)$ under β_ε are all disjoint. Thus, each connected component of $\mathcal{Q}_\varepsilon$ is the image of exactly one connected component of $\mathcal{D}_0\backslash f^{-1}(0)$ under β_ε. Finally, if ε is small enough so that $V^{-1}(\varepsilon)\cap\mathcal{D}_0$ is nonempty, then $V^{-1}(\varepsilon)\cap\mathcal{D}_0\subseteq\mathcal{Q}_\varepsilon$, and hence, every connected component of $\mathcal{Q}_\varepsilon$ has a nonempty intersection with $\mathcal{D}_0\backslash f^{-1}(0)$.

We claim that $\mathcal{Q}_\varepsilon$ is bounded for every $\varepsilon > 0$. It is easy to verify that, for every $\varepsilon_1,\varepsilon_2\in(0,\infty)$, $\mathcal{Q}_{\varepsilon_2}=\psi_h(\mathcal{Q}_{\varepsilon_1})$ with $h=l^{-1}\ln(\varepsilon_2/\varepsilon_1)$. Hence, it suffices to prove that there exists $\varepsilon>0$ such that $\mathcal{Q}_\varepsilon$ is bounded. To arrive at a contradiction, suppose, *ad absurdum*, $\mathcal{Q}_\varepsilon$ is unbounded for every $\varepsilon>0$. Choose a bounded open neighborhood $\mathcal{V}$ of $\overline{\mathcal{D}}_0$ and a sequence $\{\varepsilon_i\}_{i=1}^\infty$ in $(0,\infty)$ converging to 0. By our assumption, for every $i=1,2,\ldots$, at least one connected component of $\mathcal{Q}_{\varepsilon_i}$ must contain a point in $\mathbb{R}^n\backslash\mathcal{V}$. On the other hand, for i sufficiently large, every connected component of $\mathcal{Q}_{\varepsilon_i}$ has a nonempty intersection with $\mathcal{D}_0\subset\mathcal{V}$. It follows that $\mathcal{Q}_{\varepsilon_i}$ has a nonempty intersection with the boundary of $\mathcal{V}$ for every i sufficiently large. Hence, there exists a sequence $\{x_i\}_{i=1}^\infty$ in $\mathcal{D}_0$, and a sequence $\{t_i\}_{i=1}^\infty$ in $(0,\infty)$ such that $y_i\triangleq\psi_{t_i}(x_i)\in V^{-1}(\varepsilon_i)\cap\partial\mathcal{V}$ for every $i=1,2,\ldots$. Since $\mathcal{V}$ is bounded, we can assume that the sequence $\{y_i\}_{i=1}^\infty$ converges to $y\in\partial\mathcal{V}$. Continuity implies that $V(y)=\lim_{i\to\infty}V(y_i)=\lim_{i\to\infty}\varepsilon_i=0$. Since $V^{-1}(0)=f^{-1}(0)=\nu^{-1}(0)$, it follows that y is Lyapunov stable under $-\nu$. Since $y\notin\overline{\mathcal{D}}_0$, there exists an open neighborhood $\mathcal{U}$ of y such that $\mathcal{U}\cap\mathcal{D}_0=\emptyset$. The sequence $\{y_i\}_{i=1}^\infty$ converges to y while $\psi_{-t_i}(y_i)=x_i\in\mathcal{D}_0\subset\mathbb{R}^n\backslash\mathcal{U}$, which contradicts Lyapunov stability. This contradiction implies that there exists $\varepsilon>0$ such that $\mathcal{Q}_\varepsilon$ is bounded. It now follows that $\mathcal{Q}_\varepsilon$ is bounded for every $\varepsilon>0$.

Finally, consider $x\in\mathcal{D}_0\backslash f^{-1}(0)$. Choose $\varepsilon>0$ and note that $\psi_{\tau_\varepsilon(x)}(x)\in\mathcal{Q}_\varepsilon$. Furthermore, note that $V'(x)f(x)<0$ for all $x\in\mathbb{R}^n\backslash f^{-1}(0)$, $V'(x)f(x)$ is continuous on $\mathbb{R}^n\backslash f^{-1}(0)$, and $\overline{\mathcal{Q}}_\varepsilon\cap f^{-1}(0)=\emptyset$. Then, by homogeneity, $V(\psi_{\tau_\varepsilon(x)}(x))=\varepsilon$, and hence,

$$\min_{z\in\overline{\mathcal{Q}}_\varepsilon}V'(z)f(z)\le V'(\psi_{\tau_\varepsilon(x)}(x))f(\psi_{\tau_\varepsilon(x)}(x))\le\max_{z\in\overline{\mathcal{Q}}_\varepsilon}V'(z)f(z).\tag{8.29}$$

Since $V'(\psi_{\tau_\varepsilon(x)}(x))f(\psi_{\tau_\varepsilon(x)}(x))$ is homogeneous of degree $l+k$, it follows that

$$V'(\psi_{\tau_\varepsilon(x)}(x))f(\psi_{\tau_\varepsilon(x)}(x)) = e^{(l+k)\tau_\varepsilon(x)}V'(x)f(x) = \varepsilon^{\frac{l+k}{l}}V(x)^{-\frac{l+k}{l}}V'(x)f(x).$$

Let $c_1 \triangleq -\varepsilon^{-\frac{l+k}{l}} \min_{z\in\overline{\mathcal{Q}}_\varepsilon} V'(z)f(z)$ and $c_2 \triangleq -\varepsilon^{-\frac{l+k}{l}} \max_{z\in\overline{\mathcal{Q}}_\varepsilon} V'(z)f(z)$. Note that c_1 and c_2 are positive and well defined since $\overline{\mathcal{Q}}_\varepsilon$ is compact. Hence, the theorem is proved. $\square$

The following result represents the main application of homogeneity [33] to finite-time semistability and finite-time stabilization.

Theorem 8.7. Suppose f is homogeneous of degree $k \in \mathbb{R}$ with respect to ν. Then (8.1) is finite-time-semistable if and only if (8.1) is semistable and $k < 0$. In addition, if (8.1) is finite-time-semistable, then the settling-time function $T(\cdot)$ is homogeneous of degree $-k$ with respect to ν and $T(\cdot)$ is continuous on $\mathbb{R}^n$.

Proof. Since finite-time semistability implies semistability, it suffices to prove that if (8.1) is semistable, then (8.1) is finite-time-semistable if and only if $k < 0$. Suppose (8.1) is finite-time-semistable and let $l > \max\{-k, 0\}$. Then for each $x_{\rm e} \in f^{-1}(0)$, it follows from Theorem 8.6 that there exist a bounded, open, and positively invariant set $\mathcal{S}$ containing $x_{\rm e}$, and a continuous nonnegative function $V : \mathcal{S} \to \overline{\mathbb{R}}_+$ that is homogeneous of degree $l+k$ and is such that $V'(x)f(x)$ is continuous, negative on $\mathcal{S}\backslash f^{-1}(0)$, homogeneous of degree $l+k$, and (8.24) holds. Now, *ad absurdum*, if $k \geq 0$ and $x \in \mathcal{S}\backslash f^{-1}(0)$, then applying the comparison lemma (Theorem 4.2 in [243]) to the first inequality in (8.24) yields $V(s(t,x)) \geq \pi(t, V(x))$, where π is given by

$$\pi(t,x) = \begin{cases} \operatorname{sign}(x)\left(\frac{1}{|x|^{\alpha-1}} + c_1(\alpha-1)t\right)^{-\frac{1}{\alpha-1}}, & \alpha > 1, \\ e^{-c_1 t}x, & \alpha = 1, \end{cases} \tag{8.30}$$

and where $\operatorname{sign}(x) \triangleq x/|x|$, $x \neq 0$, and $\operatorname{sign}(0) \triangleq 0$, with $\alpha = l + k/l \geq 1$. Since, in this case, $\pi(t, V(x)) > 0$ for all $t \geq 0$, we have $s(t,x) \notin \mathcal{S} \cap f^{-1}(0)$ for every $t \geq 0$; that is, $x_{\rm e}$ is not a finite-time-semistable equilibrium under f, which is a contradiction. Hence, $k < 0$.

Conversely, if $k < 0$, pick $x_{\rm e} \in f^{-1}(0)$. Choose an open neighborhood $\mathcal{D}_0$ of $x_{\rm e}$ such that (8.25) holds. Next, $\mathcal{S}_{x_{\rm e}}$ is chosen to be a bounded, positively invariant neighborhood of $x_{\rm e}$ contained in $\mathcal{D}_0$. Then it follows from Theorem 8.6 that there exists a continuous nonnegative function $V(\cdot)$ such that (8.24) holds on $\mathcal{S}_{x_{\rm e}}$. Now, with $c = c_2 > 0$, $0 < \alpha = 1 + k/l < 1$, $\mathcal{D}_0 = \mathcal{S}_{x_{\rm e}}$, and $w(x) = -c\mathrm{sign}(x)|x|^{\alpha}$, it follows from Proposition 8.3 and Theorem 8.4 that $x_{\rm e}$ is finite-time-semistable on $\mathcal{S}_{x_{\rm e}}$. Define $\mathcal{S} \triangleq \bigcup_{x_{\rm e}\in f^{-1}(0)} \mathcal{S}_{x_{\rm e}}$. Then $\mathcal{S}$ is an open neighborhood of $f^{-1}(0)$ such that every solution in $\mathcal{S}$ converges in finite time to a Lyapunov stable equilibrium. Hence, (8.1) is finite-time-semistable. Lemma 8.3 then implies that (8.1) is globally finite-time-semistable, and $T(\cdot)$ is defined on $\mathbb{R}^n$. By Proposition 8.3 with $\mathcal{D}_0 = \mathcal{S}_{x_{\rm e}}$, and Theorem 8.4, it follows that $T(\cdot)$ is continuous on $\mathcal{S}_{x_{\rm e}}$. Next, since $x_{\rm e} \in f^{-1}(0)$ was chosen arbitrarily, it follows from Lemma 8.1 that $T(\cdot)$ is continuous on $\mathbb{R}^n$.

Finally, let $x \in \mathbb{R}^n$ and note that since every point in $\nu^{-1}(0) = f^{-1}(0)$ is a globally semistable equilibrium under $-\nu$, there exists $\tau > 0$ such that $z \triangleq \psi_{-\tau}(x) \in \mathcal{S}$. Then it follows from (8.23) that $s(t,x) = s(t,\psi_\tau(z)) = \psi_\tau(s(e^{k\tau}t, z))$, and hence, $f(s(t,x)) = 0$ if and only if $f(s(e^{k\tau}t,z)) = 0$. Now, it follows that for $x \in \mathcal{S}$, $T(\psi_{-\tau}(x)) = T(z) = e^{k\tau}T(x)$. By definition, it follows that $T(\cdot)$ is homogeneous of degree $-k$ with respect to ν. □

In order to use Theorem 8.7 to prove finite-time semistability of a homogeneous system, *a priori* information of semistability for the system is needed, which is not easy to obtain. To overcome this, we need to develop some sufficient conditions to establish finite-time semistability. Recall that a function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be *weakly proper* if and only if for every $c \in \mathbb{R}$, every connected component of the set $\{x \in \mathbb{R}^n : V(x) \leq c\} = V^{-1}((-\infty, c])$ is compact [32].

Proposition 8.4. Assume f is homogeneous of degree $k < 0$ with respect to ν. Furthermore, assume that there exists a weakly proper, continuous function $V : \mathbb{R}^n \to \mathbb{R}$ such that $\dot{V}$ is defined on $\mathbb{R}^n$ and satisfies $\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$. If every point in the largest invariant subset $\mathcal{N}$ of $\dot{V}^{-1}(0)$ is a Lyapunov stable equilibrium point of (8.1), then (8.1) is finite-time-semistable.

Proof. Since $V(\cdot)$ is weakly proper, it follows from Proposition 3.1 of [32] that the positive orbit $s^x([0,\infty))$ of $x \in \mathbb{R}^n$ is bounded in $\mathbb{R}^n$. Since every solution is bounded, it follows from the hypotheses on $V(\cdot)$ that for every $x \in \mathbb{R}^n$, the omega limit set $\omega(x)$ is nonempty and contained in the largest invariant subset $\mathcal{N}$ of $\dot{V}^{-1}(0)$. Since every point in $\mathcal{N}$ is a Lyapunov stable equilibrium point, it follows from Proposition 8.1 that the omega limit set $\omega(x)$ contains a single point for every $x \in \mathbb{R}^n$. And since $\lim_{t\to\infty} s(t,x) \in \mathcal{N}$ is Lyapunov stable for every $x \in \mathbb{R}^n$, by definition, the system (8.1) is semistable. Hence, it follows from Theorem 8.7 that (8.1) is finite-time-semistable. $\square$

Example 8.4. Consider the nonlinear dynamical system given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} (x_2(t)-x_1(t))^{\frac{1}{3}} + (x_3(t)-x_1(t))^{\frac{1}{3}} \\ (x_1(t)-x_2(t))^{\frac{1}{3}} + (x_3(t)-x_2(t))^{\frac{1}{3}} \\ (x_1(t)-x_3(t))^{\frac{1}{3}} + (x_2(t)-x_3(t))^{\frac{1}{3}} \end{bmatrix},$$
$$x_1(0) = x_{10}, \quad x_2(0) = x_{20}, \quad x_3(0) = x_{30}, \quad t \geq 0, \tag{8.31}$$

where $x_i \in \mathbb{R}$, $i = 1,2,3$. For each $a \in \mathbb{R}$, $x_1 = x_2 = x_3 = a$ is the equilibrium point of (8.31). We show that all the equilibrium points in (8.31) are finite-time-semistable. Note that the vector field f of (8.31) is homogeneous of degree -2 with respect to the semi-Euler vector field $\nu(x) = (2x_1-x_2-x_3)\frac{\partial}{\partial x_1}+(2x_2-x_1-x_3)\frac{\partial}{\partial x_2}+(2x_3-x_1-x_2)\frac{\partial}{\partial x_3}$. Next, consider $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2$. Then $\dot{V}(x(t)) \leq 0$, $t \geq 0$, and $\mathcal{N} = \{x \in \mathbb{R}^4 : x_1 = x_2 = x_3 = a\}$. Now, it follows from the Lyapunov function candidate $V(x-a\mathbf{e}) = \frac{1}{2}(x_1-a)^2+\frac{1}{2}(x_2-a)^2+\frac{1}{2}(x_3-a)^2$ that $\dot{V}(x-a\mathbf{e}) = -(x_1-x_2)^{\frac{4}{3}}-(x_2-x_3)^{\frac{4}{3}}-(x_3-x_1)^{\frac{4}{3}} \leq 0$, which implies that every point in $\mathcal{N}$ is a Lyapunov stable equilibrium point of (8.31). Hence, it follows from Proposition 8.4 that the system (8.31) is finite-time-semistable. In fact, $x_1(t) = x_2(t) = x_3(t) = \frac{1}{3}(x_{10}+x_{20}+x_{30})$ for $t \geq T(x_0)$. Figure 8.2 shows the state trajectories versus time. $\triangle$

Note that in Proposition 8.4 Lyapunov stability is needed for finite-time semistability. However, finding the corresponding Lyapunov function can be a difficult task. To overcome

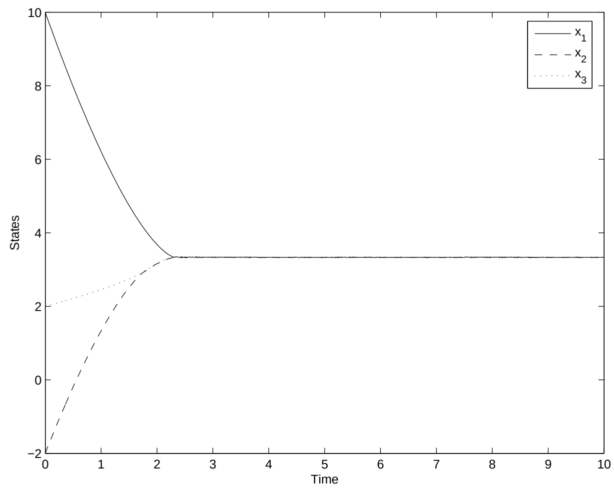

Figure 8.2: State trajectories versus time for Example 8.4

this drawback, we use the nontangency-based approach [32] to guarantee finite-time semistability by testing a condition on the vector field f, which avoids proving Lyapunov stability. Before we state this result, we need some new notation and definitions which can be found in [32].

Given a set $\mathcal{E} \subseteq \mathbb{R}^n$, let $\operatorname{co}\mathcal{E}$ denote the union of the convex hulls of the connected components of $\mathcal{E}$, and let $\operatorname{coco}\mathcal{E}$ denote the cone generated by $\operatorname{co}\mathcal{E}$. Given $x \in \mathbb{R}^n$, the *direction cone* $\mathcal{F}_x$ of f at x relative to $\mathbb{R}^n$ is the intersection of all sets of the form $\overline{\operatorname{coco}(f(\mathcal{U})\backslash\{0\})}$, where $\mathcal{U} \subseteq \mathbb{R}^n$ is an open neighborhood of x. Let $z \in \mathcal{E} \subseteq \mathbb{R}^n$. A vector $v \in \mathbb{R}^n$ is *tangent* to $\mathcal{E}$ at $z \in \mathcal{E}$ if and only if there exist a sequence $\{z_i\}_{i=1}^{\infty}$ in $\mathcal{E}$ converging to z and a sequence $\{h_i\}_{i=1}^{\infty}$ of positive real numbers converging to zero such that $\lim_{i\to\infty} \frac{1}{h_i}(z_i - z) = v$. The *tangent cone* to $\mathcal{E}$ at z is the closed cone $T_z\mathcal{E}$ of all vectors tangent to $\mathcal{E}$ at z. Finally, the vector field f is *nontangent* to the set $\mathcal{E}$ at the point $z \in \mathcal{E}$ if and only if $T_z\mathcal{E} \cap \mathcal{F}_z \subseteq \{0\}$.

Proposition 8.5. Assume f is homogeneous of degree $k < 0$ with respect to ν. Furthermore, assume that there exists a weakly proper, continuous function $V : \mathbb{R}^n \to \mathbb{R}$ such that $\dot{V}$ is defined on $\mathbb{R}^n$ and satisfies $V(x) \geq 0$, $x \in \mathbb{R}^n$, $V(z) = 0$ for $z \in f^{-1}(0)$, and $\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$. For every $z \in f^{-1}(0)$, let $\mathcal{N}_z$ denote the largest negatively invariant connected subset of $\overline{\dot{V}^{-1}(0)}$ containing z. If f is nontangent to $\mathcal{N}_z$ at the point $z \in f^{-1}(0)$, then (8.1) is finite-time-semistable.

Proof. Since $V(x) \geq 0 = V(z)$ and $\dot{V}(x) \leq 0 = \dot{V}(z)$ for all $x \in \mathbb{R}$ and $z \in f^{-1}(0)$, with all the given conditions, it follows from *ii*) of Theorem 7.1 of [32] that x is Lyapunov stable. Now, it follows from Proposition 8.4 that (8.1) is finite-time-semistable. $\square$

Example 8.5. Consider the dynamical system given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} (x_3(t)-x_4(t))^{\frac{1}{3}} \\ (x_4(t)-x_3(t))^{\frac{1}{3}} \\ \mathrm{sign}(x_4(t)-x_3(t))(x_4(t)-x_3(t))^{\frac{2}{3}} + x_2(t) - x_1(t) \\ \mathrm{sign}(x_3(t)-x_4(t))(x_3(t)-x_4(t))^{\frac{2}{3}} + x_1(t) - x_2(t) \end{bmatrix},$$
$$x_1(0) = x_{10}, \quad x_2(0) = x_{20}, \quad x_3(0) = x_{30}, \quad x_4(0) = x_{40}, \quad t \geq 0, \quad (8.32)$$

where $x_i \in \mathbb{R}$, $i = 1, 2, 3, 4$. For each $a, b \in \mathbb{R}$, $x_1 = x_2 = a$ and $x_3 = x_4 = b$ are the equilibrium points of (8.32). We show that all the equilibrium points in (8.32) are finite-time-semistable. Note that the vector field f of (8.32) is homogeneous of degree -2 with respect to the semi-Euler vector field $\nu(x) = 2(x_1-x_2)\frac{\partial}{\partial x_1} + 2(x_2-x_1)\frac{\partial}{\partial x_2} + 3(x_3-x_4)\frac{\partial}{\partial x_3} + 3(x_4-x_3)\frac{\partial}{\partial x_4}$. Now, consider $V(x) = \frac{1}{2}(x_1-x_2)^2 + \frac{3}{4}(x_3-x_4)^{\frac{4}{3}}$. Then $\dot{V}(x) = -2|x_3 - x_4| \leq 0$. Let $\mathcal{R} \triangleq \{x \in \mathbb{R}^4 : \dot{V}(x) = 0\} = \{x \in \mathbb{R}^4 : x_3 = x_4\}$ and let $\mathcal{N}$ denote the largest negatively invariant set contained in $\mathcal{R}$. On $\mathcal{N}$, it follows from (8.32) that $\dot{x}_1 = \dot{x}_2 = 0$, $\dot{x}_3 = \dot{x}_4 = 0$, and $x_1 = x_2$. Hence, $\mathcal{N} = \{x \in \mathbb{R}^4 : x_1 = x_2 = a, x_3 = x_4 = b\}$, $a, b \in \mathbb{R}$, which implies that $\mathcal{N}$ is the set of equilibrium points.

Next, we show that f for (8.32) is nontangent to $\mathcal{N}$ at the point $z \in \mathcal{N}$. To see this, note that the tangent cone $T_z\mathcal{N}$ to the equilibrium set $\mathcal{N}$ is orthogonal to the vectors $\mathbf{u}_1 \triangleq [1, -1, 0, 0]^{\mathrm{T}}$ and $\mathbf{u}_2 \triangleq [0, 0, 1, -1]^{\mathrm{T}}$. On the other hand, since $f(z) \in \mathrm{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ for all $z \in \mathbb{R}^4$, it follows that the direction cone $\mathcal{F}$ of f at $z \in \mathcal{N}$ relative to $\mathbb{R}^4$ satisfies $\mathcal{F}_z \subseteq \mathrm{span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Hence, $T_z\mathcal{N} \cap \mathcal{F}_z = \{0\}$, which implies that the vector field f is nontangent to the set of equilibria $\mathcal{N}$ at the point $z \in \mathcal{N}$. Note that for every $z \in \mathcal{N}$, the set $\mathcal{N}_z$ required by Proposition 8.5 is contained in $\mathcal{N}$. Since nontangency to $\mathcal{N}$ implies nontangency to $\mathcal{N}_z$ at the

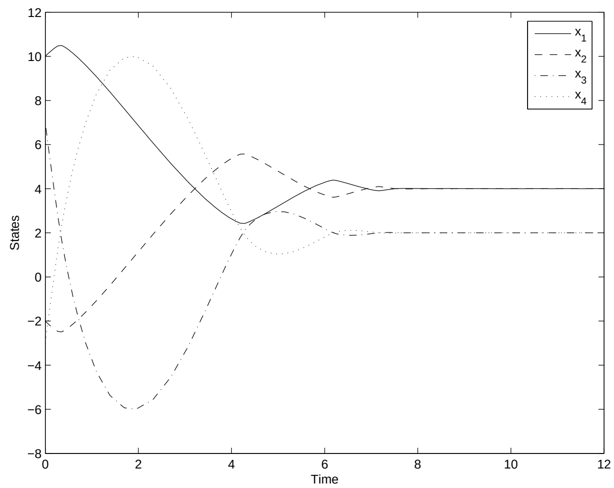

Figure 8.3: State trajectories versus time for Example 8.5

point $z \in \mathcal{N}$, it follows from Proposition 8.5 that the system (8.32) is finite-time-semistable. In particular, $x_1(t) = x_2(t) = \frac{1}{2}(x_{10} + x_{20})$ and $x_3(t) = x_4(t) = \frac{1}{2}(x_{30} + x_{40})$ for $t \geq T(x_0)$. Figure 8.3 shows the state trajectories versus time. $\triangle$

8.6. The Consensus Problem in Dynamical Networks

In this section, we address a nonlinear consensus problem in dynamical networks [187]. The information consensus problem appears frequently in coordination of multiagent systems and involves finding a dynamic algorithm that enables a group of agents in a network to agree upon certain quantities of interest with undirected or directed information flow. In this section, we use graph-theoretic notions to represent a dynamical network and present solutions to the consensus problem for networks with undirected graph *topologies* (or information flows) [187]. We begin by establishing some notation and definitions. Specifically, let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a *directed graph* (or digraph) denoting the dynamical network (or dynamic graph) with the set of *nodes* (or vertices) $\mathcal{V} = \{1, \ldots, q\}$ involving a finite nonempty set denoting the agents, the set of *edges* $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow, and an *adjacency matrix* $\mathcal{A} \in \mathbb{R}^{q \times q}$ such that $\mathcal{A}_{(i,j)} = 1$, $i, j = 1, \ldots, q$, if $(j, i) \in \mathcal{E}$, while $\mathcal{A}_{(i,j)} = 0$ if $(j, i) \notin \mathcal{E}$. The edge $(j, i) \in \mathcal{E}$ denotes that agent j can obtain information from agent i, but not necessarily vice versa. Moreover, we assume $\mathcal{A}_{(i,i)} = 0$ for all $i \in \mathcal{V}$. A *graph* or *undirected graph* $\mathfrak{G}$ associated with the adjacency matrix $\mathcal{A} \in \mathbb{R}^{q\times q}$ is a directed graph for which the *arc set* is symmetric, that is, $\mathcal{A} = \mathcal{A}^{\mathrm{T}}$. Weighted graphs can also be considered here; however, since this extension does not alter any of the conceptual results in this section we do not consider this extension for simplicity of exposition. Finally, we denote the *value* of the node $i \in \{1, \ldots, q\}$ at time t by $x_i(t) \in \mathbb{R}$. The consensus problem involves the design of a dynamic algorithm that guarantees information state equipartition, that is, $\lim_{t\to\infty} x_i(t) = \alpha \in \mathbb{R}$ for $i = 1, \ldots, q$.

The consensus problem is a dynamic graph involving the trajectories of the dynamical network characterized by the multiagent dynamical system $\mathcal{G}$ given by

$$\dot{x}_i(t) = \sum_{j=1,\, j\neq i}^{q} \phi_{ij}(x_i(t), x_j(t)), \quad x_i(t_0) = x_{i0}, \quad t \geq t_0, \quad i = 1, \ldots, q, \tag{8.33}$$

or, in vector form,

$$\dot{x}(t) = f(x(t)), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{8.34}$$

where $x(t) \triangleq [x_1(t), \ldots, x_q(t)]^{\mathrm{T}}$, $t \geq t_0$, and $f = [f_1, \ldots, f_q]^{\mathrm{T}} : \mathbb{R}^q \to \mathbb{R}^q$ is such that $f_i(x) = \sum_{j=1,\, j\neq i}^{q} \phi_{ij}(x_i, x_j)$. This nonlinear model is proposed in [104] and is called a *power balance equation*. Here, however, we address a more general model in that $\phi_{ij}(\cdot, \cdot)$ has no special structure and x need not be constrained to the nonnegative orthant of the state space. For the statement of the main results of this section the following definition is needed.

Definition 8.5 [19]. A directed graph $\mathfrak{G}$ is *strongly connected* if for any ordered pair of vertices (i, j), $i \neq j$, there exists a *path* (i.e., sequence of arcs) leading from i to j.

Recall that $\mathcal{A} \in \mathbb{R}^{q\times q}$ is *irreducible*, that is, there does not exist a permutation matrix such that $\mathcal{A}$ is cogredient to a lower-block triangular matrix, if and only if $\mathfrak{G}$ is strongly connected (see Theorem 2.7 of [19]).

Assumption 1: For the *connectivity matrix* $\mathcal{C} \in \mathbb{R}^{q \times q}$ associated with the multiagent dynamical system $\mathcal{G}$ defined by

$$\mathcal{C}_{(i,j)} \triangleq \begin{cases} 0, & \text{if } \phi_{ij}(x_i, x_j) \equiv 0, \\ 1, & \text{otherwise}, \end{cases} \quad i \neq j, \quad i, j = 1, \ldots, q, \tag{8.35}$$

and $\mathcal{C}_{(i,i)} = -\sum_{k=1,\, k\neq i}^{q} \mathcal{C}_{(i,k)}$, $i = 1, \ldots, q$, $\operatorname{rank} \mathcal{C} = q - 1$, and for $\mathcal{C}_{(i,j)} = 1$, $i \neq j$, $\phi_{ij}(x_i, x_j) = 0$ if and only if $x_i = x_j$.

Assumption 2: For $i, j = 1, \ldots, q$, $(x_i - x_j)\phi_{ij}(x_i, x_j) \leq 0$, $x_i, x_j \in \mathbb{R}$.

The fact that $\phi_{ij}(x_i, x_j) = 0$ if and only if $x_i = x_j$, $i \neq j$, implies that agents $\mathcal{G}_i$ and $\mathcal{G}_j$ are *connected*, and hence can share information; alternatively, $\phi_{ij}(x_i, x_j) \equiv 0$ implies that agents $\mathcal{G}_i$ and $\mathcal{G}_j$ are *disconnected* and hence cannot share information. Assumption 1 implies that if the energies or information in the connected agents $\mathcal{G}_i$ and $\mathcal{G}_j$ are equal, then energy or information exchange between these agents is not possible. This statement is reminiscent of the *zeroth law of thermodynamics*, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium. Furthermore, if $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$ and $\operatorname{rank} \mathcal{C} = q-1$, then it follows that the adjacency matrix $\mathcal{A}$ is irreducible, which implies that for any pair of agents $\mathcal{G}_i$ and $\mathcal{G}_j$, $i \neq j$, of $\mathcal{G}$ there exists a sequence of information connectors (information arcs) of $\mathcal{G}$ that connect $\mathcal{G}_i$ and $\mathcal{G}_j$. Assumption 2 implies that energy or information flows from more energetic or information rich agents to less energetic or information poor agents and is reminiscent of the *second law of thermodynamics*, which states that heat (energy) must flow in the direction of lower temperatures. For further details, see [104].

For the statement of the next result, let $\mathbf{e} \in \mathbb{R}^q$ denote the ones vector of order q, that is, $\mathbf{e} \triangleq [1, \ldots, 1]^{\mathrm{T}}$.

Proposition 8.6. Consider the multiagent dynamical system (8.34) and assume that Assumptions 1 and 2 hold. Then $f_i(x) = 0$ for all $i = 1, \ldots, q$ if and only if $x_1 = \cdots = x_q$. Furthermore, $\alpha\mathbf{e}$, $\alpha \in \mathbb{R}$, is an equilibrium state of (8.34).

Proof. If $x_i = x_j$ for all $(i, j) \in \mathcal{E}$, then $f_i(x) = 0$ for all $i = 1, \ldots, q$ is immediate from

Assumption 1. Next, we show that $f_i(x) = 0$ for all $i = 1, \ldots, q$ implies that $x_1 = \cdots = x_q$. If the values of all nodes are equal, then the result is immediate. Hence, assume there exists a node i^* such that $x_{i^*} \geq x_j$ for all $j \neq i^*$, $j \in \{1, \ldots, q\}$. If $(i, j) \in \mathcal{E}$, then we define a *neighbor* of node i to be node j and *vice versa*.

Define the initial node set $\mathcal{J}^{(0)} \triangleq \{i^*\}$ and denote the indices of all the first neighbors of node i^* by $\mathcal{J}^{(1)} = \mathcal{N}_{i^*}$. Then, $f_{i^*}(x) = 0$ implies that $\sum_{j \in \mathcal{N}_{i^*}} \phi_{i^*j}(x_{i^*}, x_j) = 0$. Since $x_j \leq x_{i^*}$ for all $j \in \mathcal{N}_{i^*}$ and, by Assumption 2, $\phi_{ij}(z_i, z_j) \leq 0$ for all $z_i \geq z_j$, it follows that $x_{i^*} = x_j$ for all the first neighbors $j \in \mathcal{J}^{(1)}$. Next, we define the kth neighbor of node i^* and show that the value of node i^* is equal to the values of all kth neighbors of node i^* for $k = 1, \ldots, q-1$. The set of kth neighbors of node i^* is defined by

$$\mathcal{J}^{(k)} \triangleq \mathcal{J}^{(k-1)} \cup \mathcal{N}_{\mathcal{J}^{(k-1)}}, \quad k \geq 1, \quad \mathcal{J}^{(0)} = \{i^*\}, \tag{8.36}$$

where $\mathcal{N}_{\mathcal{J}}$ denotes the set of neighbors of the node set $\mathcal{J} \subseteq \mathcal{V}$. By definition, $\{i^*\} \subset \mathcal{J}^{(k)} \subseteq \mathcal{V}$ for all $k \geq 1$ and $\mathcal{J}^{(k)}$ is a monotonically increasing sequence of node sets in the sense of inclusions.

Next, we show that $\mathcal{J}^{(q-1)} = \mathcal{V}$. Suppose, *ad absurdum*, $\mathcal{V} \backslash \mathcal{J}^{(q-1)} \neq \emptyset$. Then, by definition, there exists one node $m \in \{1, \ldots, q\}$, disconnected from all the other nodes. Hence, $\mathcal{C}_{(m,i)} = \mathcal{C}_{(i,m)} = 0$, $i = 1, \ldots, q$, which implies that the connectivity matrix $\mathcal{C}$ has a row and a column of zeros. Without loss of generality, assume that $\mathcal{C}$ has the form $\mathcal{C} = \begin{bmatrix} \mathcal{C}_{\mathrm{s}} & 0_{(q-1)\times 1} \\ 0_{1\times(q-1)} & 0 \end{bmatrix}$, where $\mathcal{C}_{\mathrm{s}} \in \mathbb{R}^{(q-1)\times(q-1)}$ denotes the connectivity matrix for the new directed graph $\mathbb{G}$ which excludes node m from the directed graph $\mathfrak{G}$. In this case, since $\operatorname{rank} \mathcal{C}_{\mathrm{s}} \leq q-2$, it follows that $\operatorname{rank} \mathcal{C} < q-1$, which contradicts Assumption 1.

Using mathematical induction, we show that the values of all the nodes in $\mathcal{J}^{(k)}$ are equal for $k \geq 1$. This statement holds for $k = 1$. Assuming that the values of all the nodes in $\mathcal{J}^{(k)}$ are equal to the value of node i^*, we show that the values of all the nodes in $\mathcal{J}^{(k+1)}$ are equal to the value of node i^* as well. Note that since $\mathfrak{G}$ is strongly connected, $\mathcal{N}_i \neq \emptyset$ for all $i \in \mathcal{V}$. If $\mathcal{N}_i \cap (\mathcal{J}^{(k+1)} \backslash \mathcal{J}^{(k)}) = \emptyset$ for all i, then it follows that $\mathcal{J}^{(k+1)} = \mathcal{J}^{(k)}$, and hence, the statement holds. Thus, it suffices to show $x_i = x_{i^*}$ for an arbitrary node $i \in \mathcal{J}^{(k)}$ with $\mathcal{N}_i \cap (\mathcal{J}^{(k+1)} \backslash \mathcal{J}^{(k)}) \neq \emptyset$. For node i, note that $\sum_{j \in \mathcal{N}_i} \phi_{ij}(x_i, x_j) = 0$. Furthermore, note that $\mathcal{N}_i = (\mathcal{N}_i \cap \mathcal{J}^{(k)}) \cup (\mathcal{N}_i \cap (\mathcal{V} \backslash \mathcal{J}^{(k)}))$, $\mathcal{V} \backslash \mathcal{J}^{(k)} = \mathcal{V} \backslash \mathcal{J}^{(k+1)} \cup (\mathcal{J}^{(k+1)} \backslash \mathcal{J}^{(k)})$, $\mathcal{J}^{(k)} \subseteq \mathcal{V}$ for all k, and $J^{(k+1)}$ contains the set of first neighbors of node i, or $\mathcal{N}_i \subseteq \mathcal{J}^{(k+1)}$. Then it follows that $\mathcal{N}_i \cap (\mathcal{V} \backslash \mathcal{J}^{(k)}) = \mathcal{N}_i \cap (\mathcal{J}^{(k+1)} \backslash \mathcal{J}^{(k)})$ and

$$\sum_{j \in \mathcal{N}_i \cap \mathcal{J}^{(k)}} \phi_{ij}(x_i, x_j) + \sum_{j \in \mathcal{N}_i \cap (\mathcal{J}^{(k+1)} \backslash J^{(k)})} \phi_{ij}(x_i, x_j) = 0. \tag{8.37}$$

Since $x_j = x_i$ for all nodes $j \in \mathcal{N}_i \cap \mathcal{J}^{(k)} \subseteq \mathcal{J}^{(k)}$, it follows that $\sum_{j \in \mathcal{N}_i \cap \mathcal{J}^{(k)}} \phi_{ij}(x_i, x_j) = 0$, and hence, $\sum_{j \in \mathcal{N}_i \cap (\mathcal{J}^{(k+1)} \backslash J^{(k)})} \phi_{ij}(x_i, x_j) = 0$. However, since $x_{i^*} = x_i \geq x_j$ for all $i \in \mathcal{J}^{(k)}$ and $j \in \mathcal{V} \backslash \mathcal{J}^{(k)}$, it follows that the values of all nodes in $\mathcal{N}_i \cap (\mathcal{J}^{(k+1)} \backslash \mathcal{J}^{(k)})$ are equal to x_{i^*}. Hence, the values of all nodes i in the node set $\bigcup_{i \in \mathcal{J}^{(k)}} \mathcal{N}_i \cap (\mathcal{J}^{(k+1)} \backslash \mathcal{J}^{(k)}) = \mathcal{J}^{(k+1)} \cap (\mathcal{J}^{(k+1)} \backslash \mathcal{J}^{(k)}) = \mathcal{J}^{(k+1)} \backslash \mathcal{J}^{(k)}$ are equal to x_{i^*}, that is, the values of all the nodes in $\mathcal{J}^{(k+1)}$ are equal. Combining this result with the fact that $\mathcal{J}^{(q-1)} = \mathcal{V}$, it follows that the values of all the nodes in $\mathcal{V}$ are equal.

The second conclusion is a direct consequence of the first conclusion. $\square$

Theorem 8.8. Consider the multiagent dynamical system (8.34) and assume that Assumptions 1 and 2 hold. Furthermore, assume that $\phi_{ij}(x_i, x_j) = -\phi_{ji}(x_j, x_i)$ for all $i, j = 1, \ldots, q$, $i \neq j$. Then for every $\alpha \in \mathbb{R}$, $\alpha \mathbf{e}$ is a semistable equilibrium state of (8.34). Furthermore, $x(t) \to \frac{1}{q} \mathbf{e}\mathbf{e}^{\mathrm{T}} x(t_0)$ as $t \to \infty$ and $\frac{1}{q} \mathbf{e}\mathbf{e}^{\mathrm{T}} x(t_0)$ is a semistable equilibrium state.

Proof. It follows from Proposition 8.6 that $\alpha \mathbf{e}$, $\alpha \in \mathbb{R}$, is an equilibrium state of (8.34). To show Lyapunov stability of the equilibrium state $\alpha \mathbf{e}$, consider the Lyapunov function candidate $V(x - \alpha \mathbf{e}) = \frac{1}{2}(x - \alpha \mathbf{e})^{\mathrm{T}}(x - \alpha \mathbf{e})$. Now, since $\phi_{ij}(x_i, x_j) = -\phi_{ji}(x_j, x_i)$, $x_i \in \mathbb{R}$, $i \neq j$, $i, j = 1, \ldots, q$, and $\mathbf{e}^{\mathrm{T}} f(x) = 0$, $x \in \mathbb{R}^q$, it follows from Assumption 2 that

$$\dot{V}(x - \alpha \mathbf{e}) = \sum_{i=1}^{q} x_i \left[\sum_{j=1, j \neq i}^{q} \phi_{ij}(x_i, x_j) \right] = \sum_{i=1}^{q} \sum_{j \in \mathcal{K}_i} (x_i - x_j) \phi_{ij}(x_i, x_j) \leq 0, \quad x \in \mathbb{R}^q,$$

where $\mathcal{K}_i \triangleq \mathcal{N}_i \backslash \bigcup_{l=1}^{i-1}\{l\}$ and $\mathcal{N}_i \triangleq \{j \in \{1,\ldots,q\} : \phi_{ij}(x_i,x_j) = 0$ if and only if $x_i = x_j\}$, $i = 1,\ldots,q$, which establishes Lyapunov stability of the equilibrium state $\alpha\mathbf{e}$.

To show that $\alpha\mathbf{e}$ is semistable, let $\mathcal{R} \triangleq \{x \in \mathbb{R}^q : \dot{V}(x-\alpha\mathbf{e}) = 0\} = \{x \in \mathbb{R}^q : (x_i - x_j)\phi_{ij}(x_i,x_j) = 0, i = 1,\ldots,q, j \in \mathcal{K}_i\}$. Now, by Assumption 1 and the fact that $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$, the undirected graph associated with the adjacency matrix $\mathcal{A}$ for the multiagent dynamical system (8.34) is strongly connected, which implies that $\mathcal{R} = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$. Since the set $\mathcal{R}$ consists of the equilibrium states of (8.34), it follows that the largest invariant set $\mathcal{M}$ contained in $\mathcal{R}$ is given by $\mathcal{M} = \mathcal{R}$. Hence, it follows from the Krasovskii-LaSalle invariant set theorem and boundedness of solutions that for any initial condition $x(t_0) \in \mathbb{R}^q$, $x(t) \to \mathcal{M}$ as $t \to \infty$. Thus, it follows from Lyapunov stability of $\alpha\mathbf{e}$ and Proposition 5.4 of [32] that $\alpha\mathbf{e}$ is a semistable equilibrium state of (8.34). Next, note that since $\mathbf{e}^{\mathrm{T}}x(t) = \mathbf{e}^{\mathrm{T}}x(t_0)$ and $x(t) \to \mathcal{M}$ as $t \to \infty$, it follows that $x(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(t_0)$ as $t \to \infty$. Hence, with $\alpha = \frac{1}{q}\mathbf{e}^{\mathrm{T}}x(t_0)$, $\alpha\mathbf{e} = \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(t_0)$ is a semistable equilibrium state of (8.34). $\square$

Theorem 8.8 implies that the steady-state value of the information state in each agent $\mathcal{G}_i$ of the multiagent dynamical system $\mathcal{G}$ is equal, that is, the steady-state value of the multiagent dynamical system $\mathcal{G}$ given by $x_\infty = \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(t_0) = \left[\frac{1}{q}\sum_{i=1}^{q} x_i(t_0)\right]\mathbf{e}$ is uniformly distributed over all multiagents of $\mathcal{G}$. This phenomenon is known as *equipartition of energy* [104] in system thermodynamics and *information consensus* or *protocol agreement* [187] in cooperative network dynamical systems.

8.7. Distributed Control Algorithms for Finite-Time Consensus

In this section, we combine the thermodynamically motivated information consensus framework for multiagent dynamic networks developed in Section 8.6 with the finite-time semistability and homogeneity theory developed in Sections 8.3–8.5 to design distributed finite-time consensus protocols for cooperative network systems. Specifically, consider q

continuous-time integrator agents with dynamics

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \tag{8.38}$$

where for each $i \in \{1, \ldots, q\}$, $x_i(t) \in \mathbb{R}$ denotes the information state and $u_i(t) \in \mathbb{R}$ denotes the information control input for all $t \geq 0$. The general consensus protocol is given by

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \phi_{ij}(x_i(t), x_j(t)), \tag{8.39}$$

where $\phi_{ij}(\cdot,\cdot)$ satisfies Assumptions 1 and 2, and $\phi_{ij}(x_i, x_j) = -\phi_{ji}(x_j, x_i)$ for all $i, j = 1, \ldots, q$, $i \neq j$. Note that (8.38) and (8.39) describes an interconnected network where information states are updated using a distributed controller involving neighbor-to-neighbor interaction between agents.

Theorem 8.9. Consider the closed-loop multiagent system $\mathcal{G}$ given by (8.38) and (8.39). Assume that Assumptions 1 and 2 hold, and $\phi_{ij}(x_i, x_j) = -\phi_{ji}(x_j, x_i)$ for all $i, j = 1, \ldots, q$, $i \neq j$. Furthermore, assume that the vector field f of the closed-loop system (8.38) and (8.39) is homogenous of degree $k \in \mathbb{R}$ with respect to $\nu(x) = -\sum_{i=1}^{q} \left[\sum_{j=1, j\neq i}^{q} \mu_{ij}(x_i, x_j)\right] \frac{\partial}{\partial x_i}$, where $x \triangleq [x_1, \ldots, x_q]^{\mathrm{T}} \in \mathbb{R}^q$ and $\mu_{ij}(\cdot,\cdot)$ satisfies Assumption 2, $\mu_{ij}(x_i, x_j) = -\mu_{ji}(x_j, x_i)$, and $\mu_{ij}(x_i, x_j) = 0$ if and only if $x_i = x_j$ for all $i, j = 1, \ldots, q$, $i \neq j$. Then for every $x_{\mathrm{e}} \in \mathbb{R}$, $x_{\mathrm{e}}\mathbf{e}$ is a finite-time-semistable equilibrium state of $\mathcal{G}$ if and only if $k < 0$. Furthermore, if $k < 0$, then $x(t) = \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}} x(0)$ for all $t \geq T(x(0))$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}} x(0)$ is a finite-time-semistable equilibrium state, where $T(x(0)) \geq 0$.

Proof. Suppose $k < 0$. It follows from Theorem 8.8 that $x_{\mathrm{e}}\mathbf{e} \in \mathbb{R}^q$, $x_{\mathrm{e}} \in \mathbb{R}$, is a semistable equilibrium state of the closed-loop homogeneous system (8.38) and (8.39). Furthermore, $x(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}} x(0)$ as $t \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}} x(0)$ is a semistable equilibrium state. Next, it can be shown using similar arguments as in the proof of Theorem 8.8 that (8.22) is globally semistable with $\nu(x) = -\sum_{i=1}^{q} \left[\sum_{j=1, j\neq i}^{q} \mu_{ij}(x_i, x_j)\right] \frac{\partial}{\partial x_i}$. Now, it follows from Theorem 8.7 that $x_{\mathrm{e}}\mathbf{e}$ is a finite-time-semistable equilibrium state by noting that the vector field

$\sum_{j=1,j\neq i}^{q}\phi_{ij}(x_i,x_j)$ is homogeneous of degree $k<0$ with respect to the semi-Euler vector field $\nu(x)=-\sum_{i=1}^{q}\left[\sum_{j=1,j\neq i}^{q}\mu_{ij}(x_i,x_j)\right]\frac{\partial}{\partial x_i}$. Hence, with $x_{\mathrm{e}}=\frac{1}{q}\mathbf{e}^{\mathrm{T}}x(0)$, $x_{\mathrm{e}}\mathbf{e}=\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0)$ is a finite-time-semistable equilibrium state. The converse follows as a direct consequence of Theorem 8.7. $\square$

The following corollary to Theorem 8.9 gives a concrete form for $\phi_{ij}(x_i,x_j)$, $i,j=1,\ldots,q$, $i\neq j$.

Corollary 8.1. Consider the closed-loop multiagent system $\mathcal{G}$ given by (8.38) and (8.39) with

$$\phi_{ij}(x_i,x_j)=\mathcal{C}_{(i,j)}\,\mathrm{sign}(x_j-x_i)|x_j-x_i|^{\alpha},\tag{8.40}$$

where $\alpha>0$ and $\mathcal{C}_{(i,j)}$ is as in (8.35) with $\mathcal{C}=\mathcal{C}^{\mathrm{T}}$. Assume that Assumptions 1 and 2 hold. Then for every $x_{\mathrm{e}}\in\mathbb{R}$, $x_{\mathrm{e}}\mathbf{e}$ is a finite-time-semistable equilibrium state of $\mathcal{G}$ if and only if $\alpha<1$. Furthermore, if $\alpha<1$, then $x(t)=\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0)$ for all $t\geq T(x(0))$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0)$ is a finite-time-semistable equilibrium state, where $T(x(0))\geq 0$.

Proof. The Lie bracket of $\nu(x)=-\sum_{i=1}^{q}\left[\sum_{j=1,j\neq i}^{q}(x_j-x_i)\right]\frac{\partial}{\partial x_i}$ and the vector field f of the closed-loop system (8.38) and (8.39) with (8.40) is given by

$$[\nu,f]=\left[\sum_{i=1}^{q}\frac{\partial f_1}{\partial x_i}\nu_i-\frac{\partial \nu_1}{\partial x_i}f_i,\ldots,\sum_{i=1}^{q}\frac{\partial f_q}{\partial x_i}\nu_i-\frac{\partial \nu_q}{\partial x_i}f_i\right]^{\mathrm{T}}.$$

Since for each $i,j=1,\ldots,q$,

$$\frac{\partial f_j}{\partial x_i}\nu_i-\frac{\partial \nu_j}{\partial x_i}f_i=\begin{cases}\mathcal{C}_{(j,i)}\alpha|x_i-x_j|^{\alpha-1}\left[\sum_{s=1,s\neq i}^{q}(x_i-x_s)\right] & \\ \qquad+\sum_{k=1,k\neq i}^{q}\mathcal{C}_{(i,k)}\mathrm{sign}(x_k-x_i)|x_k-x_i|^{\alpha}, & i\neq j,\\ \left[\sum_{k=1,k\neq j}^{q}\mathcal{C}_{(j,k)}\alpha|x_k-x_j|^{\alpha-1}\right]\left[\sum_{s=1,s\neq j}^{q}(x_s-x_j)\right] & \\ \qquad-(q-1)\sum_{k=1,k\neq j}^{q}\mathcal{C}_{(j,k)}\mathrm{sign}(x_k-x_j)|x_k-x_j|^{\alpha}, & i=j,\end{cases}$$

and noting that $\mathcal{C}_{(i,j)}=\mathcal{C}_{(j,i)}$, $i,j=1,\ldots,q$, $i\neq j$, it follows that for each $j=1,\ldots,q$,

$$\sum_{i=1}^{q}\frac{\partial f_j}{\partial x_i}\nu_i-\frac{\partial \nu_j}{\partial x_i}f_i=\frac{\partial f_j}{\partial x_j}\nu_j-\frac{\partial \nu_j}{\partial x_j}f_j+\sum_{i=1,i\neq j}^{q}\frac{\partial f_j}{\partial x_i}\nu_i-\frac{\partial \nu_j}{\partial x_i}f_i$$

$$
\begin{aligned}
&= \alpha \sum_{k=1,k\neq j}^{q} \mathcal{C}_{(j,k)}\mathrm{sign}(x_k - x_j)|x_k - x_j|^{\alpha} \\
&\quad + \sum_{k=1,k\neq j}^{q} \sum_{s=1,s\neq j,k}^{q} \mathcal{C}_{(j,k)}\alpha|x_k - x_j|^{\alpha-1}(x_s - x_j) \\
&\quad -(q-1) \sum_{k=1,k\neq j}^{q} \mathcal{C}_{(j,k)}\mathrm{sign}(x_k - x_j)|x_k - x_j|^{\alpha} \\
&\quad +\alpha \sum_{i=1,i\neq j}^{q} \mathcal{C}_{(j,i)}\mathrm{sign}(x_i - x_j)|x_i - x_j|^{\alpha} \\
&\quad + \sum_{i=1,i\neq j}^{q} \sum_{s=1,s\neq i,j}^{q} \mathcal{C}_{(j,i)}\alpha|x_i - x_j|^{\alpha-1}(x_i - x_s) \\
&\quad + \sum_{i=1}^{q} \sum_{k=1,k\neq i}^{q} \mathcal{C}_{(i,k)}\mathrm{sign}(x_k - x_i)|x_k - x_i|^{\alpha} \\
&\quad - \sum_{k=1,k\neq j}^{q} \mathcal{C}_{(j,k)}\mathrm{sign}(x_k - x_j)|x_k - x_j|^{\alpha} \\
&= 2\alpha \sum_{i=1,i\neq j}^{q} \mathcal{C}_{(j,i)}\mathrm{sign}(x_i - x_j)|x_i - x_j|^{\alpha} \\
&\quad +\alpha \sum_{i=1,i\neq j}^{q} \sum_{s=1,s\neq i,j}^{q} \mathcal{C}_{(j,i)}\mathrm{sign}(x_i - x_j)|x_i - x_j|^{\alpha} \\
&\quad -q \sum_{k=1,k\neq j}^{q} \mathcal{C}_{(j,k)}\mathrm{sign}(x_k - x_j)|x_k - x_j|^{\alpha} \\
&= q(\alpha-1) \sum_{i=1,i\neq j}^{q} \mathcal{C}_{(j,i)}\mathrm{sign}(x_i - x_j)|x_i - x_j|^{\alpha} \\
&= q(\alpha-1)f_j, \qquad (8.41)
\end{aligned}
$$

which implies that the vector field f is homogeneous of degree $k = q(\alpha - 1)$ with respect to the semi-Euler vector field $\nu(x) = -\sum_{i=1}^{q}\left[\sum_{j=1,j\neq i}^{q}(x_j - x_i)\right]\frac{\partial}{\partial x_i}$. Now, the result is a direct consequence of Theorem 8.9. $\square$

Note that Example 8.4 serves as a special case of Corollary 8.1. More importantly, note that the proposed protocol (8.40) is different from the protocols given in [58, 60] since (8.40) is a distributed *continuous* protocol and is not based on a nonsmooth gradient flow.

Furthermore, this protocol does not satisfy the conditions of Theorem 4 of [58] nor Theorem 5 of [58]. It is also important to note that the proposed protocol can achieve superior performance over the protocols given in [58] since the closed-loop system generated by (8.40) results in continuous closed-loop vector fields as opposed to discontinuous closed-loop vector fields based on nonsmooth gradient flows which can lead to chattering behavior. In addition, the proposed protocol tends to have a faster settling time. Finally, a key advantage of continuous (but non-Lipschitzian) closed-loop systems over Lipschitzian closed-loop systems is that continuous finite-time controllers tend to have better robustness and disturbance rejection properties [28, 30].

Thus far in the literature, only *static* consensus protocols have been addressed. A natural question regarding (8.38) is how to design finite-time *dynamic* compensators to achieve network consensus. This question is important because it can be used to design finite-time consensus protocols for multiagent coordination via output feedback. To begin to address this question, we consider q continuous-time integrator agents given by (8.38) and the dynamic compensators given by

$$\dot{x}_{\mathrm{c}i}(t) = \sum_{j=1, j\neq i}^{q} \phi_{ij}(x_{\mathrm{c}i}(t), x_{\mathrm{c}j}(t)) + \sum_{j=1, j\neq i}^{q} \eta_{ij}(x_i(t), x_j(t)), \quad x_{\mathrm{c}i}(0) = x_{\mathrm{c}i0}, \quad t \geq 0, \tag{8.42}$$

$$u_i(t) = -\sum_{j=1, j\neq i}^{q} \mu_{ij}(x_{\mathrm{c}i}(t), x_{\mathrm{c}j}(t)), \tag{8.43}$$

where $\phi_{ij}(\cdot,\cdot)$, $\eta_{ij}(\cdot,\cdot)$, and $\mu_{ij}(\cdot,\cdot)$, $i, j = 1, \ldots, q$, $i \neq j$, satisfy Assumptions 1 and 2. Furthermore, $\phi_{ij}(\cdot,\cdot)$, $\eta_{ij}(\cdot,\cdot)$, and $\mu_{ij}(\cdot,\cdot)$ are chosen such that the vector field of the closed-loop system (8.38), (8.42), and (8.43) is homogeneous with respect to given semi-Euler vector fields. Recall that if the closed-loop system is semistable and homogeneous of degree $k < 0$ with respect to a given semi-Euler vector field, then the closed-loop system is finite-time-semistable.

As an example, consider $\phi_{ij}(x_{\mathrm{c}i}, x_{\mathrm{c}j}) = \mathcal{C}_{(i,j)}\mathrm{sign}(x_{\mathrm{c}j} - x_{\mathrm{c}i})|x_{\mathrm{c}j} - x_{\mathrm{c}i}|^{\frac{1+\alpha}{2}}$, $\mu_{ij}(x_{\mathrm{c}i}, x_{\mathrm{c}j}) = \mathcal{C}_{(i,j)}\mathrm{sign}(x_{\mathrm{c}j} - x_{\mathrm{c}i})|x_{\mathrm{c}j} - x_{\mathrm{c}i}|^{\alpha}$, and $\eta_{ij}(x_i, x_j) = \mathcal{C}_{(i,j)}(x_j - x_i)$, $0 < \alpha < 1$, $i, j = 1, \ldots, q$, $i \neq j$.

Note that the dynamic compensator (8.42) has a similar structure to (8.34) with additional input supply. Thus, the proposed controller architecture can be viewed as an *interconnection of thermodynamic controllers*, for details see [104]. Finally, note that Example 8.5 is a special case of the closed-loop system given by (8.38), (8.42), and (8.43) with $\phi_{ij}(\cdot,\cdot)$, $\eta_{ij}(\cdot,\cdot)$, and $\mu_{ij}(\cdot,\cdot)$, $i,j=1,\ldots,q$, $i\neq j$, as specified above.

Theorem 8.10. Consider the closed-loop system given by (8.38), (8.42), and (8.43) with $\phi_{ij}(\cdot,\cdot)$, $\eta_{ij}(\cdot,\cdot)$, and $\mu_{ij}(\cdot,\cdot)$, $i,j=1,\ldots,q$, $i\neq j$, as specified above. Assume that Assumptions 1 and 2 hold, and $\mathcal{C}=\mathcal{C}^{\mathrm{T}}$. Then for every $a\in\mathbb{R}$ and $b\in\mathbb{R}$, $(x(t),x_{\mathrm{c}}(t))\equiv(a\mathbf{e},b\mathbf{e})$ is a finite-time-semistable equilibrium state of (8.38), (8.42), and (8.43). Furthermore, $x(t)=\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0)$ and $x_{\mathrm{c}}(t)=\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x_{\mathrm{c}}(0)$ for all $t\geq T(x(0),x_{\mathrm{c}}(0))$, and $(\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0),\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x_{\mathrm{c}}(0))$ is a finite-time-semistable equilibrium state.

Proof. Let $\lambda>0$. Using similar arguments as in the proof of Corollary 8.1 it can be shown that the closed-loop system given by (8.38), (8.42), and (8.43) is homogeneous of degree $k=q\lambda\frac{\alpha-1}{1+\alpha}<0$ with respect to the semi-Euler vector field

$$\nu(x,x_{\mathrm{c}})=-\lambda\sum_{i=1}^{q}\Big[\sum_{j=1,j\neq i}^{q}(x_j-x_i)\Big]\frac{\partial}{\partial x_i}-\frac{2\lambda}{1+\alpha}\sum_{i=1}^{q}\Big[\sum_{j=1,j\neq i}^{q}(x_{\mathrm{c}j}-x_{\mathrm{c}i})\Big]\frac{\partial}{\partial x_{\mathrm{c}i}}.$$

Next, note that for every $a,b\in\mathbb{R}$, $x(t)\equiv a\mathbf{e}$ and $x_{\mathrm{c}}(t)\equiv b\mathbf{e}$ are the equilibrium points for the closed-loop system. Consider the nonnegative function given by

$$V(\tilde{x})=\frac{1}{4}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}(x_i-x_j)^2+\frac{1}{2+2\alpha}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}|x_{\mathrm{c}i}-x_{\mathrm{c}j}|^{1+\alpha},\qquad(8.44)$$

where $\tilde{x}\triangleq[x^{\mathrm{T}},x_{\mathrm{c}}^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{2q}$. In this case, the derivative of $V(\cdot)$ along the trajectories of the closed-loop system is given by $\dot{V}(\tilde{x})=-2\sum_{i=1}^{q}\sum_{j=i+1}^{q-1}\mu_{ij}(x_{\mathrm{c}i},x_{\mathrm{c}j})\phi_{ij}(x_{\mathrm{c}i},x_{\mathrm{c}j})\leq 0$, $\tilde{x}\in\mathbb{R}^{2q}$. Let $\mathcal{R}\triangleq\{\tilde{x}\in\mathbb{R}^{2q}:\dot{V}(\tilde{x})=0\}=\{\tilde{x}\in\mathbb{R}^{2q}:x_{\mathrm{c}1}=\cdots=x_{\mathrm{c}q}\}$ and let $\mathcal{N}$ denote the largest negatively invariant set of $\mathcal{R}$. On $\mathcal{N}$, it follows from (8.38), (8.42), and (8.43) that $\dot{x}_i=0$, $\dot{x}_{\mathrm{c}i}=0$, and $x_1=\cdots=x_q$, $i=1,\ldots,q$. Hence, $\mathcal{N}=\{\tilde{x}\in\mathbb{R}^{2q}:x=a\mathbf{e},x_{\mathrm{c}}=b\mathbf{e}\}$, $a,b\in\mathbb{R}$, which implies that $\mathcal{N}$ is the set of equilibrium points.

Since the graph $\mathfrak{G}$ of the closed-loop system is strongly connected, assume, without loss of generality, that $\mathcal{C}_{(i,i+1)} = \mathcal{C}_{(q,1)} = 1$, where $i = 1,\ldots,q-1$. Now, for $q = 2$, it was shown in Example 8.5 that the vector field f of the closed-loop system given by (8.38), (8.42), and (8.43) is nontangent to $\mathcal{N}$ at a point $\tilde{x} \in \mathcal{N}$. Next, we show that for $q \geq 3$, the vector field f of the closed-loop system given by (8.38), (8.42), and (8.43) is nontangent to $\mathcal{N}$ at a point $\tilde{x} \in \mathcal{N}$. To see this, note that the tangent cone $T_{\tilde{x}}\mathcal{N}$ to the equilibrium set $\mathcal{N}$ is orthogonal to the $2q$ vectors $\mathbf{u}_i \triangleq [0_{1\times(i-1)}, \mathcal{C}_{(i,i+1)}, -\mathcal{C}_{(i,i+1)}, 0_{1\times(2q-i-1)}]^{\mathrm{T}} \in \mathbb{R}^{2q}$, $\mathbf{u}_q \triangleq [-\mathcal{C}_{(q,1)}, 0_{1\times(q-2)}, \mathcal{C}_{(q,1)}, 0_{1\times q}]^{\mathrm{T}} \in \mathbb{R}^{2q}$, $\mathbf{v}_i \triangleq [0_{1\times(q+i-1)}, -\mathcal{C}_{(i,i+1)}, \mathcal{C}_{(i,i+1)}, 0_{1\times(q-i-1)}]^{\mathrm{T}} \in \mathbb{R}^{2q}$, and $\mathbf{v}_q \triangleq [0_{1\times q}, \mathcal{C}_{(q,1)}, 0_{1\times(q-2)}, -\mathcal{C}_{(q,1)}]^{\mathrm{T}} \in \mathbb{R}^{2q}$, $i = 1,\ldots,q-1$, $q \geq 3$. Alternatively, since $f(\tilde{x}) \in \operatorname{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_q,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$ for all $\tilde{x} \in \mathbb{R}^{2q}$, it follows that the direction cone $\mathcal{F}_{\tilde{x}}$ of f at $\tilde{x} \in \mathcal{N}$ relative to $\mathbb{R}^{2q}$ satisfies $\mathcal{F}_{\tilde{x}} \subseteq \operatorname{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_q,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$. Hence, $T_{\tilde{x}}\mathcal{N} \cap \mathcal{F}_{\tilde{x}} = \{0\}$, which implies that the vector field f is nontangent to the set of equilibria $\mathcal{N}$ at the point $\tilde{x} \in \mathcal{N}$. Note that for every $z \in \mathcal{N}$, the set $\mathcal{N}_z$ required by Proposition 8.5 is contained in $\mathcal{N}$. Since nontangency to $\mathcal{N}$ implies nontangency to $\mathcal{N}_z$ at the point $z \in \mathcal{N}$, it follows from Proposition 8.5 that the closed-loop system (8.38), (8.42), and (8.43) is finite-time-semistable. □

Finally, we apply the developed theory to design finite-time distributed controllers for parallel formations [139] such as flocking [185]. Specifically, consider q continuous-time double integrator agents with dynamics

$$\ddot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad \dot{x}_i(0) = \dot{x}_{i0}, \quad t \geq 0, \tag{8.45}$$

where, for each $i \in \{1,\ldots,q\}$, $x_i(t) = [x_{1i}(t), x_{2i}(t), x_{3i}(t)]^{\mathrm{T}} \in \mathbb{R}^3$ denotes the position, $\dot{x}_i(t) = [\dot{x}_{1i}(t), \dot{x}_{2i}(t), \dot{x}_{3i}(t)]^{\mathrm{T}} \in \mathbb{R}^3$ denotes the velocity, and $u_i(t) = [u_{1i}(t), u_{2i}(t), u_{3i}(t)]^{\mathrm{T}} \in \mathbb{R}^3$ is the control input. We seek a continuous distributed feedback control law u_i involving transmission of both x_i and $\dot{x}_i$ between agents so that *finite-time parallel formation* is achieved; that is, the velocity $\dot{x}_i$ reaches to a constant vector in finite-time for all $i = 1,\ldots,q$, and the relative position between two agents reaches a constant value in finite-time.

Theorem 8.11. Consider the dynamical system given by (8.45). Then finite-time parallel formation for (8.45) is achieved under the distributed feedback control law given by the static controller

$$u_{ri} = \sum_{j=1, j\neq i}^{q} \phi_{rij}(\dot{x}_{ri}, \dot{x}_{rj}) - \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} \mathrm{sign}(\psi_\alpha(x_{ri}, x_{rj}))|\psi_\alpha(x_{ri}, x_{rj})|^{\frac{\alpha}{2-\alpha}}, \tag{8.46}$$

where $0 < \alpha < 1$, $\phi_{rij}(\dot{x}_{ri}, \dot{x}_{rj}) = \mathcal{C}_{(i,j)}\mathrm{sign}(\dot{x}_{rj} - \dot{x}_{ri})|\dot{x}_{rj} - \dot{x}_{ri}|^\alpha$ satisfies Assumptions 1 and 2, $\mathcal{C}_{(i,j)}$ is as in (8.35) with $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$, $\psi_\alpha(x_{ri}, x_{rj}) \triangleq x_{ri} - x_{rj} - d_{rij}$, and $d_{rij} = -d_{rji} \in \mathbb{R}$, $i, j = 1, \ldots, q$, $i \neq j$, $r = 1, 2, 3$.

Proof. For the distributed control law (8.46), let $z_{rij} \triangleq \psi_\alpha(x_{ri}, x_{rj})$, $i, j = 1, \ldots, q$, $i \neq j$, $r = 1, 2, 3$, and consider the augmented closed-loop system

$$\dot{z}_{rij}(t) = \dot{x}_{ri}(t) - \dot{x}_{rj}(t), \quad z_{rij}(0) = z_{rij0}, \quad t \geq 0,$$
$$i, j = 1, \ldots, q, \quad i \neq j, \quad r = 1, 2, 3, \tag{8.47}$$

$$\ddot{x}_{ri}(t) = \sum_{j=1, j\neq i}^{q} \phi_{rij}(\dot{x}_{ri}(t), \dot{x}_{rj}(t)) - \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}\mathrm{sign}(z_{rij}(t))|z_{rij}(t)|^{\frac{\alpha}{2-\alpha}},$$
$$\dot{x}_{ri}(0) = \dot{x}_{ri0}. \tag{8.48}$$

It can be shown using similar arguments as in the proof of Corollary 8.1 that the closed-loop system given by (8.47) and (8.48) is homogeneous of degree $k = q(\alpha - 1) < 0$ with respect to the semi-Euler vector field

$$\nu_r = -\sum_{i=1}^{q}\left[\sum_{j=1, j\neq i}^{q} (\dot{x}_{rj} - \dot{x}_{ri})\right]\frac{\partial}{\partial \dot{x}_{ri}} + q(2-\alpha)\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} z_{rij}\frac{\partial}{\partial z_{rij}}.$$

Next, consider the nonnegative function

$$V_r(z_r, \dot{x}_{(r)}) = \frac{1}{2}\sum_{i=1}^{q}\dot{x}_{ri}^2 + \frac{2-\alpha}{4}\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q}\mathcal{C}_{(i,j)}|z_{rij}|^{\frac{2}{2-\alpha}}, \tag{8.49}$$

where $z_r \triangleq [z_{r12}, z_{r13}, \ldots, z_{r1q}, z_{r21}, z_{r23}, \ldots, z_{r2q}, \ldots, z_{rq(q-1)}] \in \mathbb{R}^{q^2-q}$ and $x_{(r)} \triangleq [x_{r1}, \ldots, x_{rq}]^{\mathrm{T}} \in \mathbb{R}^q$, $r = 1, 2, 3$. In this case, the derivative of $V_r(\cdot)$ along the trajectories of the closed-loop system is given by

$$\dot{V}_r(z_r, \dot{x}_{(r)}) = \sum_{i=1}^{q}\dot{x}_{ri}\sum_{j=1, j\neq i}^{q}\phi_{rij}(\dot{x}_{ri}, \dot{x}_{rj}) - \sum_{i=1}^{q}\dot{x}_{ri}\sum_{j=1, j\neq i}^{q}\mathcal{C}_{(i,j)}\mathrm{sign}(\psi_\alpha(x_{ri}, x_{rj}))$$

$$
\begin{aligned}
&\cdot|\psi_\alpha(x_{ri},x_{rj})|^{\frac{\alpha}{2-\alpha}} + \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}\mathrm{sign}(\psi_\alpha(x_{ri},x_{rj}))\\
&\cdot|\psi_\alpha(x_{ri},x_{rj})|^{\frac{\alpha}{2-\alpha}}(\dot{x}_{ri}-\dot{x}_{rj})\\
=\;& \sum_{i=1}^{q-1}\sum_{j=i+1}^{q}(\dot{x}_{ri}-\dot{x}_{rj})\phi_{rij}(\dot{x}_{ri},\dot{x}_{rj})\\
\le\;& 0, \quad (z_r,\dot{x}_{(r)})\in\mathbb{R}^{q^2-q}\times\mathbb{R}^q. \qquad (8.50)
\end{aligned}
$$

Next, let $\mathcal{R}_r \triangleq \{(z_r,\dot{x}_{(r)})\in\mathbb{R}^{q^2} : \dot{V}_r(z_r,\dot{x}_{(r)})=0\} = \{(z_r,\dot{x}_{(r)})\in\mathbb{R}^{q^2} : \sum_{i=1}^{q-1}\sum_{j=i+1}^{q}(\dot{x}_{ri}-\dot{x}_{rj})\phi_{rij}(\dot{x}_{ri},\dot{x}_{rj})=0, i=1,\ldots,q-1\}$, $r=1,2,3$. Now, by assumption, $\mathcal{R}_r=\{(z_r,\dot{x}_{(r)})\in\mathbb{R}^{q^2} : \dot{x}_{r1}=\cdots=\dot{x}_{rq}\}$. Furthermore, since $\dot{x}_{r1}=\cdots=\dot{x}_{rq}$, it follows that $\dot{z}_{rij}=0$, $i,j=1,\ldots,q$, $i\neq j$, $r=1,2,3$. Let $\mathcal{M}_r$ denote the largest invariant set contained in $\mathcal{R}_r$. On $\mathcal{M}_r$, $\frac{\mathrm{d}}{\mathrm{d}t}|z_{rij}|^{\frac{2}{2-\alpha}} = \frac{2}{2-\alpha}\mathrm{sign}(z_{rij})|z_{rij}|^{\frac{\alpha}{2-\alpha}}\dot{z}_{rij}=0$, and hence, $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{q}\dot{x}_{ri}^2 = \dot{V}_r - \frac{2-\alpha}{4}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}\frac{\mathrm{d}}{\mathrm{d}t}|z_{rij}|^{\frac{2}{2-\alpha}}=0$, which implies that $\dot{x}_{r1}=\cdots=\dot{x}_{rq}=c$, where $c\in\mathbb{R}$. Finally, since $\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}\mathrm{sign}(z_{rij})|z_{rij}|^{\frac{\alpha}{2-\alpha}}=0$ on $\mathcal{M}_r$ and, for each $i\in\{1,\ldots,q\}$, $z_{rij}=-z_{rji}$ and $\dot{z}_{rij}=0$, it follows from Proposition 8.6 that $z_{rij}=0$, $k=1,2,3$.

To show Lyapunov stability of $\dot{x}_{(r)}(t)\equiv c\mathbf{e}$ and $z_r(t)\equiv 0$, consider the shifted Lyapunov function candidate

$$
\tilde{V}_r(z_r,\dot{x}_{(r)}) = \frac{1}{2}\sum_{i=1}^{q}(\dot{x}_{ri}-c)^2 + \frac{2-\alpha}{4}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}|z_{rij}|^{\frac{2}{2-\alpha}}, \qquad (8.51)
$$

where $r=1,2,3$. The rest of the proof now follows using identical arguments as above and invoking Proposition 8.4 with

$$
\nu_r(x_r,z_r) = -\sum_{i=1}^{q}\left[\sum_{j=1,j\neq i}^{q}(\dot{x}_{rj}-\dot{x}_{ri})\right]\frac{\partial}{\partial\dot{x}_{ri}} + q(2-\alpha)\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}z_{rij}\frac{\partial}{\partial z_{rij}}
$$

for the closed-loop system given by (8.47) and (8.48) for showing finite-time parallel formation. □

To illustrate the efficacy of the controller in Theorem 8.11, let $q=3$, $r=1$, $\alpha=\frac{1}{3}$, $d_{112}=2$, $d_{123}=1$, and $d_{131}=3$. The initial conditions are given by $x_{1i}(0)=[-3,2,5]^{\mathrm{T}}$

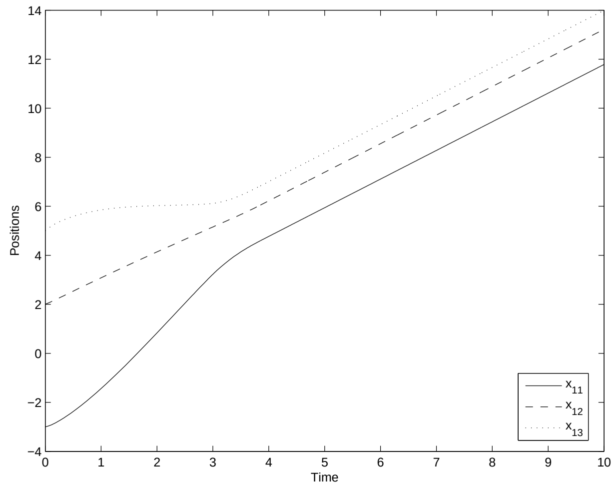

Figure 8.4: Positions versus time for finite-time parallel formation

and $\dot{x}_{1i}(0) = [0.5, 1, 2]^{\mathrm{T}}$, $i = 1, 2, 3$. Figures 8.4 and 8.5 show the positions and the velocities versus time, respectively, where $v_{1i} \triangleq \dot{x}_{1i}$, $i = 1, 2, 3$.

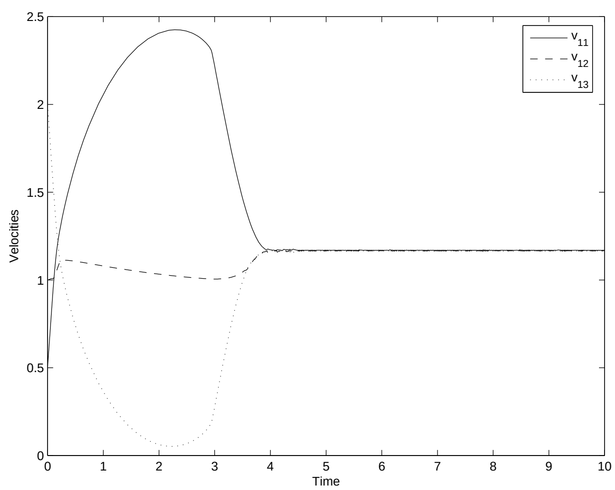

Figure 8.5: Velocities versus time for finite-time parallel formation

Chapter 9

Distributed Nonlinear Control Algorithms for Network Consensus

9.1. Introduction

Modern complex dynamical systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication networks. Distributed decision-making for coordination of networks of dynamic agents involving information flow can be naturally captured by graph-theoretic notions. These dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles (UAV's), autonomous underwater vehicles (AUV's), distributed sensor networks, air and ground transportation systems, swarms of air and space vehicle formations, and congestion control in communication networks, to cite but a few examples. Hence, it is not surprising that a considerable research effort has been devoted to control of networks and control over networks in recent years [135, 187, 205, 228].

A key application area of multiagent network coordination within aerospace systems is cooperative control of vehicle formations using distributed and decentralized controller architectures. Distributed control refers to a control architecture wherein the control is distributed via multiple computational units that are interconnected through information and communication networks, whereas decentralized control refers to a control architecture wherein local decisions are based only on local information. Vehicle formations are typically dynamically decoupled, that is, the motion of a given agent or vehicle does not directly affect the motion of the other agents or vehicles. The multiagent system is coupled via the task which the agents or vehicles are required to perform.

As discussed in Chapter 8, in many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In particular, it is important to develop information consensus protocols for networks of dynamic agents wherein a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus is the existence of a continuum of equilibria representing a state of equipartitioning or *consensus*. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well. For such systems possessing a continuum of equilibria, semistability [31,32], and not asymptotic stability, is the relevant notion of stability.

Using graph-theoretic notions, in this chapter we develop control algorithms for addressing consensus problems for nonlinear multiagent dynamical systems with fixed and switching topologies. The proposed controller architectures are predicated on the recently developed notion of system thermodynamics [104] resulting in controller architectures involving the exchange of information between agents that guarantee that the closed-loop dynamical network is consistent with basic thermodynamic principles. The proposed controllers use undirected and directed graphs to accommodate for a full range of possible graph information topologies without limitations of bidirectional communication.

9.2. The Consensus Problem in Dynamical Networks

In this chapter, we use undirected and directed graphs to represent a nonlinear dynamical network and present solutions to the consensus problem for nonlinear networks with both graph *topologies* (or information flows) [187]. Specifically, let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted *directed graph* (or digraph) denoting the dynamical network (or dynamic graph) with the set of *nodes* (or vertices) $\mathcal{V} = \{1, \ldots, q\}$ involving a finite nonempty set denoting the agents, the set of *edges* $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow, and a *weighted adjacency matrix* $\mathcal{A} \in \mathbb{R}^{q \times q}$ such that $\mathcal{A}_{(i,j)} = \alpha_{ij} > 0$, $i, j = 1, \ldots, q$, if $(j, i) \in \mathcal{E}$, while $\alpha_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. The edge $(j, i) \in \mathcal{E}$ denotes that agent $\mathcal{G}_j$ can obtain information from agent $\mathcal{G}_i$, but not necessarily vice versa. Moreover, we assume that $\alpha_{ii} = 0$ for all $i \in \mathcal{V}$. Note that if the weights α_{ij}, $i, j = 1, \ldots, q$, are not relevant, then α_{ij} is set to 1 for all $(j, i) \in \mathcal{E}$. In this case, $\mathcal{A}$ is called a *normalized adjacency matrix*. A *graph* or *undirected graph* $\mathfrak{G}$ associated with the adjacency matrix $\mathcal{A} \in \mathbb{R}^{q\times q}$ is a directed graph for which the *arc set* is symmetric, that is, $\mathcal{A} = \mathcal{A}^{\mathrm{T}}$. A graph $\mathfrak{G}$ is *balanced* if $\sum_{j=1}^{q} \alpha_{ij} = \sum_{j=1}^{q} \alpha_{ji}$ for all $i = 1, \ldots, q$. Note that for an undirected graph $\mathcal{A} = \mathcal{A}^{\mathrm{T}}$, and hence, every undirected graph is balanced. Finally, we denote the *value* of the node $i \in \{1, \ldots, q\}$ at time t by $x_i(t) \in \mathbb{R}$. The consensus problem involves the design of a dynamic algorithm that guarantees information state equipartition, that is, $\lim_{t\to\infty} x_i(t) = \alpha \in \mathbb{R}$ for $i = 1, \ldots, q$.

The consensus problem can be characterized as a dynamical network involving trajectories of a multiagent dynamical system $\mathcal{G}$ given by

$$\dot{x}_i(t) = \sum_{j=1,\, j\neq i}^{q} \phi_{ij}(x_i(t), x_j(t)), \quad x_i(t_0) = x_{i0}, \quad t \geq 0, \quad i = 1, \ldots, q, \tag{9.1}$$

where $\phi_{ij}(\cdot,\cdot)$, $i, j = 1, \ldots, q$, are locally Lipschitz continuous, or, in vector form,

$$\dot{x}(t) = f(x(t)), \quad x(t_0) = x_0, \quad t \geq 0, \tag{9.2}$$

where $x(t) \triangleq [x_1(t), \ldots, x_q(t)]^{\mathrm{T}}$, $t \geq 0$, and $f = [f_1, \ldots, f_q]^{\mathrm{T}} : \mathcal{D} \to \mathbb{R}^q$ is such that $f_i(x) = \sum_{j=1,\, j\neq i}^{q} \phi_{ij}(x_i, x_j)$, where $\mathcal{D} \subseteq \mathbb{R}^q$ is open. Here, $x_i(t)$, $t \geq 0$, represents an *information state* and $f_i(t) = u_i(t)$ is a distributed consensus algorithm involving neighbor-to-neighbor interaction between agents.

9.3. Distributed Nonlinear Control Algorithms for Consensus

In this section, we develop a thermodynamically motivated information consensus framework for multiagent nonlinear systems that achieve semistability and state equipartition. Specifically, consider q continuous-time integrator agents with dynamics

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \tag{9.3}$$

where for each $i \in \{1, \ldots, q\}$, $x_i(t) \in \mathbb{R}$ denotes the information state and $u_i(t) \in \mathbb{R}$ denotes the information control input for all $t \geq 0$. The nonlinear consensus protocol is given by

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \phi_{ij}(x_i(t), x_j(t)), \tag{9.4}$$

where $\phi_{ij}(\cdot, \cdot)$, $i, j = 1, \ldots, q$, are locally Lipschitz continuous. Here, we assume that Assumptions 1 and 2 of Chapter 8 hold.

The following lemma and definition are needed for the main result of this section. For the statement of the lemma, $(\cdot)^{\mathrm{D}}$ denotes the Drazin generalized inverse and $\mathbf{e} \in \mathbb{R}^q$ denotes the ones vector of order q, that is, $\mathbf{e} \triangleq [1, \ldots, 1]^{\mathrm{T}}$. Recall that for a diagonal matrix $A \in \mathbb{R}^{q\times q}$ the Drazin inverse $A^{\mathrm{D}} \in \mathbb{R}^{q\times q}$ is given by $A^{\mathrm{D}}_{(i,i)} = 0$ if $A_{(i,i)} = 0$ and $A^{\mathrm{D}}_{(i,i)} = 1/A_{(i,i)}$ if $A_{(i,i)} \neq 0$, $i = 1, \ldots, q$ [22, p. 227].

Lemma 9.1. Let $A \in \mathbb{R}^{q\times q}$ and $A_{\mathrm{d}i} \in \mathbb{R}^{q\times q}$, $i = 1, \ldots, n_{\mathrm{d}}$, be given by either

$$A_{(i,j)} = \begin{cases} -\sum_{k=1,k\neq i}^{q} a_{ik}, & i = j, \\ 0, & i \neq j, \end{cases}$$
$$A_{\mathrm{d}(i,j)} = \begin{cases} 0, & i = j, \\ a_{ij}, & i \neq j, \end{cases} \quad i, j = 1, \ldots, q, \tag{9.5}$$

or

$$A_{(i,j)} = \begin{cases} -\sum_{k=1,k\neq i}^{q} a_{ki}, & i = j, \\ 0, & i \neq j, \end{cases}$$
$$A_{\mathrm{d}(i,j)} = \begin{cases} 0, & i = j, \\ a_{ij}, & i \neq j, \end{cases} \quad i, j = 1, \ldots, q. \tag{9.6}$$

Assume that $\sum_{k=1,k\neq i}^{q} a_{ik} = \sum_{k=1,k\neq i}^{q} a_{ki}$ for each $i = 1, \ldots, q$. Then for every $A_{\mathrm{d}i}$, $i = 1, \ldots, n_{\mathrm{d}}$ such that $\sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i} = A_{\mathrm{d}}$ and $a_{ij} \geq 0$, $i, j = 1, \ldots, q$, $i \neq j$, there exist nonnegative definite matrices $Q_i \in \mathbb{R}^{q\times q}$, $i = 1, \ldots, n_{\mathrm{d}}$, such that

$$2A + \sum_{i=1}^{n_{\mathrm{d}}} (Q_i + A_{\mathrm{d}i}^{\mathrm{T}} Q_i^{\mathrm{D}} A_{\mathrm{d}i}) \leq 0. \tag{9.7}$$

Proof. For each $i \in \{1, \ldots, n_{\mathrm{d}}\}$, let Q_i be the diagonal matrix defined by

$$Q_{i(l,l)} \triangleq \sum_{m=1, l\neq m}^{q} A_{\mathrm{d}i(l,m)}, \quad l = 1, \ldots, q, \tag{9.8}$$

and note that $A+\sum_{i=1}^{n_{\rm d}} Q_i = 0$, $(A_{{\rm d}i}-Q_i)\mathbf{e}=0$, and $Q_iQ_i^{\rm D}A_{{\rm d}i}=A_{{\rm d}i}$, $i=1,\ldots,n_{\rm d}$. Hence, $M\mathbf{e}=0$, where

$$M \triangleq \begin{bmatrix} 2A+\sum_{i=1}^{n_{\rm d}} Q_i & A_{\rm d1}^{\rm T} & A_{\rm d2}^{\rm T} & \cdots & A_{{\rm d}n_{\rm d}}^{\rm T} \\ A_{\rm d1} & -Q_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{{\rm d}n_{\rm d}} & 0 & 0 & \cdots & -Q_{n_{\rm d}} \end{bmatrix}. \tag{9.9}$$

Now, note that $M=M^{\rm T}$ and $M_{(i,j)}\geq 0$, $i,j=1,\ldots,q$, $i\neq j$. Hence, by *ii*) of Theorem 3.2 in [94] M is *semistable*, that is, $\operatorname{Re}\lambda<0$, or $\lambda=0$ and λ is semisimple, where $\lambda\in\operatorname{spec}(M)$ and $\operatorname{spec}(M)$ denotes the spectrum of M. Thus, $M\leq 0$, and since $Q_iQ_i^{\rm D}A_{{\rm d}i}=A_{{\rm d}i}$, $i=1,\ldots,n_{\rm d}$, it follows from Proposition 8.2.3 of [22] that $M\leq 0$ if and only if (9.7) holds.

Alternatively, if $A\in\mathbb{R}^{q\times q}$ and $A_{{\rm d}i}\in\mathbb{R}^{q\times q}$, $i=1,\ldots,n_{\rm d}$, are given by (9.6), then let Q_i be the diagonal matrix defined by

$$Q_{i(l,l)} \triangleq \sum_{m=1,l\neq m}^{q} A_{{\rm d}i(m,l)}, \quad l=1,\ldots,q. \tag{9.10}$$

The result now follows using similar arguments as above. □

Next, we consider the case where (9.1) has the nonlinear structure of the form

$$\phi_{ij}(x_i,x_j)=a_{ij}(x_j)-a_{ji}(x_i), \tag{9.11}$$

where $a_{ij}:\mathbb{R}\to\mathbb{R}$, $i,j=1,\ldots,q$, $i\neq j$, are such that $a_{ij}(0)=0$ and $a_{ij}(\cdot)$, $i,j=1,\ldots,q$, $i\neq j$, is strictly increasing. For this result define $f_{{\rm c}i}(x_i)\triangleq -\sum_{j=1,j\neq i}^{q} a_{ji}(x_i)$, $f_{{\rm d}i}(x)\triangleq \mathbf{e}_i\sum_{j=1}^{q} a_{ij}(x_j)$, $i=1,\ldots,q$, and $f_{\rm c}(x)\triangleq[f_{\rm c1}(x_1),\ldots,f_{{\rm c}q}(x_q)]^{\rm T}$, where $\mathbf{e}_i\in\mathbb{R}^q$ denotes the elementary vector of order q with 1 in the ith component and 0's elsewhere.

Theorem 9.1. Consider the multiagent dynamical system given by (9.3) and (9.4) or, equivalently, (9.2) where $\phi_{ij}(x_i,x_j)$, $i,j=1,\ldots,q$, $i\neq j$, is given by (9.11) and $f_{{\rm c}i}(\cdot)$, $i=1,\ldots,q$, is strictly decreasing. Assume that $\mathbf{e}^{\rm T}[f_{\rm c}(x)+\sum_{i=1}^{q} f_{{\rm d}i}(x)]=0$, $x\in\mathbb{R}^q$, and $f_{\rm c}(x)+\sum_{i=1}^{q} f_{{\rm d}i}(x)=0$ if and only if $x=\alpha\mathbf{e}$ for some $\alpha\in\mathbb{R}$. Furthermore, assume there exist nonnegative diagonal matrices $P_i\in\mathbb{R}^{q\times q}$, $i=1,\ldots,q$, such that $P\triangleq\sum_{i=1}^{q} P_i$ is positive definite,

$$P_i^{\mathrm{D}} P_i f_{\mathrm{d}i}(x) = f_{\mathrm{d}i}(x),\ x \in \mathbb{R}^q,\ i = 1, \ldots, q, \tag{9.12}$$

$$\sum_{i=1}^{q} f_{\mathrm{d}i}^{\mathrm{T}}(x) P_i f_{\mathrm{d}i}(x) \le f_{\mathrm{c}}^{\mathrm{T}}(x) P f_{\mathrm{c}}(x), \quad x \in \mathbb{R}^q. \tag{9.13}$$

Then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a semistable equilibrium state of (9.2). Furthermore, $x(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}} x(0)$ as $t \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}} x(0)$ is a semistable equilibrium state.

Proof. Consider the nonnegative function given by

$$V(x) = -2 \sum_{i=1}^{q} \int_0^{x_i} P_{(i,i)} f_{\mathrm{c}i}(\theta) \mathrm{d}\theta. \tag{9.14}$$

Since $f_{\mathrm{c}i}(\cdot)$, $i = 1, \ldots, q$, is a strictly decreasing function it follows that

$$V(x) \ge 2 \sum_{i=1}^{q} P_{(i,i)} [-f_{\mathrm{c}i}(\delta_i x_i)] x_i > 0$$

for all $x_i \neq 0$, where $0 < \delta_i < 1$, and hence, there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ such that $V(x) \ge \alpha(\|x\|)$. Now, note that the derivative of $V(x)$ along the trajectories of (9.2) is given by

$$\begin{aligned}
\dot{V}(x) &= -2 f_{\mathrm{c}}^{\mathrm{T}}(x) P f_{\mathrm{c}}(x) - 2 \sum_{i=1}^{q} f_{\mathrm{c}}^{\mathrm{T}}(x) P f_{\mathrm{d}i}(x) \\
&\le -f_{\mathrm{c}}^{\mathrm{T}}(x) P f_{\mathrm{c}}(x) - 2 \sum_{i=1}^{q} f_{\mathrm{c}}^{\mathrm{T}}(x) P P_i^{\mathrm{D}} P_i f_{\mathrm{d}i}(x) \\
&\quad - \sum_{i=1}^{q} f_{\mathrm{d}i}(x) P_i P_i^{\mathrm{D}} P_i f_{\mathrm{d}i}(x) \\
&= -\sum_{i=1}^{q} [P f_{\mathrm{c}}(x) + P_i f_{\mathrm{d}i}(x)]^{\mathrm{T}} P_i^{\mathrm{D}} \\
&\quad \cdot [P f_{\mathrm{c}}(x) + P_i f_{\mathrm{d}i}(x)] \\
&\le 0, \quad x \in \mathbb{R}^q,
\end{aligned} \tag{9.15}$$

where the first inequality in (9.15) follows from (9.12) and (9.13), and the last equality in (9.15) follows from the fact that $f_{\mathrm{c}}^{\mathrm{T}}(x) P f_{\mathrm{c}}(x) = \sum_{i=1}^{q} f_{\mathrm{c}}^{\mathrm{T}}(x) P P_i^{\mathrm{D}} P f_{\mathrm{c}}(x)$, $x \in \mathbb{R}^q$.

Next, let $\mathcal{R} \triangleq \{x \in \mathbb{R}^q : Pf_{\rm c}(x) + P_i f_{{\rm d}i}(x) = 0, i = 1, \ldots, q\}$. Then it follows from the Krasovskii-LaSalle invariant set theorem that $x(t) \to \mathcal{M}$ as $t \to \infty$, where $\mathcal{M}$ denotes the largest invariant set contained in $\mathcal{R}$. Now, since $\mathbf{e}^{\rm T}(f_{\rm c}(x) + \sum_{i=1}^q f_{{\rm d}i}(x)) = 0, x \in \mathbb{R}^q$, it follows that

$$\begin{aligned}\mathcal{R} \subseteq \hat{\mathcal{R}} &\triangleq \left\{x \in \mathbb{R}^q : f_{\rm c}(x) + \sum_{i=1}^q f_{{\rm d}i}(x) = 0\right\} \\ &= \{x \in \mathbb{R}^q : x = \alpha\mathbf{e},\ \alpha \in \mathbb{R}\},\end{aligned} \tag{9.16}$$

which implies that $x(t) \to \hat{\mathcal{R}}$ as $t \to \infty$.

Finally, Lyapunov stability of $\alpha\mathbf{e}$, $\alpha \in \mathbb{R}$, follows by considering the Lyapunov function candidate

$$V(x) = -2\sum_{i=1}^q \int_\alpha^{x_i} P_{(i,i)}(f_{{\rm c}i}(\theta) - f_{{\rm c}i}(\alpha)){\rm d}\theta \tag{9.17}$$

and noting that

$$V(x) \geq 2\sum_{i=1}^q P_{(i,i)}[f_{{\rm c}i}(\alpha) - f_{{\rm c}i}(\alpha + \delta_i(x_i - \alpha))](x_i - \alpha) > 0,$$

for $x \neq \alpha\mathbf{e}$, where $0 < \delta_i < 1$ and $i = 1, \ldots, q$. Hence, it follows from Theorem 3.3 of [29] that for any $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a semistable equilibrium state of (9.2). Furthermore, note that since $\mathbf{e}^{\rm T}x(t) = \mathbf{e}^{\rm T}x(0)$, $t \geq 0$, and $x(t) \to \mathcal{M}$ as $t \to \infty$, it follows that $x(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\rm T}x(0)$ as $t \to \infty$. Hence, with $\alpha = \frac{1}{q}\mathbf{e}^{\rm T}x(0)$, $\alpha\mathbf{e} = \frac{1}{q}\mathbf{e}\mathbf{e}^{\rm T}x(0)$ is a semistable equilibrium state of (9.2).

□

Theorem 9.2. Consider the multiagent dynamical system (9.3) and (9.4) or, equivalently, (9.2), and assume that Assumptions 1 and 2 hold.

i) Assume that $\phi_{ij}(x_i, x_j) = -\phi_{ji}(x_j, x_i)$ for all $i, j = 1, \ldots, q$, $i \neq j$. Then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a semistable equilibrium state of (9.2). Furthermore, $x(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\rm T}x(0)$ as $t \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\rm T}x(0)$ is a semistable equilibrium state.

ii) Let $\phi_{ij}(x_i, x_j) = \mathcal{C}_{(i,j)}[\sigma(x_j) - \sigma(x_i)]$ for all $i, j = 1, \ldots, q$, $i \neq j$, where $\sigma(0) = 0$ and $\sigma(\cdot)$ is strictly increasing. Assume that $\mathcal{C}^{\mathrm{T}}\mathbf{e} = 0$. Then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a semistable equilibrium state of (9.2). Furthermore, $x(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0)$ as $t \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0)$ is a semistable equilibrium state.

Proof. *i*) The proof is identical to the proof of Theorem 8.8 and, hence, is omitted.

ii) It follows from Lemma 9.1 that there exists Q_i, $i = 1, \ldots, q$, such that (9.7) holds with Q_i given by (9.8), and A and $A_{\mathrm{d}i}$, $i = 1, \ldots, q$, are given by (9.5) with a_{ij} replaced by $\mathcal{C}_{(i,j)}$. Next, consider the nonnegative function given by $V(x) = 2\sum_{i=1}^{q}\int_0^{x_i}\sigma(\theta)\mathrm{d}\theta$. Since $\sigma(\cdot)$ is a strictly increasing function it follows that $V(x) \geq 2\sum_{i=1}^{q}\sigma(\delta_i x_i)x_i > 0$ for all $x \neq 0$, where $0 < \delta_i < 1$, and hence, there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ such that $V(x) \geq \alpha(\|x\|)$. Now, the rest of the proof is similar to the proof of Theorem 9.1. $\square$

Remark 9.1. Note that the assumption $\phi_{ij}(x_i, x_j) = -\phi_{ji}(x_j, x_i)$, $i, j = 1, \ldots, q$, $i \neq j$, in *i*) of Theorem 9.2 implies that $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$, and hence, the underlying graph for the multiagent system $\mathcal{G}$ given by (9.3) and (9.4) is undirected. Furthermore, since $\phi_{ij}(x_i, x_j)$ is not restricted to a specified structure, the consensus protocol algorithm is not restricted to a particular reference. Alternatively, in *ii*) of Theorem 9.2 the assumption $\mathcal{C}^{\mathrm{T}}\mathbf{e} = 0$ implies that the underlying directed graph of $\mathcal{G}$ is balanced. To see this, recall that for a directed graph $\mathfrak{G}$, $\mathcal{A}\mathbf{e} = \mathcal{A}^{\mathrm{T}}\mathbf{e}$ implies that $\mathfrak{G}$ is balanced. Since $\mathcal{C} = \mathcal{A} - \Delta$, where $\mathcal{A}$ denotes the normalized adjacency matrix and $\Delta \triangleq \operatorname{diag}\left[\sum_{j=1}^{q}\alpha_{1j}, \ldots, \sum_{j=1}^{q}\alpha_{qj}\right] \in \mathbb{R}^{q\times q}$, it follows that $\mathcal{A}\mathbf{e} = \mathcal{A}^{\mathrm{T}}\mathbf{e}$ if and only if $\mathcal{C}\mathbf{e} = \mathcal{C}^{\mathrm{T}}\mathbf{e}$. Hence, $\mathcal{C}^{\mathrm{T}}\mathbf{e} = 0$ implies that $\mathfrak{G}$ is balanced.

Theorem 9.2 implies that the steady-state value of the state of each agent $\mathcal{G}_i$ of the multiagent dynamical system $\mathcal{G}$ is equal; that is, the steady-state information of the multiagent dynamical system $\mathcal{G}$ given by $x_\infty = \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0) = \left[\frac{1}{q}\sum_{i=1}^{q}x_i(0)\right]\mathbf{e}$ is uniformly distributed over all multiagents of $\mathcal{G}$. This phenomenon is known as *equipartition of energy* [104] in system thermodynamics and *information consensus* or *protocol agreement* [187] in cooperative network systems.

Finally, we specialize Theorem 9.1 to the case where

$$\phi_{ij}(x_i, x_j) = a_{ij}\sigma(x_j) - a_{ji}\sigma(x_i), \tag{9.18}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is such that $\sigma(u) = 0$ if and only if $u = 0$, $a_{ij} \geq 0$, $i, j = 1, \ldots, q$, $i \neq j$. In this case, (9.2) can be rewritten as

$$\dot{x}(t) = A\hat{\sigma}(x(t)) + \sum_{i=1}^{q} A_{\mathrm{d}i}\hat{\sigma}(x(t)),\ x(0) = x_0,\ t \geq 0, \tag{9.19}$$

where $\hat{\sigma} : \mathbb{R}^q \to \mathbb{R}^q$ is given by $\hat{\sigma}(x) \triangleq [\sigma(x_1), \ldots, \sigma(x_q)]^{\mathrm{T}}$, and A and $A_{\mathrm{d}i}$, $i = 1, \ldots, q$, are given by (9.6).

Theorem 9.3. Consider the multiagent dynamical system given by (9.19) where $\sigma : \mathbb{R} \to \mathbb{R}$ is such that $\sigma(0) = 0$ and $\sigma(\cdot)$ is strictly increasing. Assume that $(A + \sum_{i=1}^{q} A_{\mathrm{d}i})^{\mathrm{T}}\mathbf{e} = (A + \sum_{i=1}^{q} A_{\mathrm{d}i})\mathbf{e} = 0$ and $\operatorname{rank}(A + \sum_{i=1}^{q} A_{\mathrm{d}i}) = q - 1$. Then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a semistable equilibrium point of (9.2). Furthermore, $x(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0)$ as $t \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0)$ is a semistable equilibrium state.

Proof. It follows from Lemma 9.1 that there exists Q_i, $i = 1, \ldots, q$, such that (9.7) holds with Q_i given by (9.10). Now, since $A = -\sum_{i=1}^{q} Q_i = -\sum_{i=1}^{q} P_i^{\mathrm{D}} = -P^{-1}$, where $P = \sum_{i=1}^{q} P_i$, it follows from (9.7) that, for all $x \in \mathbb{R}^q$,

$$\begin{aligned} 0 \geq\ & 2\hat{\sigma}^{\mathrm{T}}(x)A\hat{\sigma}(x) + \hat{\sigma}^{\mathrm{T}}(x)\sum_{i=1}^{q}(Q_i + A_{\mathrm{d}i}^{\mathrm{T}}Q_i^{\mathrm{D}}A_{\mathrm{d}i})\hat{\sigma}(x) \\ =\ & -f_{\mathrm{c}}^{\mathrm{T}}(x)Pf_{\mathrm{c}}(x) + \sum_{i=1}^{q} f_{\mathrm{d}i}^{\mathrm{T}}(x)P_i f_{\mathrm{d}i}(x), \end{aligned}$$

where $f_{\mathrm{c}}(x) = A\hat{\sigma}(x)$ and $f_{\mathrm{d}i}(x) = A_{\mathrm{d}i}\hat{\sigma}(x)$, $i = 1, \ldots, q$, $x \in \mathbb{R}^q$. Furthermore, since $P_i^{\mathrm{D}}P_iA_{\mathrm{d}i} = A_{\mathrm{d}i}$, $i = 1, \ldots, q$, it follows that $P_i^{\mathrm{D}}P_if_{\mathrm{d}i}(x) = f_{\mathrm{d}i}(x)$, $i = 1, \ldots, q$, $x \in \mathbb{R}^q$. Now, the result is an immediate consequence of Theorem 9.1 by noting that $\mathbf{e}^{\mathrm{T}}[f_{\mathrm{c}}(x) + \sum_{i=1}^{q} f_{\mathrm{d}i}(x)] = 0$ and $f_{\mathrm{c}}(x) + \sum_{i=1}^{q} f_{\mathrm{d}i}(x) = 0$ if and only if $x = \alpha\mathbf{e}$ for some $\alpha \in \mathbb{R}$. $\square$

Theorems 9.1 and 9.3 can be extended to address linear and nonlinear dynamical networks with multiple time-delays. For details, see [54] and Chapter 11. The results of this section provide a generalization to Theorems 4 and 5 of [187] which establish information consensus protocols for the special structure $\phi_{ij}(x_i, x_j) = a_{ij}(x_i - x_j)$, $i, j = 1, \ldots, q$, $i \neq j$. In particular, the nonlinear function $\sigma(\cdot)$ within $\hat{\sigma}(\cdot)$ may be used to enhance the performance of the dynamic consensus algorithm or satisfy other constraints. For example, choosing $\sigma(x_i) = \tanh(x_i)$ we can constrain bandwidth information from one agent to another.

9.4. Network Consensus with Switching Topology

Communication links among multiagent systems are often unreliable due to multipath effects and exogenous disturbances leading to dynamic information exchange topologies. In this section, we develop a switched consensus protocol to achieve agreement over a network with switching topology. A Complete theory of network consensus with switching topology is addressed in Chapter 12. In contrast to the static controllers addressed in [135], [187], and [205], the proposed controller is a dynamic compensator. This controller architecture allows us to design hybrid consensus protocols involving time and state-dependent communication links. In this case, the closed-loop system involves a nonsmooth dynamical system [55, 76].

We begin by considering the differential equation given by

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{9.20}$$

where $f : \mathbb{R}^q \to \mathbb{R}^q$ is measurable and locally essentially bounded [76]. The *Filippov solution* of (9.20) is defined by an absolutely continuous function $x : [0, \tau] \to \mathbb{R}^q$ such that

$$\dot{x}(t) \in \mathcal{K}[f](x(t)) \tag{9.21}$$

for almost all $t \in [0, \tau]$, where

$$\mathcal{K}[f](x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(\mathcal{S})=0} \overline{\mathrm{co}}\{f(\mathcal{B}_\delta(x) \backslash \mathcal{S})\}, \tag{9.22}$$

and where $\mu(\cdot)$ denotes the usual Lebesgue measure in $\mathbb{R}^q$, $\mathcal{B}_\delta(x)$, $x \in \mathbb{R}^q$, denotes the open ball centered at x with radius $\delta > 0$, and "$\overline{\mathrm{co}}$" denotes the convex closure. Since the set-valued map given by (9.22) is upper semicontinuous with nonempty, convex, and compact values, and is also locally bounded, it follows that Filippov solutions to (9.20) exist [76].

In order to state the main result of this section, we need some new notation and definitions. We say that a set $\mathcal{M}$ is *weakly invariant* (resp., *strongly invariant*) with respect to (9.20) if for every $x_0 \in \mathcal{M}$, $\mathcal{M}$ contains a maximal solution (resp., all maximal solutions) of (9.20). We use $\mathcal{L}_f V(x)$ to denote the *set-valued derivative* of V with respect to (9.20) [12,60]. In this section, we assume that $f(\cdot)$ is locally Lipschitz continuous and regular in the sense of [55]. The following definition is an extension of Definition 8.1 to Filippov dynamical systems. The definition of Lyapunov stability for the solution $x(t) \equiv z$ to (9.20) can be found in [76] and [12].

Definition 9.1. Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to the differential inclusion (9.20). An equilibrium point $z \in \mathcal{D}$ of (9.20) is *semistable* with respect to $\mathcal{D}$ if it is Lyapunov stable and there exists an open subset $\mathcal{D}_0$ of $\mathcal{D}$ containing z such that for all initial conditions in $\mathcal{D}_0$, the Filippov solutions of (9.20) converge to a Lyapunov stable equilibrium point.

Theorem 9.4. Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to (9.20) and let $V : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz continuous and regular. Assume that for each $x \in \mathcal{D}$ and each Filippov solution $\gamma(\cdot)$, $\gamma(t)$ is bounded for all $t \geq 0$ and $\gamma(0) = x$. Furthermore, assume that $\max \mathcal{L}_f V(x) \leq 0$ or $\mathcal{L}_f V(x) = \emptyset$ for all $x \in \mathcal{D}$. Let $\mathcal{Z} \triangleq \{x \in \mathbb{R}^q : 0 \in \mathcal{L}_f V(x)\}$. If every point in the largest weakly invariant subset $\mathcal{M}$ of $\overline{\mathcal{Z}} \cap \mathcal{D}$ is a Lyapunov stable equilibrium point with respect to $\mathcal{D}$, where $\overline{\mathcal{Z}}$ denotes the closure of $\mathcal{Z}$, then (9.20) is semistable with respect to $\mathcal{D}$.

Proof. The proof of this result follows as in the proofs of Theorem 3 of [12] and Theorem

3.3 of [29] and, hence, is omitted. $\square$

Next, we design a switching dynamic consensus protocol for (9.3) with $x_i \in \mathbb{R}^n$. Specifically, consider q mobile agents with the dynamics $\mathcal{G}_i$ given by (9.3). Furthermore, consider the switched dynamic compensators $\mathcal{G}_{ci}$ given by

$$\dot{x}_{ci}(t) = -\sum_{j=1,j\neq i}^{q} \mathcal{C}_{(i,j)}(t, x(t))(x_{ci}(t) - x_{cj}(t)) + \sum_{j=1,j\neq i}^{q} \mathcal{C}_{(i,j)}(t, x(t))(x_i(t) - x_j(t)),$$
$$x_{ci}(0) = x_{ci0}, \quad t \geq 0, \tag{9.23}$$

$$u_i(t) = \sum_{j=1,j\neq i}^{q} \mathcal{C}_{(j,i)}(t, x(t))(x_{cj}(t) - x_{ci}(t)), \tag{9.24}$$

where $x_{ci}(t) \in \mathbb{R}^n$, $t \geq 0$, $x \triangleq [x_1^{\mathrm{T}}, \ldots, x_q^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{nq}$, and $\mathcal{C}_{(i,j)} : [0, \infty) \times \mathbb{R}^{nq} \to \{0, 1\}$, $i, j = 1, \ldots, q$, $i \neq j$, is a piecewise constant switching signal. The motivation of the particular structure of the dynamic controller given by (9.23) and (9.24) comes from designing consensus protocols via output feedback [124] and designing distributed feedback controllers to achieve parallel and circular formations [123].

Theorem 9.5. Consider the closed-loop system $\tilde{\mathcal{G}}$ given by the multiagent dynamical system (9.3) and the switched dynamic controller (9.23) and (9.24). Assume that Assumption 1 holds and $\mathcal{C}(t, x) = \mathcal{C}^{\mathrm{T}}(t, x)$ for all $t \geq 0$ and $x \in \mathbb{R}^{nq}$. Then for every $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^n$, $x_1 = \cdots = x_q = \alpha$ and $x_{c1} = \cdots = x_{cq} = \beta$ is a semistable state of $\tilde{\mathcal{G}}$. Furthermore, $x_i(t) \to \frac{1}{q}\sum_{i=1}^{q} x_{i0}$ and $x_{ci}(t) \to \frac{1}{q}\sum_{i=1}^{q} x_{ci0}$ as $t \to \infty$ and $(\frac{1}{q}\sum_{i=1}^{q} x_{i0}, \frac{1}{q}\sum_{i=1}^{q} x_{ci0})$ is a semistable equilibrium state.

Proof. To show that the closed-loop system $\tilde{\mathcal{G}}$ is Lyapunov stable with $x_i(t) \equiv \alpha$ and $x_{ci}(t) \equiv \beta$, consider the Lyapunov function candidate

$$V(\tilde{x} - \tilde{x}_{\mathrm{e}}) = \sum_{i=1}^{q} \|x_i - \alpha\|_2^2 + \sum_{i=1}^{q} \|x_{ci} - \beta\|_2^2, \tag{9.25}$$

where $\tilde{x} \triangleq [x_1^{\mathrm{T}}, \ldots, x_q^{\mathrm{T}}, x_{c1}^{\mathrm{T}}, \ldots, x_{cq}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2nq}$ and $\tilde{x}_{\mathrm{e}} \triangleq [\alpha^{\mathrm{T}}, \ldots, \alpha^{\mathrm{T}}, \beta^{\mathrm{T}}, \ldots, \beta^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2nq}$. Note that the closed-loop system $\tilde{\mathcal{G}}$ can be rewritten as

$$\dot{\tilde{x}} = F\tilde{x}, \tag{9.26}$$

where $F \triangleq \begin{bmatrix} 0 & \mathcal{C}\otimes I_n \\ -\mathcal{C}\otimes I_n & \mathcal{C}\otimes I_n \end{bmatrix}$, $\mathcal{C} \in \{\mathcal{C}_1,\ldots,\mathcal{C}_m\}$, and $\{\mathcal{C}_1,\ldots,\mathcal{C}_m\}$ is a finite set that contains all the possible communication topologies of the connectivity matrix $\mathcal{C}$ satisfying Assumption 1. Next, the Lie derivative of $V(\tilde{x}-\tilde{x}_{\rm e})$ along the vector field of the switched closed-loop dynamics is given by

$$\begin{aligned}
\mathcal{L}_{F\tilde{x}}V(\tilde{x}-\tilde{x}_{\rm e}) &= 2\sum_{i=1}^{q}(x_i-\alpha)^{\rm T}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(j,i)}(x_{{\rm c}j}-x_{{\rm c}i}) \\
&\quad +2\sum_{i=1}^{q}(x_{{\rm c}i}-\beta)^{\rm T}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}(x_i-x_j) \\
&\quad -2\sum_{i=1}^{q}(x_{{\rm c}i}-\beta)^{\rm T}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}(x_{{\rm c}i}-x_{{\rm c}j}) \\
&= -2\sum_{i=1}^{q}x_{{\rm c}i}^{\rm T}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}(x_{{\rm c}i}-x_{{\rm c}j}) \\
&= 2x_{\rm c}^{\rm T}\left[A\otimes I_n+\sum_{i=1}^{q}A_{{\rm d}i}\otimes I_n\right]x_{\rm c} \\
&\leq -\sum_{i=1}^{q}\Big[x_{\rm c}^{\rm T}(Q_i\otimes I_n)x_{\rm c}-2x_{\rm c}^{\rm T}(A_{{\rm d}i}\otimes I_n)x_{\rm c} \\
&\quad +x_{\rm c}^{\rm T}(A_{{\rm d}i}\otimes I_n)^{\rm T}(Q_i\otimes I_n)^{\rm D}(A_{{\rm d}i}\otimes I_n)x_{\rm c}\Big] \\
&= -\sum_{i=1}^{q}[-(Q_i\otimes I_n)x_{\rm c}+(A_{{\rm d}i}\otimes I_n)x_{\rm c}]^{\rm T}(Q_i\otimes I_n)^{\rm D} \\
&\quad \cdot[-(Q_i\otimes I_n)x_{\rm c}+(A_{{\rm d}i}\otimes I_n)x_{\rm c}] \\
&\leq 0, \quad \tilde{x}\in\mathbb{R}^{2nq},
\end{aligned} \tag{9.27}$$

where $x_{\rm c} \triangleq [x_{{\rm c}1}^{\rm T},\ldots,x_{{\rm c}q}^{\rm T}]^{\rm T}$, A and $A_{{\rm d}i}$, $i=1,\ldots,q$, are given by (9.5) with a_{ij} replaced by $\mathcal{C}_{(i,j)}$, Q_i, $i=1,\ldots,q$, is given by (9.8) with a_{ij} replaced by $\mathcal{C}_{(i,j)}$, and "$\otimes$" denotes Kronecker product. Now, it follows from Theorem 1 of [12] that the closed-loop system $\tilde{\mathcal{G}}$ is Lyapunov stable.

Next, we rewrite the closed-loop system $\tilde{\mathcal{G}}$ as the differential inclusion $\dot{\tilde{x}}(t)\in\mathcal{K}[\tilde{f}](\tilde{x}(t))$ a.e., where a.e. denotes almost everywhere and $\tilde{f}$ denotes the closed-loop dynamics of $\tilde{\mathcal{G}}$. Note that $\mathcal{K}[\tilde{f}](\tilde{x})=\mathcal{K}[F\tilde{x}]$. Let $v_{\tilde{x}}$ be an arbitrary element of $\mathcal{K}[\tilde{f}]$ and recall that the *Clarke upper generalized derivative* [55] of $V(\tilde{x})$ along a vector $v\in\mathcal{K}[\tilde{f}]$ is defined by $V^{o}(\tilde{x},v)\triangleq\tilde{x}^{\rm T}v_{\tilde{x}}$.

Note that for $i, j = 1, \ldots, q$, $i \neq j$, the set $\tilde{\mathcal{D}}_{\mathrm{c}} \triangleq \{\tilde{x} \in \mathbb{R}^{2nq} : V(\tilde{x}) \leq c\}$, where $c > 0$, is a compact set. Next, consider $\max V^{o}(\tilde{x}, v) \triangleq \max_{v_{\tilde{x}} \in \mathcal{K}[\tilde{f}]} \left\{\tilde{x}^{\mathrm{T}} v_{\tilde{x}}\right\}$. It follows from Theorem 1 of [193] and (9.27) that

$$\begin{aligned}
&\tilde{x}^{\mathrm{T}} \mathcal{K}\left[\left[\begin{array}{cc} 0 & \mathcal{C} \otimes I_n \\ -\mathcal{C} \otimes I_n & \mathcal{C} \otimes I_n \end{array}\right] \tilde{x}\right] \\
&= \mathcal{K}\left[\tilde{x}^{\mathrm{T}}\left[\begin{array}{cc} 0 & \mathcal{C} \otimes I_n \\ -\mathcal{C} \otimes I_n & \mathcal{C} \otimes I_n \end{array}\right] \tilde{x}\right] \\
&= \mathcal{K}\left[2 x_{\mathrm{c}}^{\mathrm{T}}\left(A \otimes I_n + \sum_{i=1}^{q} A_{\mathrm{d}i} \otimes I_n\right) x_{\mathrm{c}}\right],
\end{aligned}$$

and hence, by definition of a differential inclusion, it follows that

$$\max V^{o}(\tilde{x}, v) = \max \overline{\mathrm{co}}\left\{2 x_{\mathrm{c}}^{\mathrm{T}}(A \otimes I_n + \sum_{i=1}^{q} A_{\mathrm{d}i} \otimes I_n) x_{\mathrm{c}}\right\}.$$

Note that since, by (9.27), $2x_{\mathrm{c}}^{\mathrm{T}}(A \otimes I_n + \sum_{i=1}^{q} A_{\mathrm{d}i} \otimes I_n) x_{\mathrm{c}} \leq 0$, $x_{\mathrm{c}} \in \mathbb{R}^{nq}$, it follows that $\max V^{o}(\tilde{x}, v)$ cannot be positive, and hence, the largest value $\max V^{o}(\tilde{x}, v)$ can achieve is zero.

Let $\mathcal{R} = \{\tilde{x} \in \tilde{\mathcal{D}}_{\mathrm{c}} : -(Q_i \otimes I_n) x_{\mathrm{c}} + (A_{\mathrm{d}i} \otimes I_n) x_{\mathrm{c}} = 0, i = 1, \ldots, q\}$ and let $\mathcal{M}$ denote the largest weakly invariant set contained in $\mathcal{R}$, where Q_i, $i = 1, \ldots, q$, is given by (9.8) with a_{ij} replaced by $\mathcal{C}_{(i,j)}$. Now, since $A \otimes I_n + \sum_{i=1}^{q}(Q_i \otimes I_n) = 0$, it follows that $\mathcal{R} \subseteq \hat{\mathcal{R}} \triangleq \{\tilde{x} \in \tilde{\mathcal{D}}_{\mathrm{c}} : (A \otimes I_n) x_{\mathrm{c}} + \sum_{i=1}^{q}(A_{\mathrm{d}i} \otimes I_n) x_{\mathrm{c}} = 0\}$. Hence, since $\mathrm{rank}(A + \sum_{i=1}^{q} A_{\mathrm{d}i}) = q - 1$ and $(A + \sum_{i=1}^{q} A_{\mathrm{d}i})\mathbf{e} = 0$, it follows that on the largest weakly invariant set $\hat{\mathcal{M}}$ contained in $\hat{\mathcal{R}}$, $x_{\mathrm{c}1} = \cdots = x_{\mathrm{c}q}$, and hence, $x_i = \alpha_i$ for all $i = 1, \ldots, q$, where $\alpha_i \in \mathbb{R}^n$. Furthermore, since $\sum_{i=1}^{q} \dot{x}_{\mathrm{c}i} = 0$, it follows from Proposition 8.6 that $x_{\mathrm{c}i} = \beta$ for all $i = 1, \ldots, q$, where $\beta \in \mathbb{R}^n$, and hence, $\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}(x_i - x_j) = 0$, which, using Proposition 8.6, implies that $\alpha_i = \alpha$ for all $i = 1, \ldots, q$, where $\alpha \in \mathbb{R}^n$. Since $\hat{\mathcal{M}} \subseteq \mathcal{R} \subseteq \hat{\mathcal{R}}$, it follows that $\mathcal{M} = \hat{\mathcal{M}}$. Now, it follows from Theorem 3 of [12] that for any initial condition $\tilde{x}_0 \in \tilde{\mathcal{D}}_{\mathrm{c}}$, the Filippov solutions $\tilde{x}(t)$ of the closed-loop system $\tilde{\mathcal{G}}$ converge to the largest weakly invariant set $\mathcal{M}$ contained in the set $\{\tilde{x} \in \tilde{\mathcal{D}}_{\mathrm{c}} : x_1 = \cdots = x_q = \alpha, x_{\mathrm{c}1} = \cdots = x_{\mathrm{c}q} = \beta\}$. Since $c > 0$ is arbitrary, it follows from Theorem 9.4 that for any $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^n$, $x_1 = \cdots = x_q = \alpha$ and $x_{\mathrm{c}1} = \cdots = x_{\mathrm{c}q} = \beta$ is a semistable state of $\tilde{\mathcal{G}}$. Finally, note that since $\sum_{i=1}^{q} x_i(t) =$

$\sum_{i=1}^{q} x_i(0)$, $\sum_{i=1}^{q} x_{ci}(t) = \sum_{i=1}^{q} x_{ci}(0)$, $t \geq 0$, and $\tilde{x}(t) \to \mathcal{M}$ as $t \to \infty$, it follows that $x_i(t) \to \frac{1}{q}\sum_{j=1}^{q} x_j(0)$ and $x_{ci}(t) \to \frac{1}{q}\sum_{j=1}^{q} x_{cj}(0)$ as $t \to \infty$, $i = 1, \ldots, q$. Hence, with $\alpha = \frac{1}{q}\sum_{j=1}^{q} x_j(0)$ and $\beta = \frac{1}{q}\sum_{j=1}^{q} x_{cj}(0)$, $x_i = \alpha$ and $x_{ci} = \beta$, $i = 1, \ldots, q$, is a semistable equilibrium state of $\tilde{\mathcal{G}}$. □

It is straightforward to extend Theorem 9.5 to the case of a switching static nonlinear consensus protocol given by

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(j,i)}(t, x)\sigma(x_j(t) - x_i(t)), \tag{9.28}$$

where $x_i(t) \in \mathbb{R}$, $t \geq 0$, $i = 1, \ldots, q$, $x \triangleq [x_1, \ldots, x_q]^{\mathrm{T}} \in \mathbb{R}^q$, and $\sigma : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz such that $\sigma(\cdot)$ is strictly increasing and $\sigma(0) = 0$. In addition, Theorem 9.5 can be extended to the nonlinear form of the switched dynamic controller (9.23) and (9.24) given by

$$\dot{x}_{ci}(t) = -\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}(x)\sigma(x_{ci}(t) - x_{cj}(t)) + \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}(x)\sigma(x_i(t) - x_j(t)),$$
$$x_{ci}(0) = x_{ci0}, \quad t \geq 0, \tag{9.29}$$

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(j,i)}(x)\sigma(x_{cj}(t) - x_{ci}(t)), \tag{9.30}$$

where $x_i(t), x_{ci}(t) \in \mathbb{R}$, $t \geq 0$, $i = 1, \ldots, q$. However, this extension requires additional machinery involving nontangency-based Lyapunov tests for semistability [124] since the proof of Theorem 9.5 fails in the case where $\sigma(x) \neq x$, $x \in \mathbb{R}$. These extensions are discussed in Chapter 12.

Chapter 10

Robust Control Algorithms for Nonlinear Network Consensus Protocols

10.1. Introduction

Due to advances in embedded computational resources over the last several years, a considerable research effort has been devoted to the control of networks and control over networks. Network systems involve distributed decision-making for coordination of networks of dynamic agents involving information flow enabling enhanced operational effectiveness via cooperative control in autonomous systems. These dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles (UAV's) and autonomous underwater vehicles (AUV's) for combat, surveillance, and reconnaissance; distributed reconfigurable sensor networks for managing power levels of wireless networks; air and ground transportation systems for air traffic control and payload transport and traffic management; swarms of air and space vehicle formations for command and control between heterogeneous air and space vehicles; and congestion control in communication networks for routing the flow of information through a network.

To enable the applications for these multiagent systems, cooperative control tasks such as formation control, rendezvous, flocking, cyclic pursuit, cohesion, separation, alignment, and consensus need to be developed [124, 126, 135, 158, 166, 185, 187, 225]. To realize these tasks, individual agents need to share information of the system objectives as well as the dynamical network. In particular, in many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. Information consensus over dynamic information-exchange topologies guarantees agreement between agents for a given coordination task. Distributed consensus algorithms involve neighbor-to-neighbor interaction between agents wherein agents update their information state based on the information states of the neighboring agents. A unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in a dynamical network is the existence of a continuum of equilibria representing a state of consensus. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial state as well. As noted in Chapter 8, in systems possessing a continuum of equilibria, *semistability*, and not asymptotic stability is the relevant notion of stability [31, 32].

It is important to note that semistability is not merely equivalent to asymptotic stability of the set of equilibria. Indeed, it is possible for a trajectory to converge to the set of equilibria without converging to any one equilibrium point as examples in [32] show. Conversely, semistability does not imply that the equilibrium set is asymptotically stable in any accepted sense. This is because stability of sets is defined in terms of distance (especially in case of noncompact sets), and it is possible to construct examples in which the dynamical system is semistable, but the domain of semistability contains no ε-neighborhood (defined in terms of the distance) of the (noncompact) equilibrium set, thus ruling out asymptotic stability of the equilibrium set. Hence, semistability and set stability of the equilibrium set are independent notions. Thus, even though the coordination protocols of [8, 157] are guaranteed to converge, the limit points are not guaranteed to be Lyapunov stable.

Even though many consensus protocol algorithms have been developed over the last several years in the literature (see [124, 126, 135, 158, 166, 178, 185, 187, 225] and the numerous references therein), and some robustness issues have been considered [8, 34, 35, 69, 89, 177, 187], robustness properties of these algorithms involving nonlinear dynamics have been largely ignored. Robustness here refers to sensitivity of the control algorithm achieving semistability and consensus in the face of model uncertainty. In this chapter, we build on the results of [124, 126] to examine the robustness of several control algorithms for network consensus protocols with information model uncertainty of a specified structure. In particular, we develop sufficient conditions for robust stability of control protocol functions involving higher-order perturbation terms that scale in a consistent fashion with respect to a scaling operation on an underlying space with the additional property that the protocol functions can be written as a sum of functions, each homogeneous with respect to a fixed scaling operation, that retain system semistability and consensus. In addition, control protocol functions containing higher-order perturbation terms involving a thermodynamic information structure are also explored. Unlike the present research, [8, 157] do not consider the effect of higher-order perturbation terms appearing in the control functions. In this sense, our work complements the work reported in [8, 157].

10.2. Mathematical Preliminaries

In this section, we consider nonlinear dynamical systems of the form

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \in \mathcal{I}_{x_0}, \tag{10.1}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \in \mathcal{I}_{x_0}$, is the system state vector, $\mathcal{D}$ is an open set, $f : \mathcal{D} \to \mathbb{R}^n$ is continuous on $\mathcal{D}$, $f^{-1}(0) \triangleq \{x \in \mathcal{D} : f(x) = 0\}$ is nonempty, and $\mathcal{I}_{x_0} = [0, \tau_{x_0})$, $0 \leq \tau_{x_0} \leq \infty$, is the maximal interval of existence for the solution $x(\cdot)$ of (10.1). A continuously differentiable function $x : \mathcal{I}_{x_0} \to \mathcal{D}$ is said to be a *solution* of (10.1) on the interval $\mathcal{I}_{x_0} \subset \mathbb{R}$ if x satisfies (10.1) for all $t \in \mathcal{I}_{x_0}$. The continuity of f implies that, for every $x_0 \in \mathcal{D}$, there exist $\tau_0 < 0 < \tau_1$ and a solution $x(\cdot)$ of (10.1) defined on (τ_0, τ_1) such that $x(0) = x_0$. A solution x is said to be *right maximally defined* if x cannot be extended on the right (either uniquely or nonuniquely) to a solution of (10.1). Here, we assume that for every initial condition $x_0 \in \mathcal{D}$, (10.1) has a unique right maximally defined solution, and this unique solution is defined on $[0, \infty)$. Furthermore, we assume that $f(\cdot)$ is locally Lipschitz continuous on $\mathcal{D}\backslash f^{-1}(0)$. Note that the local Lipschitzness of $f(\cdot)$ on $\mathcal{D}\backslash f^{-1}(0)$ implies local uniqueness in forward and backward time for nonequilibrium initial states.

Under these assumptions on f, the solutions of (10.1) define a continuous *global semiflow* on $\mathcal{D}$, that is, $s : [0, \infty) \times \mathcal{D} \to \mathcal{D}$ is a jointly continuous function satisfying the *consistency* *property* $s(0,x) = x$ and the *semi-group property* $s(t, s(\tau, x)) = s(t+\tau, x)$ for every $x \in \mathcal{D}$ and $t, \tau \in [0,\infty)$. Given $t \in [0,\infty)$ we denote the *flow* $s(t,\cdot) : \mathcal{D} \to \mathcal{D}$ of (10.1) by $s_t(x_0)$ or s_t.

A set $\mathcal{M} \subset \mathbb{R}^n$ is *positively invariant* if $s_t(\mathcal{M}) \subseteq \mathcal{M}$ for all $t \geq 0$. The set $\mathcal{M}$ is *negatively invariant* if, for every $z \in \mathcal{M}$ and every $t \geq 0$, there exists $x \in \mathcal{M}$ such that $s(t,x) = z$ and $s(\tau, x) \in \mathcal{M}$ for all $\tau \in [0,t]$. Finally, the set $\mathcal{M}$ is *invariant* if $s_t(\mathcal{M}) = \mathcal{M}$ for all $t \geq 0$. Note that a set is invariant if and only if it is positively and negatively invariant.

Definition 10.1 [32]. An equilibrium point $x \in \mathcal{D}$ of (10.1) is *Lyapunov stable under* f if for every open subset $\mathcal{N}_\varepsilon$ of $\mathcal{D}$ containing x, there exists an open subset $\mathcal{N}_\delta$ of $\mathcal{D}$ containing x such that $s_t(\mathcal{N}_\delta) \subset \mathcal{N}_\varepsilon$ for all $t \geq 0$. An equilibrium point $x \in \mathcal{D}$ of (10.1) is *semistable under* f if it is Lyapunov stable under f and there exists an open subset $\mathcal{U}$ of $\mathcal{D}$ containing x such that, for every initial condition $z \in \mathcal{U}$, the trajectory of (10.1) converges to a Lyapunov stable equilibrium point, that is, $\lim_{t\to\infty} s(t,z) = y$, where $y \in \mathcal{D}$ is a Lyapunov stable equilibrium point of (10.1). If, in addition, $\mathcal{U} = \mathcal{D} = \mathbb{R}^n$, then an equilibrium point $x \in \mathcal{D}$ of (10.1) is a *globally semistable equilibrium*. The system (10.1) is said to be *semistable* if every equilibrium point of (10.1) is semistable under f. Finally, (10.1) is said to be *globally semistable* if every equilibrium point of (10.1) is globally semistable under f.

Given a continuous function $V : \mathcal{D} \to \mathbb{R}$, the *upper right Dini derivative* of V along the solution of (10.1) is defined by

$$\dot{V}(s(t,x)) \triangleq \limsup_{h\to 0^+} \frac{1}{h}[V(s(t+h,x)) - V(s(t,x))]. \tag{10.2}$$

It is easy to see that $\dot{V}(x_{\mathrm{e}}) = 0$ for every $x_{\mathrm{e}} \in f^{-1}(0)$. In addition, note that $\dot{V}(x) = \dot{V}(s(0,x))$. Finally, if $V(\cdot)$ is continuously differentiable, then $\dot{V}(x) = V'(x)f(x)$.

In the sequel, we will need to consider a complete vector field ν on $\mathbb{R}^n$, that is, a vector field ν such that the solutions of the differential equation $\dot{y}(t) = \nu(y(t))$ define a continuous *global flow* $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$, where $\nu^{-1}(0) = f^{-1}(0)$. For each $\tau \in \mathbb{R}$, the map $\psi_\tau(\cdot) = \psi(\tau, \cdot)$ is a homeomorphism and $\psi_\tau^{-1} = \psi_{-\tau}$. Our assumptions imply that every connected component of $\mathbb{R}^n \backslash f^{-1}(0)$ is invariant under ν.

Recall that a function $V : \mathbb{R}^n \to \mathbb{R}$ is *homogeneous of degree* $l \in \mathbb{R}$ *with respect to* ν if and only if

$$(V \circ \psi_\tau)(x) = e^{l\tau} V(x), \quad \tau \in \mathbb{R}, \quad x \in \mathbb{R}^n. \tag{10.3}$$

Note that if $l \neq 0$, then it follows from (10.3) that $V(x) = 0$ if $x \in \nu^{-1}(0)$.

The following proposition provides a useful comparison between positive definite homogeneous functions with respect to an equilibrium set.

Proposition 10.1. Assume $V_1(\cdot)$ and $V_2(\cdot)$ are continuous real-valued functions on $\mathbb{R}^n$, homogeneous with respect to ν of degrees $l_1 > 0$ and $l_2 > 0$, respectively, and $V_1(\cdot)$ satisfies $V_1(x) > 0$ for $x \in \mathbb{R}^n \backslash \nu^{-1}(0)$. Then for each $x_e \in \nu^{-1}(0)$ and each bounded open neighborhood $\mathcal{D}_0$ containing x_e, there exist $c_1 = c_1(\mathcal{D}_0) \in \mathbb{R}$ and $c_2 = c_2(\mathcal{D}_0) \in \mathbb{R}$, where $c_2 \geq c_1$, such that

$$c_1 (V_1(x))^{\frac{l_2}{l_1}} \leq V_2(x) \leq c_2 (V_1(x))^{\frac{l_2}{l_1}}, \quad x \in \mathcal{D}_0. \tag{10.4}$$

If, in addition, $V_2(x) < 0$ for $x \in \mathbb{R}^n \backslash \nu^{-1}(0)$, then c_1 and c_2 in (10.4) may be chosen to additionally satisfy $c_1 \leq c_2 < 0$.

Proof. Let $x_e \in \nu^{-1}(0)$ and choose a bounded open neighborhood $\mathcal{D}_0$ of x_e. Let $\mathcal{Q} = \psi(\mathbb{R}_+ \times \mathcal{D}_0)$. For every $\varepsilon > 0$, denote $\mathcal{Q}_\varepsilon = \mathcal{Q} \cap V_1^{-1}(\varepsilon)$, define the continuous map $\tau_\varepsilon : \mathbb{R}^n \backslash \nu^{-1}(0) \to \mathbb{R}$ by $\tau_\varepsilon(x) \triangleq l^{-1} \ln(\varepsilon / V_1(x))$, and note that, for every $x \in \mathbb{R}^n \backslash \nu^{-1}(0)$, $\psi(t, x) \in V_1^{-1}(\varepsilon)$ if and only if $t = \tau_\varepsilon(x)$. Next, define $\beta_\varepsilon : \mathbb{R}^n \backslash \nu^{-1}(0) \to \mathbb{R}^n$ by $\beta_\varepsilon \triangleq \psi(\tau_\varepsilon(x), x)$. Note that, for every $\varepsilon > 0$, β_ε is continuous, and $\beta_\varepsilon(x) \in V_1^{-1}(\varepsilon)$ for every $x \in \mathbb{R}^n \backslash \nu^{-1}(0)$.

Consider $\varepsilon > 0$. $\mathcal{Q}_\varepsilon$ is the union of the images of connected components of $\mathcal{D}_0 \backslash \nu^{-1}(0)$ under the continuous map β_ε. Since every connected component of $\mathbb{R}^n \backslash \nu^{-1}(0)$ is invariant under $-\nu$, it follows that the image of each connected component $\mathcal{U}$ of $\mathbb{R}^n\backslash\nu^{-1}(0)$ under β_ε is contained in $\mathcal{U}$. In particular, the images of the connected components of $\mathcal{D}_0\backslash\nu^{-1}(0)$ under β_ε are all disjoint. Thus, each connected component of $\mathcal{Q}_\varepsilon$ is the image of exactly one connected component of $\mathcal{D}_0\backslash\nu^{-1}(0)$ under β_ε. Finally, if ε is small enough so that $V_1^{-1}(\varepsilon)\cap\mathcal{D}_0$ is nonempty, then $V_1^{-1}(\varepsilon)\cap\mathcal{D}_0\subseteq\mathcal{Q}_\varepsilon$, and hence, every connected component of $\mathcal{Q}_\varepsilon$ has a nonempty intersection with $\mathcal{D}_0\backslash\nu^{-1}(0)$.

We claim that $\mathcal{Q}_\varepsilon$ is bounded for every $\varepsilon > 0$. It is easy to verify that, for every $\varepsilon_1, \varepsilon_2 \in (0,\infty)$, $\mathcal{Q}_{\varepsilon_2} = \psi_h(\mathcal{Q}_{\varepsilon_1})$ with $h = l^{-1}\ln(\varepsilon_2/\varepsilon_1)$. Hence, it suffices to prove that there exists $\varepsilon > 0$ such that $\mathcal{Q}_\varepsilon$ is bounded. To arrive at a contradiction, suppose, *ad absurdum*, $\mathcal{Q}_\varepsilon$ is unbounded for every $\varepsilon > 0$. Choose a bounded open neighborhood $\mathcal{V}$ of $\overline{\mathcal{D}}_0$ and a sequence $\{\varepsilon_i\}_{i=1}^\infty$ in $(0,\infty)$ converging to 0. By our assumption, for every $i = 1, 2, \ldots$, at least one connected component of $\mathcal{Q}_{\varepsilon_i}$ must contain a point in $\mathbb{R}^n\backslash\mathcal{V}$. On the other hand, for i sufficiently large, every connected component of $\mathcal{Q}_{\varepsilon_i}$ has a nonempty intersection with $\mathcal{D}_0 \subset \mathcal{V}$. It follows that $\mathcal{Q}_{\varepsilon_i}$ has a nonempty intersection with the boundary of $\mathcal{V}$ for every i sufficiently large. Hence, there exist a sequence $\{x_i\}_{i=1}^\infty$ in $\mathcal{D}_0$ and a sequence $\{t_i\}_{i=1}^\infty$ in $(0,\infty)$ such that $y_i \triangleq \psi_{t_i}(x_i) \in V_1^{-1}(\varepsilon_i)\cap\partial\mathcal{V}$ for every $i = 1, 2, \ldots$. Since $\mathcal{V}$ is bounded, we can assume that the sequence $\{y_i\}_{i=1}^\infty$ converges to $y \in \partial\mathcal{V}$. Continuity implies that $V_1(y) = \lim_{i\to\infty} V_1(y_i) = \lim_{i\to\infty}\varepsilon_i = 0$. Since $V_1^{-1}(0) = \nu^{-1}(0)$, it follows that y is Lyapunov stable under $-\nu$. Since $y \notin \overline{\mathcal{D}}_0$, there exists an open neighborhood $\mathcal{W}$ of y such that $\mathcal{W}\cap\mathcal{D}_0 = \emptyset$. The sequence $\{y_i\}_{i=1}^\infty$ converges to y while $\psi_{-t_i}(y_i) = x_i \in \mathcal{D}_0 \subset \mathbb{R}^n\backslash\mathcal{W}$, which contradicts Lyapunov stability. This contradiction implies that there exists $\varepsilon > 0$ such that $\mathcal{Q}_\varepsilon$ is bounded. It now follows that $\mathcal{Q}_\varepsilon$ is bounded for every $\varepsilon > 0$.

Finally, consider $x \in \mathcal{D}_0\backslash\nu^{-1}(0)$. Choose $\varepsilon > 0$ and note that $\psi_{\tau_\varepsilon(x)}(x) \in \mathcal{Q}_\varepsilon$. Furthermore, note that $V_2(x)$ is continuous on $x \in \mathbb{R}^n\backslash\nu^{-1}(0)$ and $\overline{\mathcal{Q}}_\varepsilon \cap \nu^{-1}(0) = \emptyset$. Then, by homogeneity, $V_1(\psi_{\tau_\varepsilon(x)}(x)) = \varepsilon$, and hence,

$$\min_{z\in\overline{\mathcal{Q}}_\varepsilon} V_2(z) \leq V_2(\psi_{\tau_\varepsilon(x)}(x)) \leq \max_{z\in\overline{\mathcal{Q}}_\varepsilon} V_2(z). \tag{10.5}$$

Since $V_2(\psi_{\tau_\varepsilon(x)}(x))$ is homogeneous of degree l_2, it follows that $V_2(\psi_{\tau_\varepsilon(x)}(x)) = e^{l_2\tau_\varepsilon(x)}V_2(x) = \varepsilon^{-\frac{l_2}{l_1}}(V_1(x))^{-\frac{l_2}{l_1}}V_2(x)$. Let $c_1 \triangleq \varepsilon^{\frac{l_2}{l_1}}\min_{z\in\overline{\mathcal{Q}}_\varepsilon} V_2(z)$ and $c_2 \triangleq \varepsilon^{\frac{l_2}{l_1}}\max_{z\in\overline{\mathcal{Q}}_\varepsilon} V_2(z)$. Note that c_1 and c_2 are well defined, and hence, the first assertion is proved. Finally, if $V_2(x) < 0$ for $x \in \mathbb{R}^n\backslash\nu^{-1}(0)$, then it follows from the definitions of c_1 and c_2 that $c_1 \leq c_2 < 0$. □

The *Lie derivative* of a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ with respect to ν is given by

$$L_\nu V(x) \triangleq \lim_{t\to 0^+} \frac{1}{t}[V(\psi(t,x)) - V(x)], \tag{10.6}$$

whenever the limit on the right-hand side exists. If V is a continuous homogeneous function of degree $l > 0$, then $L_\nu V$ is defined everywhere and satisfies $L_\nu V = lV$. We assume that the vector field ν is a *semi-Euler vector field*, that is, the dynamical system

$$\dot{y}(t) = -\nu(y(t)), \quad y(0) = y_0, \quad t \geq 0, \tag{10.7}$$

is globally semistable. Thus, for each $x \in \mathbb{R}^n$, $\lim_{\tau\to\infty}\psi(-\tau, x) = x^* \in \nu^{-1}(0)$, and for each $x_{\mathrm{e}} \in \nu^{-1}(0)$, there exists $z \in \mathbb{R}^n$ such that $x_{\mathrm{e}} = \lim_{\tau\to\infty}\psi(-\tau, z)$. If $\nu^{-1}(0) = \{0\}$, then the semi-Euler vector field becomes the *Euler vector field* given in [33]. Finally, we say that the vector field f is *homogeneous of degree* $k \in \mathbb{R}$ *with respect to* ν if and only if $\nu^{-1}(0) = f^{-1}(0)$ and, for every $t \in \overline{\mathbb{R}}_+$ and $\tau \in \mathbb{R}$,

$$s_t \circ \psi_\tau = \psi_\tau \circ s_{e^{k\tau}t}. \tag{10.8}$$

Note that if $V : \mathbb{R}^n \to \mathbb{R}$ is a homogeneous function of degree l such that $L_f V(x)$ is defined everywhere, then $L_f V(x)$ is a homogeneous function of degree $l + k$. Finally, note that if ν and f are continuously differentiable in a neighborhood of $x \in \mathbb{R}^n$, then (10.8) holds at x for sufficiently small t and τ if and only if $[\nu, f](x) = kf(x)$ in a neighborhood of $x \in \mathbb{R}^n$, where the Lie bracket $[\nu, f]$ of ν and f can be computed using $[\nu, f] = \frac{\partial f}{\partial x}\nu - \frac{\partial \nu}{\partial x}f$.

10.3. Semistability and Homogeneous Dynamical Systems

Homogeneity of dynamical systems is a property whereby system vector fields scale in relation to a scaling operation or *dilation* on the state space. In this section, we present a robustness result for a vector field that can be written as a sum of several vector fields, each of which is homogeneous with respect to a certain fixed dilation. First, however, we present a result that shows that a semistable homogeneous system admits a homogeneous Lyapunov function. This is a weaker version of Theorem 6.2 of [33] which considers asymptotically stable homogeneous systems.

Theorem 10.1 [125]. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree $k \in \mathbb{R}$ with respect to ν and (10.1) is semistable under f. Then for every $l > \max\{-k, 0\}$, there exists a continuous nonnegative function $V : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ that is homogeneous of degree l with respect to ν, continuously differentiable on $\mathbb{R}^n \backslash f^{-1}(0)$, $V^{-1}(0) = f^{-1}(0)$, and $V'(x)f(x) < 0$ for $x \in \mathbb{R}^n \backslash f^{-1}(0)$.

Next, we state the main theorem of this section involving a robustness result of a vector field that can be written as a sum of several vector fields.

Theorem 10.2. Let $f = g_1 + \cdots + g_p$, where, for each $i = 1, \ldots, p$, the vector field g_i is continuous, homogeneous of degree m_i with respect to ν, and $m_1 < m_2 < \cdots < m_p$. If every equilibrium point in $g_1^{-1}(0)$ is semistable under g_1 and is Lyapunov stable under f, then every equilibrium point in $g_1^{-1}(0)$ is semistable under f.

Proof. Let every point in $g_1^{-1}(0)$ be a semistable equilibrium under g_1. Choose $l > \max\{-m_1, 0\}$. Then it follows from Theorem 10.1 that there exists a continuous homogeneous function $V : \mathbb{R}^n \to \mathbb{R}$ of degree l such that $V(x) = 0$ for $x \in g_1^{-1}(0)$, $V(x) > 0$ for $x \in \mathbb{R}^n \backslash g_1^{-1}(0)$, and $L_{g_1}V$ satisfies $L_{g_1}V(x) = 0$ for $x \in g_1^{-1}(0)$ and $L_{g_1}V(x) < 0$ for $x \in \mathbb{R}^n \backslash g_1^{-1}(0)$. For each $i \in \{1, \ldots, p\}$, $L_{g_i}V$ is continuous and homogeneous of degree

$l+m_i>0$ with respect to ν. Let $x_{\rm e}\in g_1^{-1}(0)$ and $\mathcal{U}$ be a bounded neighborhood of $x_{\rm e}$. Then it follows from Proposition 10.1 and Theorem 10.1 that there exist $c_1>0$, $c_2,\ldots,c_p\in\mathbb{R}$ such that

$$L_{g_i}V(x)\leq -c_i(V(x))^{\frac{l+m_i}{l}},\quad x\in\mathcal{U},\quad i=1,\ldots,p. \tag{10.9}$$

Hence, for every $x\in\mathcal{U}$,

$$L_fV(x)\leq -\sum_{i=1}^{p}c_i(V(x))^{\frac{l+m_i}{l}}=(V(x))^{\frac{l+m_1}{l}}(-c_1+U(x)), \tag{10.10}$$

where $U(x)\triangleq -\sum_{i=2}^{p}c_i(V(x))^{\frac{m_i-m_1}{l}}$.

Since $m_i-m_1>0$ for every $i\geq 2$, it follows that the function $U(\cdot)$, which takes the value 0 on the set $g_1^{-1}(0)\cap\mathcal{U}$, is continuous. Hence, there exists an open neighborhood $\mathcal{V}\subseteq\mathcal{U}$ of $x_{\rm e}$ such that $U(x)<c_1/2$ for all $x\in\mathcal{V}$. Now, it follows from (10.10) that

$$L_fV(x)\leq -\frac{c_1}{2}(V(x))^{\frac{l+m_1}{l}},\quad x\in\mathcal{V}. \tag{10.11}$$

Since $x_{\rm e}$ is Lyapunov stable, it follows that one can find a bounded neighborhood $\mathcal{W}$ of $x_{\rm e}$ such that solutions in $\mathcal{W}$ remain in $\mathcal{V}$. Take an initial condition in $\mathcal{W}$. Since the solution is bounded (remains in $\mathcal{U}$), it follows from the Krasovskii-LaSalle invariance theorem that this solution converges to its compact positive limit set in $f^{-1}(0)$. Since all points in $f^{-1}(0)$ are Lyapunov stable, it follows from Proposition 8.1 that the positive limit set is a singleton involving a Lyapunov stable equilibrium in $f^{-1}(0)$. Since $x_{\rm e}$ was chosen arbitrarily, it follows that all equilibria in $g_1^{-1}(0)$ are semistable. $\square$

10.4. Robust Control Algorithms for Network Consensus Protocols

In this section, we apply the results of Chapter 9 (see also [126]) and the results of Section 10.3 to develop sufficient conditions for robust stability of protocol consensus for dynamical networks [164, 187, 240]. In particular, using the thermodynamically motivated information consensus framework for multiagent nonlinear systems that achieve semistability and consensus developed in [126], we develop sufficient conditions for robust stability of control protocol functions involving higher-order perturbation terms. These higher-order terms involve control functions that scale in a consistent fashion with respect to a scaling operation on an underlying space with the additional property that the control functions can be written as a sum of homogeneous functions with respect to a fixed scaling operation. In addition, we develop control protocol functions containing higher-order perturbation terms involving thermodynamic information structures.

The information consensus problem appears frequently in coordination of multiagent systems and involves finding a dynamic algorithm that enables a group of agents in a network to agree upon certain quantities of interest with directed information flow. In this research, we use undirected and directed graphs to represent a nonlinear dynamical network and present solutions to the consensus problem for nonlinear networks with both graph *topologies* (or information flows) [187]. Specifically, let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a *directed graph* (or digraph) denoting the dynamical network (or dynamic graph) with the set of *nodes* (or vertices) $\mathcal{V} = \{1, \ldots, q\}$ involving a finite nonempty set denoting the agents, the set of *edges* $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow, and an *adjacency matrix* $\mathcal{A} \in \mathbb{R}^{q \times q}$ such that $\mathcal{A}_{(i,j)} = 1$, $i, j = 1, \ldots, q$, if $(j, i) \in \mathcal{E}$, and 0 otherwise. The edge $(j, i) \in \mathcal{E}$ denotes that agent $\mathcal{G}_j$ can obtain information from agent $\mathcal{G}_i$, but not necessarily vice versa. Moreover, we assume that $\mathcal{A}_{(i,i)} = 0$ for all $i \in \mathcal{V}$. A *graph* or *undirected graph* $\mathfrak{G}$ associated with the adjacency matrix $\mathcal{A} \in \mathbb{R}^{q \times q}$ is a directed graph for which the *arc set* is symmetric, that is, $\mathcal{A} = \mathcal{A}^{\mathrm{T}}$. A graph $\mathfrak{G}$ is *balanced* if $\sum_{j=1}^{q} \mathcal{A}_{(i,j)} = \sum_{j=1}^{q} \mathcal{A}_{(j,i)}$ for all $i = 1, \ldots, q$. Finally, we denote the *value* of the node $i \in \{1, \ldots, q\}$ at time t by $x_i(t) \in \mathbb{R}$. The consensus problem involves the design of a dynamic algorithm that guarantees information state equipartition, that is, $\lim_{t \to \infty} x_i(t) = \alpha \in \mathbb{R}$ for $i = 1, \ldots, q$.

Next, consider q continuous-time integrator agents with dynamics

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \tag{10.12}$$

where for each $i \in \{1, \ldots, q\}$, $x_i(t) \in \mathbb{R}$ denotes the information state and $u_i(t) \in \mathbb{R}$ denotes the information control input for all $t \geq 0$. The consensus protocol is given by

$$u_i(t) = f_i(x(t)) = \sum_{j=1, j\neq i}^{q} \phi_{ij}(x_i(t), x_j(t)), \tag{10.13}$$

where $\phi_{ij}(\cdot, \cdot)$ satisfies the conditions in Theorem 10.3. Note that (10.12) and (10.13) describes an interconnected network where information states are updated using a distributed controller involving neighbor-to-neighbor interaction between agents. Hence, the consensus problem involves the trajectories of the dynamical network characterized by the multiagent dynamical system $\mathcal{G}$ given by

$$\dot{x}_i(t) = \sum_{j=1,\, j\neq i}^{q} \phi_{ij}(x_i(t), x_j(t)), \quad x_i(0) = x_{i0}, \quad t \geq 0, \quad i = 1, \ldots, q, \tag{10.14}$$

or, in vector form,

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{10.15}$$

where $x(t) \triangleq [x_1(t), \ldots, x_q(t)]^{\mathrm{T}}$, $t \geq 0$, and $f = [f_1, \ldots, f_q]^{\mathrm{T}} : \mathcal{D} \to \mathbb{R}^q$ is such that

$$f_i(x) = \sum_{j=1,\, j\neq i}^{q} \phi_{ij}(x_i, x_j), \tag{10.16}$$

where $\mathcal{D} \subseteq \mathbb{R}^q$ is open. Here, we assume that Assumptions 1 and 2 of Chapter 8 hold.

For the statement of the next result, let $\mathbf{e} \in \mathbb{R}^q$ denote the ones vector of order q, that is, $\mathbf{e} \triangleq [1, \ldots, 1]^{\mathrm{T}}$.

Theorem 10.3 [125]. Consider the multiagent dynamical system (10.15) and assume that Assumptions 1 and 2 of Chapter 8 hold. Then the following statements hold:

i) Assume that $\phi_{ij}(x_i, x_j) = -\phi_{ji}(x_j, x_i)$ for all $i, j = 1, \ldots, q$, $i \neq j$. Then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a semistable equilibrium state of (10.15). Furthermore, $x(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x_0$ as $t \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x_0$ is a semistable equilibrium state.

ii) Let $\phi_{ij}(x_i, x_j) = \mathcal{C}_{(i,j)}[\sigma(x_j) - \sigma(x_i)]$ for all $i, j = 1, \ldots, q$, $i \neq j$, where $\sigma(0) = 0$ and $\sigma(\cdot)$ is strictly increasing, and assume that $\mathcal{C}^{\mathrm{T}}\mathbf{e} = 0$. Then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a semistable equilibrium state of (10.15). Furthermore, $x(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x_0$ as $t \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x_0$ is a semistable equilibrium state.

Remark 10.1. Note that the assumption $\phi_{ij}(x_i, x_j) = -\phi_{ji}(x_j, x_i)$, $i, j = 1, \ldots, q$, $i \neq j$, in *i*) of Theorem 10.3 implies that $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$, and hence, the underlying graph for the multiagent system $\mathcal{G}$ given by (10.12) and (10.13) is undirected. Furthermore, since $\phi_{ij}(x_i, x_j)$ is not restricted to a specified structure, the consensus protocol algorithm is not restricted to a particular reference. Alternatively, in *ii*) of Theorem 10.3 the assumption $\mathcal{C}^{\mathrm{T}}\mathbf{e} = 0$ implies that the underlying directed graph of $\mathcal{G}$ is balanced. To see this, recall that for a directed graph $\mathfrak{G}$, $\mathcal{A}\mathbf{e} = \mathcal{A}^{\mathrm{T}}\mathbf{e}$ implies that $\mathfrak{G}$ is balanced. Since $\mathcal{C} = \mathcal{A} - \mathcal{N}$, where $\mathcal{A}$ denotes the normalized adjacency matrix and $\mathcal{N} \triangleq \operatorname{diag}\left[\sum_{j=1}^{q}\mathcal{A}_{(1,j)}, \ldots, \sum_{j=1}^{q}\mathcal{A}_{(q,j)}\right] \in \mathbb{R}^{q\times q}$, it follows that $\mathcal{A}\mathbf{e} = \mathcal{A}^{\mathrm{T}}\mathbf{e}$ if and only if $\mathcal{C}\mathbf{e} = \mathcal{C}^{\mathrm{T}}\mathbf{e}$. Hence, $\mathcal{C}^{\mathrm{T}}\mathbf{e} = 0$ implies that $\mathfrak{G}$ is balanced.

Theorem 10.3 implies that the steady-state value of the information state in each agent $\mathcal{G}_i$ of the multiagent dynamical system $\mathcal{G}$ is equal, that is, the steady-state value of the multiagent dynamical system $\mathcal{G}$ given by

$$x_\infty = \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x_0 = \left[\frac{1}{q}\sum_{i=1}^{q}x_{i0}\right]\mathbf{e} \tag{10.17}$$

is uniformly distributed over all multiagents of $\mathcal{G}$.

Next, consider (10.12) and (10.13), and assume that the vector field $f = [f_1, \ldots, f_q]$ is homogeneous of degree $k \in \mathbb{R}$ with respect to ν. Finally, consider the generalized (or perturbed) consensus protocol architecture

$$\dot{z}_i(t) = \sum_{j=1, j\neq i}^{q}\phi_{ij}(z_i(t), z_j(t)) + \Delta_i(z), \quad z_i(0) = z_{i0}, \quad i = 1, \ldots, q, \quad t \geq 0, \tag{10.18}$$

where $\Delta = [\Delta_1, \ldots, \Delta_q]^{\mathrm{T}} : \mathbb{R}^q \to \mathbb{R}$ is a continuous function such that Δ is homogeneous of degree $l \in \mathbb{R}$ with respect to ν and (10.18) possesses unique solutions in forward time for initial conditions in $\mathbb{R}^q\backslash\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$.

Theorem 10.4. Consider the nominal consensus protocol (10.12) and (10.13), and the generalized consensus protocol (10.18). If $\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\} = \Delta^{-1}(0)$, every equilibrium point in $\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$ is a Lyapunov stable equilibrium of (10.18), and $k < l$, then every equilibrium point in $\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$ is a semistable equilibrium of (10.12) and (10.13), and (10.18).

Proof. It follows from Proposition 5.1 of [124] that for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is an equilibrium point of (10.12) and (10.13). Next, it follows from Theorem 10.3 that $\alpha\mathbf{e}$ is a semistable equilibrium state of (10.12) and (10.13). Now, the result is a direct consequence of Theorem 10.2. $\square$

As a special case of Theorem 10.4, consider the nominal linear consensus protocol given by

$$\dot{x}_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[x_j(t) - x_i(t)], \quad x_i(0) = x_{i0}, \quad i = 1, \ldots, q, \quad t \geq 0, \tag{10.19}$$

where for each $i \in \{1, \ldots, q\}$, $x_i \in \mathbb{R}$, $\mathcal{C}$ satisfies Assumption 1, and $\mathcal{C}^{\mathrm{T}} = \mathcal{C}$. Next, consider the generalized consensus protocol given by

$$\dot{z}_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[z_j(t) - z_i(t)] + \sum_{j=1, j\neq i}^{q} \delta_{ij}(z_j(t) - z_i(t)),$$
$$z_i(0) = z_{i0}, \quad i = 1, \ldots, q, \quad t \geq 0, \tag{10.20}$$

and assume $\delta_{ij} : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies $\delta_{ij} \equiv 0$ if $\mathcal{C}_{(i,j)} = 0$, $\delta_{ij}(\lambda z) = \lambda^{1+r}\delta_{ij}(z)$ for all $\lambda > 0$ and for some $r \geq 0$, and $\delta_{ij}(z) = -\delta_{ji}(-z)$ for $z \in \mathbb{R}$ and $i, j = 1, \ldots, q$, $i \neq j$. Finally, let $\Delta = [\Delta_1, \ldots, \Delta_q]^{\mathrm{T}}$, where $\Delta_i = \sum_{j=1, j\neq i}^{q} \delta_{ij}(z_j - z_i)$, $i = 1, \ldots, q$.

Proposition 10.2. For $i, j = 1, \ldots, q$, $i \neq j$, let $\delta_{ij} : \mathbb{R} \to \mathbb{R}$ be continuously differentiable such that $\delta_{ij} \equiv 0$ if $\mathcal{C}_{(i,j)} = 0$ and $\delta_{ij}(\lambda z) = \lambda^{1+r}\delta_{ij}(z)$ for all $\lambda > 0$ and some $r \geq 0$, and $\delta_{ij}(z) = -\delta_{ji}(-z)$ for all $z \in \mathbb{R}$. Furthermore, let $\Delta = [\Delta_1, \ldots, \Delta_q]^{\mathrm{T}}$, where $\Delta_i = \sum_{j=1, j\neq i}^{q} \delta_{ij}(z_j - z_i)$, $i = 1, \ldots, q$. Then Δ is homogeneous of degree qr with respect to the semi-Euler vector field $\nu(x) = -\sum_{i=1}^{q} \left[\sum_{j=1, j\neq i}^{q}(x_j - x_i)\right] \frac{\partial}{\partial x_i}$.

Proof. First, note that the Lie bracket of $\nu(x) = -\sum_{i=1}^{q}\left[\sum_{j=1,j\neq i}^{q}(x_j - x_i)\right]\frac{\partial}{\partial x_i}$ and the vector field Δ is given by

$$[\nu, \Delta] = \left[\sum_{i=1}^{q}\frac{\partial \Delta_1}{\partial x_i}\nu_i - \sum_{i=1}^{q}\frac{\partial \nu_1}{\partial x_i}\Delta_i, \ldots, \sum_{i=1}^{q}\frac{\partial \Delta_q}{\partial x_i}\nu_i - \sum_{i=1}^{q}\frac{\partial \nu_q}{\partial x_i}\Delta_i\right]^{\mathrm{T}}.$$

Now, it follows from (10.8) and the assumptions on δ_{ij} that Δ_i, $i = 1, \ldots, q$, is homogeneous of degree r with respect to the standard dilation of the form $\Delta_\lambda(x_1, \ldots, x_q) = (\lambda x_1, \ldots, \lambda x_q)$ or, equivalently, the Euler vector field $\tilde{\nu}(x) = x_1\frac{\partial}{\partial x_1} + \cdots + x_q\frac{\partial}{\partial x_q}$ [33]. Hence, $[\tilde{\nu}, \Delta_i] = r\Delta_i$, $i = 1, \ldots, q$, or, equivalently,

$$\sum_{i=1}^{q}\frac{\partial \Delta_j}{\partial x_i}x_i = (r+1)\Delta_j, \quad j = 1, \ldots, q. \tag{10.21}$$

Next, note that

$$\nu_i = -\sum_{j=1,j\neq i}^{q}(x_j - x_i) = qx_i - \sum_{j=1}^{q}x_j, \quad i = 1, \ldots, q, \tag{10.22}$$

and

$$\sum_{i=1}^{q}\frac{\partial \Delta_j}{\partial x_i} = \sum_{i=1}^{q}\sum_{s=1,s\neq j}^{q}\frac{\partial \delta_{js}(x_s - x_j)}{\partial x_i} = 0, \quad j = 1, \ldots, q. \tag{10.23}$$

Hence, it follows that

$$\begin{aligned}
\sum_{i=1}^{q}\frac{\partial \Delta_j}{\partial x_i}\nu_i &= \sum_{i=1}^{q}\frac{\partial \Delta_j}{\partial x_i}\left(qx_i - \sum_{j=1}^{q}x_j\right)\\
&= q\sum_{i=1}^{q}\frac{\partial \Delta_j}{\partial x_i}x_i - \left(\sum_{i=1}^{q}\frac{\partial \Delta_j}{\partial x_i}\right)\left(\sum_{j=1}^{q}x_j\right)\\
&= q(r+1)\Delta_j, \quad j = 1, \ldots, q.
\end{aligned} \tag{10.24}$$

Alternatively, note that

$$\sum_{i=1}^{q}\Delta_i = \sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\delta_{ij}(x_j - x_i) = 0, \tag{10.25}$$

and hence,

$$\sum_{i=1}^{q}\frac{\partial \nu_j}{\partial x_i}\Delta_i = (q-1)\Delta_j - \sum_{i=1,i\neq j}^{q}\Delta_i = q\Delta_j - \sum_{i=1}^{q}\Delta_i = q\Delta_j, \quad j = 1, \ldots, q. \tag{10.26}$$

Thus,

$$\sum_{i=1}^{q} \frac{\partial \Delta_j}{\partial x_i}\nu_i - \sum_{i=1}^{q} \frac{\partial \nu_j}{\partial x_i}\Delta_i = qr\Delta_j, \quad j = 1, \ldots, q, \tag{10.27}$$

or, equivalently, $[\nu, \Delta] = qr\Delta$, which implies that the vector field Δ is homogeneous of degree qr with respect to the semi-Euler vector field $\nu(x) = -\sum_{i=1}^{q}\left[\sum_{j=1,j\neq i}^{q}(x_j - x_i)\right]\frac{\partial}{\partial x_i}$. □

Corollary 10.1. The vector field of (10.19) is homogeneous of degree $k = 0$ with respect to the semi-Euler vector field $\nu(x) = -\sum_{i=1}^{q}\left[\sum_{j=1,j\neq i}^{q}(x_j - x_i)\right]\frac{\partial}{\partial x_i}$.

Proof. The result is a direct consequence of Proposition 10.2 by setting $r = 0$. □

Corollary 10.2. Consider the linear nominal consensus protocol (10.19) and the generalized nonlinear consensus protocol (10.20). Then every equilibrium point in $\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$ is a semistable equilibrium of (10.19) and (10.20). Furthermore, $z(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}z_0$ as $t \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}z_0$ is a semistable equilibrium state.

Proof. It follows from $i)$ of Theorem 10.3 that $\alpha\mathbf{e}$, $\alpha \in \mathbb{R}$, is a semistable equilibrium of (10.19). Next, it follows from Corollary 10.1 that the right-hand side of (10.19) is homogeneous of degree $k = 0$ with respect to the semi-Euler vector field

$$\nu(x) = -\sum_{i=1}^{q}\left[\sum_{j=1,j\neq i}^{q}(x_j - x_i)\right]\frac{\partial}{\partial x_i}.$$

To show that every point in $\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$ is a Lyapunov stable equilibrium of (10.20), consider the Lyapunov function candidate given by $V(z-\alpha\mathbf{e}) = \frac{1}{2}\|z-\alpha\mathbf{e}\|^2$. Then it follows that

$$\begin{aligned}\dot{V}(z-\alpha\mathbf{e}) &= (z-\alpha\mathbf{e})^{\mathrm{T}}\dot{z}\\ &= \sum_{i=1}^{q}(z_i-\alpha)\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}[z_j - z_i] + \sum_{i=1}^{q}(z_i-\alpha)\sum_{j=1,j\neq i}^{q}\delta_{ij}(z_j - z_i)\end{aligned}$$

$$= -\sum_{i=1}^{q-1}\sum_{j=i+1}^{q} \mathcal{C}_{(i,j)}[z_i - z_j]^2 + \sum_{i=1}^{q-1}\sum_{j=i+1}^{q} (z_i - z_j)\delta_{ij}(z_j - z_i)$$
$$= -\sum_{i=1}^{q-1}\sum_{j=i+1}^{q} \mathcal{C}_{(i,j)}[z_i - z_j]^2 + \sum_{i=1}^{q-1}\sum_{j=i+1}^{q} \mathcal{C}_{(i,j)}[z_i - z_j]\delta_{ij}(z_j - z_i), \quad z \in \mathbb{R}^q. \tag{10.28}$$

Next, since, by homogeneity of δ_{ij}, $\delta_{ij}(\cdot)$ is such that $\lim_{z\to 0} \delta_{ij}(z)/z = 0$, it follows that for every $\gamma > 0$, there exists $\varepsilon_{ij} > 0$ such that $|\delta_{ij}(z)| \le \gamma |z|$ for all $|z| < \varepsilon_{ij}$. Hence,

$$\sum_{i=1}^{q-1}\sum_{j=i+1}^{q} \mathcal{C}_{(i,j)}[z_i - z_j]\delta_{ij}(z_j - z_i) \le \sum_{i=1}^{q-1}\sum_{j=i+1}^{q} \gamma\mathcal{C}_{(i,j)}[z_i - z_j]^2, \quad |z_i - z_j| < \varepsilon_{ij}. \tag{10.29}$$

Now, choosing $\gamma \le 1$, it follows from (10.28) and (10.29) that

$$\begin{aligned}\dot{V}(z - \alpha\mathbf{e}) &\le -\sum_{i=1}^{q-1}\sum_{j=i+1}^{q} (1-\gamma)\mathcal{C}_{(i,j)}[z_i - z_j]^2 \\ &\le 0, \quad |z_i - z_j| < \varepsilon_{ij},\end{aligned} \tag{10.30}$$

which establishes Lyapunov stability of the equilibrium state $\alpha\mathbf{e}$. Now, the result follows from Theorem 10.4. □

It is important to note that Corollary 10.2 still holds for the case where the generalized consensus protocol has the nonlinear form

$$\dot{z}(t) = \mathcal{C}z(t) + \sum_{i=1}^{p} g_i(z(t)), \quad z(0) = z_0, \quad t \ge 0, \tag{10.31}$$

where for each $i \in \{1, \ldots, q\}$, $g_i(z)$ is homogeneous of degree $l_i > 0$ with respect to $\nu(x) = -\sum_{i=1}^{q}[\sum_{j=1, j\ne i}^{q}(x_j - x_i)]\frac{\partial}{\partial x_i}$ and $l_1 < \cdots < l_p$.

As an application of Corollary 10.2, consider the Kuramoto model [224] given by

$$\dot{x}_1(t) = \sin(x_2(t) - x_1(t)), \quad x_1(0) = x_{10}, \quad t \ge 0, \tag{10.32}$$
$$\dot{x}_2(t) = \sin(x_1(t) - x_2(t)), \quad x_2(0) = x_{20}. \tag{10.33}$$

Note that for sufficiently small x, $\sin x$ can be approximated by $x - x^3/3! + \cdots + (-1)^{p-1}x^{2p-1}/(2p-1)!$, where p is a positive integer. The truncated system associated with (10.32) and

(10.33) is given by

$$\dot{x}_1 = x_2 - x_1 - \frac{1}{3!}(x_2 - x_1)^3 + \cdots + \frac{(-1)^{p-1}}{(2p-1)!}(x_2 - x_1)^{2p-1}, \tag{10.34}$$

$$\dot{x}_2 = x_1 - x_2 - \frac{1}{3!}(x_1 - x_2)^3 + \cdots + \frac{(-1)^{p-1}}{(2p-1)!}(x_1 - x_2)^{2p-1}, \tag{10.35}$$

or, equivalently,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \sum_{i=1}^{p-1} g_i(x_1, x_2), \tag{10.36}$$

where

$$g_i(x_1, x_2) \triangleq \frac{(-1)^i}{(2i+1)!} \begin{bmatrix} (x_2 - x_1)^{2i+1} \\ (x_1 - x_2)^{2i+1} \end{bmatrix}, \quad i = 1, \ldots, p-1. \tag{10.37}$$

It can be easily shown that all the conditions of Corollary 10.2 hold for (10.36). Hence, it follows from Corollary 10.2 that every equilibrium point in $\{\alpha[1,1]^{\mathrm{T}} : \alpha \in \mathbb{R}\}$ is a local semistable equilibrium of (10.34) and (10.35), which implies that the equilibrium set $\{\alpha[1,1]^{\mathrm{T}} : \alpha \in \mathbb{R}\}$ of (10.34) and (10.35) has the same stability properties as the linear nominal system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{10.38}$$

It should be noted that while our analysis above holds for every p, it does not imply that the exact model (10.32) and (10.33) is semistable.

Note that Corollary 10.2 deals with the undirected graph $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{A}$ is a symmetric adjacency matrix. Next, we consider the case where $\mathfrak{G}$ is a directed graph and the control protocol functions involving higher-order perturbation terms are not homogeneous. The following lemma is needed for the next result.

Lemma 10.1. Suppose $A \in \mathbb{R}^{q \times q}$ and $A_{\mathrm{d}} \in \mathbb{R}^{q \times q}$ satisfy

$$A_{(i,j)} = \begin{cases} \mathcal{C}_{(i,i)}, & i = j, \\ 0, & i \neq j, \end{cases} \quad A_{\mathrm{d}(i,j)} = \begin{cases} 0, & i = j, \\ \mathcal{C}_{(i,j)}, & i \neq j, \end{cases} \quad i, j = 1, \ldots, q, \tag{10.39}$$

Assume that $\mathcal{C}^{\mathrm{T}}\mathbf{e} = 0$. Then for every $A_{\mathrm{d}i}$, $i = 1, \ldots, n_{\mathrm{d}}$, such that $\sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i} = A_{\mathrm{d}}$, there exist nonnegative definite matrices $Q_i \in \mathbb{R}^{q\times q}$, $i = 1, \ldots, n_{\mathrm{d}}$, such that

$$2A + \sum_{i=1}^{n_{\mathrm{d}}} (Q_i + A_{\mathrm{d}i}^{\mathrm{T}} Q_i^{\mathrm{D}} A_{\mathrm{d}i}) \leq 0. \tag{10.40}$$

Proof. For each $i \in \{1, \ldots, n_{\mathrm{d}}\}$, let Q_i be the diagonal matrix defined by

$$Q_{i(l,l)} \triangleq \sum_{m=1, m\neq l}^{q} A_{\mathrm{d}i(l,m)}, \quad l = 1, \ldots, q, \tag{10.41}$$

and note that $A + \sum_{i=1}^{n_{\mathrm{d}}} Q_i = 0$, $(A_{\mathrm{d}i} - Q_i)\mathbf{e} = 0$, and $Q_i Q_i^{\mathrm{D}} A_{\mathrm{d}i} = A_{\mathrm{d}i}$, $i = 1, \ldots, n_{\mathrm{d}}$. Hence, $M\mathbf{e} = 0$, where

$$M \triangleq \begin{bmatrix} 2A + \sum_{i=1}^{n_{\mathrm{d}}} Q_i & A_{\mathrm{d}1}^{\mathrm{T}} & A_{\mathrm{d}2}^{\mathrm{T}} & \cdots & A_{\mathrm{d}n_{\mathrm{d}}}^{\mathrm{T}} \\ A_{\mathrm{d}1} & -Q_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{\mathrm{d}n_{\mathrm{d}}} & 0 & 0 & \cdots & -Q_{n_{\mathrm{d}}} \end{bmatrix}. \tag{10.42}$$

Now, note that $M = M^{\mathrm{T}}$ and $M_{(i,j)} \geq 0$, $i, j = 1, \ldots, q$, $i \neq j$. Hence, by *ii*) of Theorem 3.2 in [94] M is *semistable*, that is, $\operatorname{Re}\lambda < 0$, or $\lambda = 0$ and λ is semisimple, where $\lambda \in \operatorname{spec}(A)$. Thus, $M \leq 0$, and since $Q_i Q_i^{\mathrm{D}} A_{\mathrm{d}i} = A_{\mathrm{d}i}$, $i = 1, \ldots, n_{\mathrm{d}}$, it follows from Proposition 8.2.3 of [22] that $M \leq 0$ if and only if (10.40) holds. □

Theorem 10.5. Consider the linear nominal consensus protocol (10.19), where $\mathcal{C}$ satisfies Assumption 1 and $\mathcal{C}^{\mathrm{T}}\mathbf{e} = 0$, and the generalized nonlinear consensus protocol given by

$$\dot{z}_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[z_j(t) - z_i(t)] + \sum_{j=1, j\neq i}^{q} \mathcal{H}_{(i,j)}[\sigma(z_j(t)) - \sigma(z_i(t))],$$
$$z_i(0) = z_{i0}, \quad i = 1, \ldots, q, \quad t \geq 0, \tag{10.43}$$

where $\sigma(\cdot)$ satisfies $\sigma(0) = 0$, $\sigma : \mathbb{R} \to \mathbb{R}$ is strictly increasing, and the matrix $\mathcal{H} = [\mathcal{H}_{(i,j)}]$ satisfies Assumption 1, $\mathcal{H}^{\mathrm{T}}\mathbf{e} = 0$, $\mathcal{H}_{(i,j)} = 0$ whenever $\mathcal{C}_{(i,j)} = 0$, $i, j = 1, \ldots, q$, $i \neq j$, and $\mathcal{H} = \mathcal{C} - \mathcal{L}$, where $\mathcal{L}^{\mathrm{T}} = \mathcal{L} \in \mathbb{R}^{q\times q}$. Then every equilibrium point in $\{\alpha\mathbf{e} : \alpha \in \mathbb{R}\}$ is a semistable equilibrium of (10.19) and (10.43). Furthermore, $z(t) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}} z_0$ as $t \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}} z_0$ is a semistable equilibrium state.

Proof. It follows from *ii*) of Theorem 10.3 that $\alpha\mathbf{e}$, $\alpha \in \mathbb{R}$, is a semistable equilibrium of (10.19). Next, note that (10.43) can be rewritten as

$$\begin{aligned}\dot{z}_i(t) = &\sum_{j=1,j\neq i}^{q} \mathcal{H}_{(i,j)}[(z_j(t)+\sigma(z_j(t))) - (z_i(t)+\sigma(z_i(t)))]\\ &+\sum_{j=1,j\neq i}^{q} \mathcal{L}_{(i,j)}[z_j(t)-z_i(t)], \quad z_i(0)=z_{i0}, \quad i=1,\ldots,q, \quad t\geq 0. \quad (10.44)\end{aligned}$$

Define $\hat{\sigma}:\mathbb{R}^q \to \mathbb{R}^q$ by $\hat{\sigma}(z) \triangleq [\sigma(z_1),\ldots,\sigma(z_q)]^{\mathrm{T}}$. Now, for $C \in \mathbb{R}^{q\times q}$ and $C_{\mathrm{d}} \in \mathbb{R}^{q\times q}$ satisfying

$$C_{(i,j)} = \begin{cases} \mathcal{H}_{(i,i)}, & i=j,\\ 0, & i\neq j,\end{cases} \qquad C_{\mathrm{d}(i,j)} = \begin{cases} 0, & i=j,\\ \mathcal{H}_{(i,j)}, & i\neq j,\end{cases} \qquad i,j=1,\ldots,q, \quad (10.45)$$

it follows from Lemma 10.1 that, for every $C_{\mathrm{d}i}$, $i=1,\ldots,n_{\mathrm{d}}$, such that $\sum_{i=1}^{n_{\mathrm{d}}} C_{\mathrm{d}i} = C_{\mathrm{d}}$, there exist nonnegative definite matrices $Q_i \in \mathbb{R}^{q\times q}$, $i=1,\ldots,q$, such that

$$2C + \sum_{i=1}^{q}(Q_i + C_{\mathrm{d}i}^{\mathrm{T}} Q_i^{\mathrm{D}} C_{\mathrm{d}i}) \leq 0. \quad (10.46)$$

To show that every equilibrium point $\alpha\mathbf{e}$, $\alpha \in \mathbb{R}$, of (10.43) is Lyapunov stable, consider the Lyapunov function candidate given by

$$V(z-\alpha\mathbf{e}) = \|z-\alpha\mathbf{e}\|^2 + 2\sum_{i=1}^{q}\int_{\alpha}^{z_i}[\sigma(\theta)-\sigma(\alpha)]\mathrm{d}\theta. \quad (10.47)$$

Now, the derivative of $V(z-\alpha\mathbf{e})$ along the trajectories of (10.43) is given by

$$\begin{aligned}\dot{V}(z-\alpha\mathbf{e}) = &\ 2[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]^{\mathrm{T}}C[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]\\ &+2\sum_{i=1}^{q}[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]^{\mathrm{T}}C_{\mathrm{d}i}[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]\\ &+2\sum_{i=1}^{q}[z_i-\alpha+\sigma(z_i)-\sigma(\alpha)]\sum_{j=1,j\neq i}^{q}\mathcal{L}_{(i,j)}(z_j-z_i)\\ \leq &-\sum_{i=1}^{q}[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]^{\mathrm{T}}Q_i[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]\\ &+\sum_{i=1}^{q}2[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]^{\mathrm{T}}C_{\mathrm{d}i}[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{q}[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]^{\mathrm{T}}C_{\mathrm{d}i}^{\mathrm{T}}Q_i^{\mathrm{D}}C_{\mathrm{d}i}[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})] \\
&-2\sum_{i=1}^{q-1}\sum_{j=i+1}^{q}\mathcal{L}_{(i,j)}(z_i-z_j)[\sigma(z_i)-\sigma(z_j)]-2\sum_{i=1}^{q-1}\sum_{j=i+1}^{q}\mathcal{L}_{(i,j)}(z_i-z_j)^2 \\
=&-\sum_{i=1}^{q}\left(-Q_i[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]+C_{\mathrm{d}i}[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]\right)^{\mathrm{T}}Q_i^{\mathrm{D}} \\
&\cdot\left(-Q_i[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]+C_{\mathrm{d}i}[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]\right) \\
&-\sum_{i=1}^{q}[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})]^{\mathrm{T}}C_{\mathrm{d}i}^{\mathrm{T}}Q_i^{\mathrm{D}}C_{\mathrm{d}i}[z-\alpha\mathbf{e}+\hat{\sigma}(z)-\hat{\sigma}(\alpha\mathbf{e})] \\
&-2\sum_{i=1}^{q-1}\sum_{j=i+1}^{q}\mathcal{L}_{(i,j)}(z_i-z_j)[\sigma(z_i)-\sigma(z_j)]-2\sum_{i=1}^{q-1}\sum_{j=i+1}^{q}\mathcal{L}_{(i,j)}(z_i-z_j)^2 \\
\leq&\ 0,\quad z\in\mathbb{R}^q, \qquad (10.48)
\end{aligned}
$$

which establishes Lyapunov stability of $\alpha\mathbf{e}$.

Finally, let $\mathcal{R}\triangleq\{x\in\mathbb{R}^q:\dot{V}(x)=0\}$ and $\tilde{\mathcal{R}}\triangleq\{x\in\mathbb{R}^q:-Q_i[x+\hat{\sigma}(x)]+C_{\mathrm{d}i}[x+\hat{\sigma}(x)]=0, i=1,\ldots,q\}$, and note that $\mathcal{R}\subseteq\tilde{\mathcal{R}}$. Then it follows from the Krasovskii-LaSalle invariant set theorem that $x(t)\to\mathcal{M}$ as $t\to\infty$, where $\mathcal{M}$ denotes the largest invariant set contained in $\mathcal{R}$. Now, since $C+\sum_{i=1}^{q}Q_i=0$, it follows that $\mathcal{R}\subseteq\tilde{\mathcal{R}}\subseteq\hat{\mathcal{R}}\triangleq\{x\in\mathbb{R}^q:C\hat{\sigma}(x)+\sum_{i=1}^{q}C_{\mathrm{d}i}\hat{\sigma}(x)=0\}$. Hence, since $C+\sum_{i=1}^{q}C_{\mathrm{d}i}=\mathcal{H}$, $\operatorname{rank}\mathcal{H}=q-1$, and $\mathcal{H}\mathbf{e}=0$, it follows that the largest invariant set $\hat{\mathcal{M}}$ contained in $\hat{\mathcal{R}}$ is given by $\hat{\mathcal{M}}=\{x\in\mathbb{R}^q:x=\alpha\mathbf{e},\alpha\in\mathbb{R}\}$. Furthermore, since $\hat{\mathcal{M}}\subseteq\mathcal{R}\subseteq\hat{\mathcal{R}}$, it follows that $\mathcal{M}=\hat{\mathcal{M}}$. Hence, using similar arguments as in the proof of *iii*) $\Rightarrow$ *i*) of Proposition 8.2, it follows that every equilibrium point in $\{\alpha\mathbf{e}:\alpha\in\mathbb{R}\}$ is a semistable equilibrium of (10.19) and (10.43). $\square$

As an illustrative example for Theorem 10.5, consider the generalized consensus protocol given by

$$
\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\\ \dot{x}_5(t)\end{bmatrix}=\begin{bmatrix}x_2(t)-x_1(t)+x_3(t)-x_1(t)\\ x_3(t)-x_2(t)\\ x_4(t)-x_3(t)+x_1(t)-x_3(t)\\ x_5(t)-x_4(t)\\ x_1(t)-x_5(t)\end{bmatrix}+a\begin{bmatrix}\sigma(x_2(t))-\sigma(x_1(t))\\ \sigma(x_3(t))-\sigma(x_2(t))\\ \sigma(x_4(t))-\sigma(x_3(t))\\ \sigma(x_5(t))-\sigma(x_4(t))\\ \sigma(x_1(t))-\sigma(x_5(t))\end{bmatrix},
$$
$$
x_1(0)=x_{10},\quad x_2(0)=x_{20},\quad x_3(0)=x_{30},\quad t\geq 0, \qquad (10.49)
$$

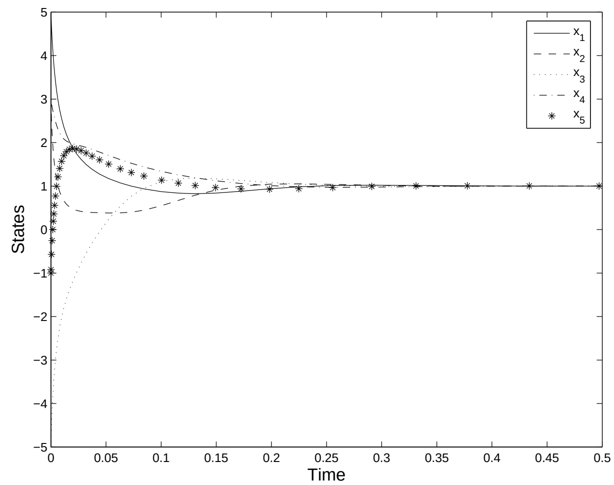

Figure 10.1: State trajectories versus time for (10.49)

where $\sigma(x) = \mathrm{sign}(x)|x|^{\alpha+1}$, $\mathrm{sign}(x) \triangleq x/|x|$ for $x \neq 0$, $\mathrm{sign}(0) \triangleq 0$, and $\alpha \geq 0$. Note that (10.49) can be rewritten in the form of (10.43) with

$$\mathcal{C} = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}, \tag{10.50}$$

$$\mathcal{L} = \mathcal{C} - \mathcal{H} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{10.51}$$

Then it follows from Theorem 10.5 that every point in $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = x_2 = x_3 = x_4 = x_5 = c, c \in \mathbb{R}\}$ is a semistable equilibrium state of (10.49) with $a > 0$ and $a = 0$. Let $[x_{10}, x_{20}, x_{30}, x_{40}, x_{50}]^{\mathrm{T}} = [5, 3, -5, 3, -1]^{\mathrm{T}}$, $a = 6$, and $\alpha = 2$. Figure 10.1 shows the state trajectories versus time.

Chapter 11

System State Equipartitioning and Semistability in Network Dynamical Systems with Arbitrary Time-Delays

11.1. Introduction

Modern complex dynamical systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication networks. By properly formulating these systems in terms of subsystem interaction involving energy/mass transfer, the dynamical models of many of these systems can be derived from mass, energy, and information balance considerations that involve dynamic states whose values are nonnegative. Hence, it follows from physical considerations that the state trajectory of such systems remains in the nonnegative orthant of the state space for nonnegative initial conditions. Such systems are commonly referred to as *nonnegative dynamical systems* in the literature [71, 94]. A subclass of nonnegative dynamical systems are *compartmental systems* [26, 94, 132, 216]. Compartmental systems involve dynamical models that are characterized by conservation laws (e.g., mass and energy) capturing the exchange of material between coupled macroscopic subsystems known as compartments. Each compartment is assumed to be kinetically homogeneous, that is, any material entering the compartment is instantaneously mixed with the material of the compartment. The range of applications of nonnegative systems and compartmental systems includes biological and physiological systems [132, 133], chemical reaction systems [74, 146], queuing systems [234], large-scale systems [50], stochastic systems (whose state variables represent probabilities) [234], ecological systems [182], economic systems [19], demographic systems [132], telecommunications systems [79], transportation systems, power systems, thermodynamic systems [104], and structural vibration systems, to cite but a few examples.

A key physical limitation of compartmental systems is that transfers between compartments are not instantaneous and realistic models for capturing the dynamics of such systems should account for material, energy, or information in transit between compartments. Hence, to accurately describe the evolution of the aforementioned systems, it is necessary to include in any mathematical model of the system dynamics some information of the past system states. In this case, the state of the system at a given time involves a piece of trajectories in the space of continuous functions defined on an interval in the nonnegative orthant of the state space. This of course leads to (infinite-dimensional) delay dynamical systems [115,144].

Nonnegative and compartmental models are also widespread in agreement problems in dynamical networks with directed graphs and switching topologies [186, 187]. Specifically, distributed decision-making for coordination of networks of dynamic agents involving information flow can be naturally captured by compartmental models. These dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles, distributed sensor networks, swarms of air and space vehicle formations [72,231], and congestion control in communication networks [194]. In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In particular, it is important to develop consensus protocols for networks of dynamic agents with directed information flow, switching network topologies, and possible system time-delays.

In this chapter, we use compartmental dynamical system models to characterize dynamic algorithms for linear and nonlinear networks of dynamic agents in the presence of inter-agent communication delays that possess a continuum of semistable equilibria, that is, protocol algorithms that guarantee convergence to Lyapunov stable equilibria. In addition, we show that the steady-state distribution of the dynamic network is uniform, leading to system state equipartitioning or consensus. These results extend the results in the literature on consensus protocols for linear balanced networks to linear and nonlinear unbalanced networks with time-delays.

11.2. Mathematical Preliminaries

In the first part of this chapter, we consider linear, time-delay dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n_{\rm d}} A_{{\rm d}i}x(t-\tau_i), \quad x(\theta) = \eta(\theta), \quad -\bar{\tau} \le \theta \le 0, \quad t \ge 0, \tag{11.1}$$

where $x(t) \in \mathbb{R}^n$, $t \ge 0$, $A \in \mathbb{R}^{n\times n}$, $A_{{\rm d}i} \in \mathbb{R}^{n\times n}$, $\tau_i \in \mathbb{R}$, $i = 1,\ldots,n_{\rm d}$, $\bar{\tau} = \max_{i\in\{1,\ldots,n_{\rm d}\}}\tau_i$, $\eta(\cdot) \in \mathcal{C}_+ \triangleq \{\psi(\cdot) \in \mathcal{C}([-\bar{\tau},0],\mathbb{R}^n) : \psi(\theta) \ge\ge 0, \theta \in [-\bar{\tau},0]\}$ is a continuous vector-valued function specifying the initial state of the system, and $\mathcal{C}([-\bar{\tau},0],\mathbb{R}^n)$ denotes a Banach space of continuous functions mapping the interval $[-\bar{\tau},0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. Note that the state of (11.1) at time t is the *piece of trajectories* x between $t-\tau$ and t, or, equivalently, the *element* x_t in the space of continuous functions defined on the interval $[-\bar{\tau},0]$ and taking values in $\mathbb{R}^n$, that is, $x_t \in \mathcal{C}([-\bar{\tau},0],\mathbb{R}^n)$, where $x_t(\theta) \triangleq x(t+\theta)$, $\theta \in [-\bar{\tau},0]$. Furthermore, since for a given time t the piece of the trajectories x_t is defined on $[-\bar{\tau},0]$, the uniform norm $|||x_t||| = \sup_{\theta\in[-\bar{\tau},0]} \|x(t+\theta)\|$, where $\|\cdot\|$ denotes the Euclidean vector norm, is used for the definitions of Lyapunov and asymptotic stability of (11.1). For further details, see [115, 144]. In addition, note that since $\eta(\cdot)$ is continuous it follows from Theorem 2.1 of [115, p. 14] that there exists a unique solution $x(\eta)$ defined on $[-\bar{\tau},\infty)$ that coincides with η on $[-\bar{\tau},0]$ and satisfies (11.1) for all $t \ge 0$. Finally, recall that if the positive orbit $\gamma^+(\eta(\theta))$ of (11.1) is bounded, then $\gamma^+(\eta(\theta))$ is *precompact* [113], that is, $\gamma^+(\eta(\theta))$ can be enclosed in the union of a finite number of ε-balls around elements of $\gamma^+(\eta(\theta))$.

The following theorem gives necessary and sufficient conditions for asymptotic stability of a linear time-delay nonnegative dynamical system $\mathcal{G}$ given by (11.1). For this result, the following definition and proposition are needed.

Definition 11.1. The linear time-delay dynamical system given by (11.1) is *nonnegative* if for every $\eta(\cdot) \in \mathcal{C}_+$, the solution $x(t)$, $t \ge 0$, to (11.1) is nonnegative.

Proposition 11.1 [93, 106]. The linear time-delay dynamical system $\mathcal{G}$ given by (11.1) is nonnegative if and only if $A \in \mathbb{R}^{n\times n}$ is essentially nonnegative and $A_{\mathrm{d}i} \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, n_{\mathrm{d}}$, is nonnegative.

Theorem 11.1 [93, 106]. Consider the linear time-delay dynamical system $\mathcal{G}$ given by (11.1) where $A \in \mathbb{R}^{n\times n}$ is essentially nonnegative and $A_{\mathrm{d}i} \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, n_{\mathrm{d}}$, is nonnegative. If there exist $p, r \in \mathbb{R}^n$ such that $p >> 0$ and $r \geq\geq 0$ (resp., $r >> 0$) satisfying

$$0 = \left(A + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i}\right)^{\mathrm{T}} p + r, \tag{11.2}$$

then $\mathcal{G}$ is Lyapunov (resp., asymptotically) stable for all $\bar{\tau} \in [0, \infty)$. Conversely, if $\mathcal{G}$ is asymptotically stable for all $\bar{\tau} \in [0, \infty)$, then there exist $p, r \in \mathbb{R}^n$ such that $p >> 0$ and $r >> 0$ satisfying (11.2).

Next, we consider a subclass of nonnegative systems, namely, compartmental systems. As noted in the Introduction, compartmental dynamical systems are of major importance in biological systems, physiological systems, chemical reaction systems, ecological systems, economic systems, power systems, telecommunications systems, and network systems.

Definition 11.2 [93, 106]. The linear time-delay dynamical system (11.1) is called a *compartmental dynamical system* if $A + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i}$ is a compartmental matrix.

Note that the linear time-delay dynamical system (11.1) is compartmental if A and $A_{\mathrm{d}} \triangleq \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i}$ are given by

$$A_{(i,j)} = \begin{cases} -\sum_{k=1}^{n} a_{ki}, & i = j, \\ 0, & i \neq j, \end{cases} \qquad A_{\mathrm{d}(i,j)} = \begin{cases} 0, & i = j, \\ a_{ij}, & i \neq j, \end{cases} \tag{11.3}$$

where $a_{ii} \geq 0$, $i \in \{1, \ldots, n\}$, denotes the loss coefficients of the ith compartment and $a_{ij} \geq 0$, $i \neq j$, $i, j \in \{1, \ldots, n\}$, denotes the transfer coefficients from the jth compartment to the ith compartment. The following results are necessary for developing some of the main results of this section.

Proposition 11.2 [94]. Let $A \in \mathbb{R}^{n\times n}$ be essentially nonnegative and assume there exists $p \in \mathbb{R}_+^n$ such that $A^{\mathrm{T}}p \leq\leq 0$. Then A is semistable, that is, $\mathrm{Re}\,\lambda < 0$, or $\lambda = 0$ and λ is semisimple, where $\lambda \in \mathrm{spec(A)}$.

Corollary 11.1. Let $A \in \mathbb{R}^{n\times n}$ be an essentially nonnegative matrix such that $A = A^{\mathrm{T}}$. If there exists $p \in \mathbb{R}_+^n$ such that $A^{\mathrm{T}}p \leq\leq 0$, then $A \leq 0$.

Proof. The proof is a direct consequence of Proposition 11.2 by noting that if A is symmetric, then semistability implies that $A \leq 0$. $\square$

Lemma 11.1. Let $X \in \mathbb{R}^{n\times n}$ and $Z \in \mathbb{R}^{m\times m}$ be such that $X = X^{\mathrm{T}}$ and $Z = Z^{\mathrm{T}}$, and let $Y \in \mathbb{R}^{n\times m}$ be such that $Y = YZ^{\mathrm{D}}Z$. Then

$$M \triangleq \begin{bmatrix} X & Y \\ Y^{\mathrm{T}} & Z \end{bmatrix} \leq 0 \tag{11.4}$$

if and only if $Z \leq 0$ and $X - YZ^{\mathrm{D}}Y^{\mathrm{T}} \leq 0$.

Proof. Define $T \triangleq \begin{bmatrix} I_n & -YZ^{\mathrm{D}} \\ 0 & I_m \end{bmatrix}$ and note that $\det T \neq 0$. Now, noting that $TMT^{\mathrm{T}} \leq 0$ if and only if $M \leq 0$, and

$$\begin{aligned} TMT^{\mathrm{T}} &= \begin{bmatrix} I_n & -YZ^{\mathrm{D}} \\ 0 & I_m \end{bmatrix} \begin{bmatrix} X & Y \\ Y^{\mathrm{T}} & Z \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -Z^{\mathrm{D}}Y^{\mathrm{T}} & I_m \end{bmatrix} \\ &= \begin{bmatrix} X - YZ^{\mathrm{D}}Y^{\mathrm{T}} & 0 \\ 0 & Z \end{bmatrix} \\ &\leq 0, \end{aligned}$$

the result follows immediately. $\square$

11.3. Semistability and Equipartition of Linear Compartmental Systems with Time-Delay

In this section, we present sufficient conditions for semistability and system state equipartition for linear compartmental dynamical systems with time delay. Note that for addressing the stability of the zero solution of a time delay nonnegative system, the usual stability definitions given in [115] need to be slightly modified. In particular, stability notions for nonnegative dynamical systems need to be defined with respect to relatively open subsets of $\overline{\mathbb{R}}_+^n$ containing the equilibrium solution $x_t \equiv 0$. For a similar definition see [94]. In this case, standard Lyapunov-Krasovskii stability theorems for linear and nonlinear time delay systems [115] can be used directly with the required sufficient conditions verified on $\overline{\mathbb{R}}_+^n$. The following lemma is needed for the main theorem of this section.

Lemma 11.2. Let $A \in \mathbb{R}^{n\times n}$ and $A_{\mathrm{d}i} \in \mathbb{R}^{n\times n}$, $i = 1,\ldots,n_{\mathrm{d}}$, be given by (11.3). Assume that $(A + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i})\mathbf{e} = 0$. Then there exist nonnegative definite matrices $Q_i \in \mathbb{R}^{n\times n}$, $i = 1,\ldots,n_{\mathrm{d}}$, such that

$$A + A^{\mathrm{T}} + \sum_{i=1}^{n_{\mathrm{d}}} (Q_i + A_{\mathrm{d}i}^{\mathrm{T}} Q_i^{\mathrm{D}} A_{\mathrm{d}i}) \leq 0. \tag{11.5}$$

Proof. For each $i \in \{1,\ldots,n_{\mathrm{d}}\}$, let Q_i be the diagonal matrix defined by

$$Q_{i(l,l)} \triangleq \sum_{m=1, l\neq m}^{n_{\mathrm{d}}} A_{\mathrm{d}i(l,m)}, \tag{11.6}$$

and note that it follows from (11.6) and the definition of the Drazin inverse that $(A_{\mathrm{d}i} - Q_i)\mathbf{e} = 0$ and $Q_i Q_i^{\mathrm{D}} A_{\mathrm{d}i} = A_{\mathrm{d}i}$, $i = 1,\ldots,n_{\mathrm{d}}$. Since A and Q_i, $i = 1,\ldots,n_{\mathrm{d}}$, are diagonal and $(A + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i})\mathbf{e} = 0$ it follows that $A + \sum_{i=1}^{n_{\mathrm{d}}} Q_i = 0$. Hence, $M\mathbf{e} = 0$, where

$$M \triangleq \begin{bmatrix} A + A^{\mathrm{T}} + \sum_{i=1}^{n_{\mathrm{d}}} Q_i & A_{\mathrm{d}1}^{\mathrm{T}} & A_{\mathrm{d}2}^{\mathrm{T}} & \cdots & A_{\mathrm{d}n_{\mathrm{d}}}^{\mathrm{T}} \\ A_{\mathrm{d}1} & -Q_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{\mathrm{d}n_{\mathrm{d}}} & 0 & 0 & \cdots & -Q_{n_{\mathrm{d}}} \end{bmatrix}. \tag{11.7}$$

Now, it follows from Corollary 11.1 that $M \leq 0$ and since $Q_i Q_i^{\mathrm{D}} A_{\mathrm{d}i} = A_{\mathrm{d}i}$, $i = 1,\ldots,n_{\mathrm{d}}$, it follows from Lemma 11.1 that $M \leq 0$ if and only if (11.5) holds. $\square$

For the next result, recall that the equilibrium solution $x_t \equiv x_{\mathrm{e}}$ to (11.1) is *semistable* if and only if x_{e} is Lyapunov stable and $\lim_{t\to\infty} x(t)$ exists.

Theorem 11.2. Consider the linear time-delay dynamical system given by (11.1) where A and $A_{\mathrm{d}i}$, $i = 1, \ldots, n_{\mathrm{d}}$, are given by (11.3). Assume that $(A + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i})^{\mathrm{T}}\mathbf{e} = (A + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i})\mathbf{e} = 0$ and $\mathrm{rank}(A + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i}) = n - 1$. Then for every $\alpha \geq 0$, $\alpha\mathbf{e}$ is a semistable equilibrium point of (11.1). Furthermore, $x(t) \to \alpha^*\mathbf{e}$ as $t \to \infty$, where

$$\alpha^* = \frac{\mathbf{e}^{\mathrm{T}}\eta(0) + \sum_{i=1}^{n_{\mathrm{d}}} \int_{-\tau_i}^{0} \mathbf{e}^{\mathrm{T}} A_{\mathrm{d}i}\eta(\theta)\mathrm{d}\theta}{n + \sum_{i=1}^{n_{\mathrm{d}}} \tau_i \mathbf{e}^{\mathrm{T}} A_{\mathrm{d}i}\mathbf{e}}. \tag{11.8}$$

Proof. It follows from Lemma 11.2 that there exist nonnegative matrices Q_i, $i = 1, \ldots, n_{\mathrm{d}}$, such that (11.5) holds. Now, consider the Lyapunov-Krasovskii functional $V : \mathcal{C}_+ \to \mathbb{R}$ given by

$$V(\psi(\cdot)) = \psi^{\mathrm{T}}(0)\psi(0) + \sum_{i=1}^{n_{\mathrm{d}}} \int_{-\tau_i}^{0} \psi^{\mathrm{T}}(\theta) A_{\mathrm{d}i}^{\mathrm{T}} Q_i^{\mathrm{D}} A_{\mathrm{d}i}\psi(\theta)\mathrm{d}\theta, \tag{11.9}$$

and note that the directional derivative of $V(x_t)$ along the trajectories of (11.1) is given by

$$\begin{aligned}
\dot{V}(x_t) =&\ 2x^{\mathrm{T}}(t)\dot{x}(t) + \sum_{i=1}^{n_{\mathrm{d}}} x^{\mathrm{T}}(t) A_{\mathrm{d}i}^{\mathrm{T}} Q_i^{\mathrm{D}} A_{\mathrm{d}i} x(t) - \sum_{i=1}^{n_{\mathrm{d}}} x^{\mathrm{T}}(t-\tau_i) A_{\mathrm{d}i}^{\mathrm{T}} Q_i^{\mathrm{D}} A_{\mathrm{d}i} x(t-\tau_i) \\
=&\ 2x^{\mathrm{T}}(t)Ax(t) + 2x^{\mathrm{T}}(t)\sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i}x(t-\tau_i) + \sum_{i=1}^{n_{\mathrm{d}}} x^{\mathrm{T}}(t) A_{\mathrm{d}i}^{\mathrm{T}} Q_i^{\mathrm{D}} A_{\mathrm{d}i} x(t) \\
&- \sum_{i=1}^{n_{\mathrm{d}}} x^{\mathrm{T}}(t-\tau_i) A_{\mathrm{d}i}^{\mathrm{T}} Q_i^{\mathrm{D}} A_{\mathrm{d}i} x(t-\tau_i) \\
\leq&\ -\sum_{i=1}^{n_{\mathrm{d}}} [x^{\mathrm{T}}(t) Q_i x(t) - 2x^{\mathrm{T}}(t) A_{\mathrm{d}i} x(t-\tau_i) + x^{\mathrm{T}}(t-\tau_i) A_{\mathrm{d}i}^{\mathrm{T}} Q_i^{\mathrm{D}} A_{\mathrm{d}i} x(t-\tau_i)] \\
=&\ -\sum_{i=1}^{n_{\mathrm{d}}} [-Q_i x(t) + A_{\mathrm{d}i}x(t-\tau_i)]^{\mathrm{T}} Q_i^{\mathrm{D}} [-Q_i x(t) + A_{\mathrm{d}i}x(t-\tau_i)] \\
\leq&\ 0, \qquad t \geq 0.
\end{aligned} \tag{11.10}$$

Next, let $\mathcal{R} \triangleq \{\psi(\cdot) \in \mathcal{C}_+ : -Q_i\psi(0) + A_{\mathrm{d}i}\psi(-\tau_i) = 0, i = 1, \ldots, n_{\mathrm{d}}\}$ and note that since the positive orbit $\gamma^+(\eta(\theta))$ of (11.1) is bounded, $\gamma^+(\eta(\theta))$ belongs to a compact subset of $\mathcal{C}_+$, and hence, it follows from Theorem 3.2 of [115] that $x_t \to \mathcal{M}$, where $\mathcal{M}$ denotes the largest invariant set contained in $\mathcal{R}$. Now, since $A + \sum_{i=1}^{n_{\mathrm{d}}} Q_i = 0$, it follows that $\mathcal{R} \subset \hat{\mathcal{R}} \triangleq \{\psi(\cdot) \in \mathcal{C}_+ : A\psi(0) + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i}\psi(-\tau_i) = 0\}$. Hence, since $\mathrm{rank}(A + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i}) = n - 1$ and

$(A+\sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i})\mathbf{e}=0$, it follows that the largest invariant set $\hat{\mathcal{M}}$ contained in $\hat{\mathcal{R}}$ is given by $\hat{\mathcal{M}}=\{\psi\in\mathcal{C}_+ : \psi(\theta)=\alpha\mathbf{e}, \theta\in[-\bar{\tau},0], \alpha\geq 0\}$. Furthermore, since $\hat{\mathcal{M}}\subset\mathcal{R}\subset\hat{\mathcal{R}}$, it follows that $\mathcal{M}=\hat{\mathcal{M}}$.

Next, define the functional $E:\mathcal{C}_+\to\mathbb{R}$ by

$$E(\psi(\cdot))=\mathbf{e}^{\mathrm{T}}\psi(0)+\sum_{i=1}^{n_{\mathrm{d}}}\int_{-\tau_i}^{0}\mathbf{e}^{\mathrm{T}}A_{\mathrm{d}i}\psi(\theta)\mathrm{d}\theta, \tag{11.11}$$

and note that $\dot{E}(x_t)\equiv 0$ along the trajectories of (11.1). Thus, for all $t\geq 0$,

$$E(x_t)=E(\eta(\cdot))=\mathbf{e}^{\mathrm{T}}\eta(0)+\sum_{i=1}^{n_{\mathrm{d}}}\int_{-\tau_i}^{0}\mathbf{e}^{\mathrm{T}}A_{\mathrm{d}i}\eta(\theta)\mathrm{d}\theta, \tag{11.12}$$

which implies that $x_t\to\mathcal{M}\cap\mathcal{E}$, where $\mathcal{E}\triangleq\{\psi(\cdot)\in\mathcal{C}_+ : E(\psi(\cdot))=E(\eta(\cdot))\}$. Hence, since $\mathcal{M}\cap\mathcal{E}=\{\alpha^*\mathbf{e}\}$, it follows that $x(t)\to\alpha^*\mathbf{e}$, where α^* is given by (11.8).

Finally, Lyapunov stability of $\alpha\mathbf{e}$, $\alpha\geq 0$, follows by considering the Lyapunov-Krasovskii functional

$$V(\psi(\cdot))=(\psi(0)-\alpha\mathbf{e})^{\mathrm{T}}(\psi(0)-\alpha\mathbf{e})+\sum_{i=1}^{n_{\mathrm{d}}}\int_{-\tau_i}^{0}(\psi(\theta)-\alpha\mathbf{e})^{\mathrm{T}}A_{\mathrm{d}i}^{\mathrm{T}}Q_i^{\mathrm{D}}A_{\mathrm{d}i}(\psi(\theta)-\alpha\mathbf{e})\mathrm{d}\theta$$

and noting that $V(\psi)\geq\|\psi(0)-\alpha\mathbf{e}\|_2^2$. $\square$

Note that if $n_{\mathrm{d}}=n^2-n$, $A_{\mathrm{d}}=A_{\mathrm{d}}^{\mathrm{T}}$, and $(A+A_{\mathrm{d}})\mathbf{e}=0$, then (11.1) can be rewritten as

$$\dot{x}_i(t)=-\sum_{j=1,j\neq i}^{n}a_{ij}[x_i(t)-x_j(t-\tau_{ij})],\quad x(\theta)=\eta(\theta),\quad -\bar{\tau}\leq\theta\leq 0,\quad t\geq 0, \tag{11.13}$$

where $i=1,\ldots,n$, and $\tau_{ij}\in[0,\bar{\tau}]$, $i\neq j$, $i,j=1,\ldots,n$, which implies that the rate of material transfer from the ith compartment to the jth compartment is proportional to the difference $x_j(t-\tau_{ij})-x_i(t)$. Hence, the rate of material transfer is positive (resp., negative) if $x_j(t-\tau_{ij})>x_i(t)$ (resp., $x_j(t-\tau_{ij})<x_i(t)$). Equation (11.13) is an information flow balance equation that governs the information exchange among coupled subsystems and is completely analogous to the equations of thermal transfer with subsystem information playing the role of temperatures. Furthermore, note that since $a_{ij}\geq 0$, $i\neq j$, $i,j=1,\ldots,n$,

information energy flows from more energetic (information rich) subsystems to less energetic (information poor) subsystems, which is consistent with the second law of thermodynamics requiring that heat (energy) must flow in the direction of lower temperatures.

11.4. Semistability and Equipartition of Nonlinear Compartmental Systems with Time-Delay

In this section, we extend the results of Section 11.3 to nonlinear compartmental systems with time delay. Specifically, we consider nonlinear time-delay dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f(x(t)) + f_{\mathrm{d}}(x(t-\tau_1), \ldots, x(t-\tau_{n_{\mathrm{d}}})), \quad x(\theta) = \eta(\theta), \quad -\bar{\tau} \leq \theta \leq 0, \quad t \geq 0, \tag{11.14}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous and $f(0) = 0$, $f_{\mathrm{d}} : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous and $f_{\mathrm{d}}(0, \ldots, 0) = 0$, $\bar{\tau} = \max_{i \in \{1,\ldots,n_{\mathrm{d}}\}} \tau_i$, $\tau_i \geq 0$, $i = 1, \ldots, n_{\mathrm{d}}$, and $\eta(\cdot) \in \mathcal{C} = \mathcal{C}([-\bar{\tau}, 0], \mathbb{R}^n)$ is a continuous vector-valued function specifying the initial state of the system. Note that since $\eta(\cdot)$ is continuous it follows from Theorem 2.3 of [115, p. 44] that there exists a unique solution $x(\eta)$ defined on $[-\bar{\tau}, \infty)$ that coincides with η on $[-\bar{\tau}, 0]$ and satisfies (11.14) for all $t \geq 0$. In addition, recall that if the positive orbit $\gamma^+(\eta(\theta))$ of (11.14) is bounded, then $\gamma^+(\eta(\theta))$ is precompact [113]. The following definitions generalize the notions of essential nonnegativity and nonnegativity to vector fields.

Definition 11.3 [94]. Let $f = [f_1, \ldots, f_n]^{\mathrm{T}} : \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ that contains $\overline{\mathbb{R}}_+^n$. Then f is *essentially nonnegative* if $f_i(x) \geq 0$ for all $i = 1, \ldots, n$ and $x \in \overline{\mathbb{R}}_+^n$ such that $x_i = 0$, where x_i denotes the ith element of x. f is *compartmental* if f is essentially nonnegative and $\mathbf{e}^{\mathrm{T}} f(x) \leq 0$, $x \in \overline{\mathbb{R}}_+^n$.

Definition 11.4 [97]. Let $f = [f_1, \ldots, f_n]^{\mathrm{T}} : \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ that contains $\overline{\mathbb{R}}_+^n$. Then f is *nonnegative* if $f_i(x) \geq 0$ for all $i = 1, \ldots, n$ and $x \in \overline{\mathbb{R}}_+^n$.

Note that if $f(x) = Ax$, where $A \in \mathbb{R}^{n\times n}$, then $f(\cdot)$ is essentially nonnegative if and only if A is essentially nonnegative, and $f(\cdot)$ is nonnegative if and only if A is nonnegative.

Definition 11.5 [93]. The nonlinear time-delay dynamical system $\mathcal{G}$ given by (11.14) is *nonnegative* if for every $\eta(\cdot) \in \mathcal{C}_+$, where $\mathcal{C}_+ \triangleq \{\psi(\cdot) \in \mathcal{C} : \psi(\theta) \geq\geq 0,\, \theta \in [-\bar{\tau}, 0]\}$, the solution $x(t)$, $t \geq 0$, to (11.14) is nonnegative.

Proposition 11.3 [93]. Consider the nonlinear time-delay dynamical system $\mathcal{G}$ given by (11.14). If $f(\cdot)$ is essentially nonnegative and $f_\mathrm{d}(\cdot)$ is nonnegative, then $\mathcal{G}$ is nonnegative.

For the remainder of this research, we assume that $f(\cdot)$ is essentially nonnegative and $f_\mathrm{d}(\cdot)$ is nonnegative so that for every $\eta(\cdot) \in \mathcal{C}_+$, the nonlinear time-delay dynamical system $\mathcal{G}$ given by (11.14) is nonnegative. Next, we consider a subclass of nonlinear nonnegative systems, namely, nonlinear compartmental systems.

Definition 11.6. The nonlinear time-delay dynamical system (11.14) is called a *compartmental dynamical system* if $F(\cdot)$ is compartmental, where $F(x) \triangleq f(x) + f_\mathrm{d}(x, x, \ldots, x)$.

Note that the nonlinear time-delay dynamical system is compartmental if $f(\cdot)$ and $f_\mathrm{d} = [f_{\mathrm{d}1}, \ldots, f_{\mathrm{d}n}]^\mathrm{T}$ are given by

$$f_i(x(t)) = -\sum_{j=1, j\neq i}^{n} a_{ji}(x(t)), \quad f_{\mathrm{d}i}(x(t-\tau_1), \ldots, x(t-\tau_{n_\mathrm{d}})) = \sum_{j=1, j\neq i}^{n} a_{ij}(x(t-\tau_{ij})), \tag{11.15}$$

where $a_{ii}(x(\cdot)) \geq 0$, $x(\cdot) \in \mathcal{C}_+$, $a_{ii}(0) = 0$, $i \in \{1, \ldots, n\}$, denotes the instantaneous rate of flow of material loss of the ith compartment, $a_{ij}(x(\cdot)) \geq 0$, $x(\cdot) \in \mathcal{C}_+$, $i \neq j$, $i, j \in \{1, \ldots, n\}$, denotes the instantaneous rate of material flow from the jth compartment to the ith compartment, τ_{ij}, $i \neq j$, $i, j \in \{1, \ldots, n\}$, denotes the transfer time of material flow from the jth compartment to the ith compartment, and $a_{ii}(\cdot)$ and $a_{ij}(\cdot)$ are such that if $x_i = 0$, then $a_{ii}(x) = 0$ and $a_{ji}(x) = 0$ for all $i, j = 1, \ldots, n$, and $x \in \overline{\mathbb{R}}_+^n$. Note that the above constraints imply that $f(\cdot)$ is essentially nonnegative and $f_{\rm d}(\cdot)$ is nonnegative. The next result generalizes Theorem 11.2 to nonlinear time-delay compartmental systems of the form

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{n_{\rm d}} f_{{\rm d}i}(x(t-\tau_i)), \quad x(\theta) = \eta(\theta), \quad -\bar{\tau} \le \theta \le 0, \quad t \ge 0, \tag{11.16}$$

where $f : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+^n$ is given by $f(x) = [f_1(x_1), \ldots, f_n(x_n)]^{\rm T}$, $f(0) = 0$, $f_{{\rm d}i} : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+^n$, $i = 1, \ldots, n_{\rm d}$, and $f_{\rm d}(0) = 0$. Furthermore, we assume that $f_i(\cdot)$, $i = 1, \ldots, n$, is a strictly decreasing function.

Theorem 11.3. Consider the nonlinear time-delay dynamical system given by (11.16) where $f_i(\cdot)$, $i = 1, \ldots, n$, is strictly decreasing and $f_i(0) = 0$. Assume that $\mathbf{e}^{\rm T}[f(x) + \sum_{i=1}^{n_{\rm d}} f_{{\rm d}i}(x)] = 0$, $x \in \overline{\mathbb{R}}_+^n$, and $f(x) + \sum_{i=1}^{n_{\rm d}} f_{{\rm d}i}(x) = 0$ if and only if $x = \alpha\mathbf{e}$ for some $\alpha \ge 0$. Furthermore, assume there exist nonnegative diagonal matrices $P_i \in \overline{\mathbb{R}}_+^{n\times n}$, $i = 1, \ldots, n_{\rm d}$, such that $P \triangleq \sum_{i=1}^{n_{\rm d}} P_i > 0$,

$$P_i^{\rm D} P_i f_{{\rm d}i}(x) = f_{{\rm d}i}(x), \quad x \in \overline{\mathbb{R}}_+^n, \quad i = 1, \ldots, n_{\rm d}, \tag{11.17}$$

$$\sum_{i=1}^{n_{\rm d}} f_{{\rm d}i}^{\rm T}(x) P_i f_{{\rm d}i}(x) \le f^{\rm T}(x) P f(x), \quad x \in \overline{\mathbb{R}}_+^n. \tag{11.18}$$

Then, for every $\alpha \ge 0$, $\alpha\mathbf{e}$ is a semistable equilibrium point of (11.16). Furthermore, $x(t) \to \alpha^*{\rm e}$ as $t \to \infty$, where α^* satisfies

$$n\alpha^* + \sum_{i=1}^{n_{\rm d}} \tau_i \mathbf{e}^{\rm T} f_{{\rm d}i}(\alpha^*\mathbf{e}) = \mathbf{e}^{\rm T}\eta(0) + \sum_{i=1}^{n_{\rm d}} \int_{-\tau_i}^0 \mathbf{e}^{\rm T} f_{{\rm d}i}(\eta(\theta)){\rm d}\theta. \tag{11.19}$$

Proof. Consider the Lyapunov-Krasovskii functional $V : \mathcal{C}_+ \to \mathbb{R}$ given by

$$V(\psi(\cdot)) = -2\sum_{i=1}^{n} \int_0^{\psi_i(0)} P_{(i,i)} f_i(\zeta){\rm d}\zeta + \sum_{i=1}^{n_{\rm d}} \int_{-\tau_i}^0 f_{{\rm d}i}^{\rm T}(\psi(\theta)) P_i f_{{\rm d}i}(\psi(\theta)){\rm d}\theta. \tag{11.20}$$

Since, $f_i(\cdot)$, $i = 1, \ldots, n$, is a strictly decreasing function it follows that

$$V(\psi) \ge 2\sum_{i=1}^{n} P_{(i,i)}[-$$
$$f_i(\delta_i\psi_i(0))]\psi_i(0) > 0$$

for all $\psi(0) \neq 0$, where $0 < \delta_i < 1$, and hence, there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ such that $V(\psi) \geq \alpha(\|\psi(0)\|)$. Now, note that the directional derivative of $V(x_t)$ along the trajectories of (11.16) is given by

$$\begin{aligned}
\dot{V}(x_t) = & -2f^{\mathrm{T}}(x(t))P\dot{x}(t) + \sum_{i=1}^{n_{\mathrm{d}}} f_{\mathrm{d}i}^{\mathrm{T}}(x(t))P_i f_{\mathrm{d}i}(x(t)) \\
& -\sum_{i=1}^{n_{\mathrm{d}}} f_{\mathrm{d}i}^{\mathrm{T}}(x(t-\tau_i))P_i f_{\mathrm{d}i}(x(t-\tau_i)) \\
= & -2f^{\mathrm{T}}(x(t))Pf(x(t)) - 2\sum_{i=1}^{n_{\mathrm{d}}} f^{\mathrm{T}}(x(t))Pf_{\mathrm{d}i}(x(t-\tau_i)) \\
& +\sum_{i=1}^{n_{\mathrm{d}}} f_{\mathrm{d}i}^{\mathrm{T}}(x(t))P_i f_{\mathrm{d}i}(x(t)) - \sum_{i=1}^{n_{\mathrm{d}}} f_{\mathrm{d}i}^{\mathrm{T}}(x(t-\tau_i))P_i f_{\mathrm{d}i}(x(t-\tau_i)) \\
\leq & -f^{\mathrm{T}}(x(t))Pf(x(t)) - 2\sum_{i=1}^{n_{\mathrm{d}}} f^{\mathrm{T}}(x(t))PP_i^{\mathrm{D}}P_i f_{\mathrm{d}i}(x(t-\tau_i)) \\
& -\sum_{i=1}^{n_{\mathrm{d}}} f_{\mathrm{d}i}^{\mathrm{T}}(x(t-\tau_i))P_i P_i^{\mathrm{D}} P_i f_{\mathrm{d}i}(x(t-\tau_i)) \\
= & -\sum_{i=1}^{n_{\mathrm{d}}} [Pf(x(t)) + P_i f_{\mathrm{d}i}(x(t-\tau_i))]^{\mathrm{T}} P_i^{\mathrm{D}} [Pf(x(t)) + P_i f_{\mathrm{d}i}(x(t-\tau_i))] \\
\leq & \ 0, \quad t \geq 0,
\end{aligned} \tag{11.21}$$

where the first inequality in (11.21) follows from (11.17) and (11.18), and the last equality in (11.21) follows from the fact that $f^{\mathrm{T}}(x)Pf(x) = \sum_{i=1}^{n_{\mathrm{d}}} f^{\mathrm{T}}(x)PP_i^{\mathrm{D}}Pf(x)$, $x \in \overline{\mathbb{R}}_+^n$.

Next, let $\mathcal{R} \triangleq \{\psi(\cdot) \in \mathcal{C}_+ : Pf(\psi(0)) + P_i f_{\mathrm{d}i}(\psi(-\tau_i)) = 0, i = 1, \ldots, n_{\mathrm{d}}\}$ and note that since the positive orbit $\gamma^+(\eta(\theta))$ of (11.16) is bounded, $\gamma^+(\eta(\theta))$ belongs to a compact subset of $\mathcal{C}_+$, and hence, it follows from Theorem 3.2 of [115] that $x_t \to \mathcal{M}$, where $\mathcal{M}$ denotes the largest invariant set (with respect to (11.16)) contained in $\mathcal{R}$. Now, since $\mathbf{e}^{\mathrm{T}}(f(x) + \sum_{i=1}^{n_{\mathrm{d}}} f_{\mathrm{d}i}(x)) = 0, x \in \overline{\mathbb{R}}_+^n$, it follows that

$$\begin{aligned}
\mathcal{R} \subset \hat{\mathcal{R}} \triangleq & \ \{\psi(\cdot) \in \mathcal{C}_+ : f(\psi(0)) + \sum_{i=1}^{n_{\mathrm{d}}} f_{\mathrm{d}i}(\psi(-\tau_i)) = 0\} \\
= & \ \{\psi(\cdot) \in \mathcal{C}_+ : \psi(\theta) = \alpha\mathbf{e},\ \theta \in [-\bar{\tau}, 0],\ \alpha \geq 0\},
\end{aligned}$$

which implies that $x_t \to \hat{\mathcal{R}}$ as $t \to \infty$.

Next, define the functional $E : \mathcal{C}_+ \to \mathbb{R}$ by

$$E(\psi(\cdot)) = \mathbf{e}^{\mathrm{T}}\psi(0) + \sum_{i=1}^{n_{\mathrm{d}}} \int_{-\tau_i}^{0} \mathbf{e}^{\mathrm{T}} f_{\mathrm{d}i}(\psi(\theta))\mathrm{d}\theta, \tag{11.22}$$

and note that $\dot{E}(x_t) \equiv 0$ along the trajectories of (11.16). Thus, for all $t \geq 0$,

$$E(x_t) = E(\eta(\cdot)) = \mathbf{e}^{\mathrm{T}}\eta(0) + \sum_{i=1}^{n_{\mathrm{d}}} \int_{-\tau_i}^{0} \mathbf{e}^{\mathrm{T}} f_{\mathrm{d}i}(\eta(\theta))\mathrm{d}\theta, \tag{11.23}$$

which implies that $x_t \to \hat{\mathcal{R}} \cap \mathcal{E}$, where $\mathcal{E} \triangleq \{\psi(\cdot) \in \mathcal{C}_+ : E(\psi(\cdot)) = E(\eta(\cdot))\}$. Hence, $\hat{\mathcal{R}} \cap \mathcal{E} = \{\alpha^* \mathrm{e}\}$, it follows that $x(t) \to \alpha^*\mathbf{e}$, where α^* satisfies (11.19).

Finally, Lyapunov stability of $\alpha\mathbf{e}$, $\alpha \geq 0$, follows by considering the Lyapunov-Krasovskii functional

$$\begin{aligned} V(\psi(\cdot)) = & -2\sum_{i=1}^{n} \int_{\alpha}^{\psi_i(0)} P_{(i,i)}(f_i(\zeta) - f_i(\alpha))\mathrm{d}\zeta \\ & + \sum_{i=1}^{n_{\mathrm{d}}} \int_{-\tau_i}^{0} [f_{\mathrm{d}i}(\psi(\theta)) - f_{\mathrm{d}i}(\alpha\mathbf{e})]^{\mathrm{T}} P_i [f_{\mathrm{d}i}(\psi(\theta)) - f_{\mathrm{d}i}(\alpha\mathbf{e})]\mathrm{d}\theta, \end{aligned}$$

and noting that $V(\psi) \geq 2\sum_{i=1}^{n} P_{(i,i)}[f_i(\alpha) - f_i(\alpha + \delta_i(\psi_i(0) - \alpha))](\psi_i(0) - \alpha) > 0$, for all $\psi_i(0) \neq \alpha$, where $0 < \delta_i < 1$. $\square$

Theorem 11.3 establishes semistability and state equipartition for the special case of nonlinear compartmental systems of the form (11.15) where $f(\cdot)$ and $f_{\mathrm{d}i}(\cdot)$, $i = 1, \ldots, n$, satisfy (11.17) and (11.18). For general n-dimensional nonlinear compartmental systems with time-delay and vector fields given by (11.16) it is not possible to guarantee semistability and state equipartition. However, semistability without state equipartition may be shown. For example, consider the nonlinear time-delay compartmental dynamical system given by

$$\dot{x}_1(t) = -a_{21}(x_1(t)) + a_{12}(x_2(t - \tau_{12})), \quad x_1(\theta) = \eta_1(\theta), \quad -\bar{\tau} \leq \theta \leq 0, \quad t \geq 0, \tag{11.24}$$

$$\dot{x}_2(t) = -a_{12}(x_2(t)) + a_{21}(x_1(t - \tau_{21})), \quad x_2(\theta) = \eta_2(\theta), \quad -\bar{\tau} \leq \theta \leq 0, \quad t \geq 0, \tag{11.25}$$

where $x_1(t), x_2(t) \in \mathbb{R}$, $t \geq 0$, $a_{12} : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ and $a_{21} : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ satisfy $a_{12}(0) = a_{21}(0) = 0$ and $a_{12}(\cdot)$ and $a_{21}(\cdot)$ are strictly increasing, $\tau_{12}, \tau_{21} > 0$, $\bar{\tau} = \max\{\tau_{12}, \tau_{21}\}$, and $\eta_1(\cdot), \eta_2(\cdot) \in$

$\mathcal{C}_+ = \mathcal{C}([-\bar{\tau}, 0], \overline{\mathbb{R}}_+)$. Note that (11.24) and (11.25) can have multiple equilibria with all the equilibria lying on the curve $a_{21}(u) = a_{12}(v)$, $u, v \geq 0$. It follows from the conditions on $a_{12}(\cdot)$ and $a_{21}(\cdot)$ that all system equilibria lie on the curve $y = a_{12}^{-1}(a_{21}(x))$ in the (x, y) plane, where $a_{12}^{-1}(\cdot)$ denotes the inverse function of $a_{12}(\cdot)$.

Consider the functional $E : \mathcal{C}_+ \times \mathcal{C}_+ \to \mathbb{R}$ given by

$$E(\psi_1, \psi_2) = \psi_1(0) + \psi_2(0) + \int_{-\tau_{12}}^{0} a_{12}(\psi_2(\theta))\mathrm{d}\theta + \int_{-\tau_{21}}^{0} a_{21}(\psi_1(\theta))\mathrm{d}\theta. \tag{11.26}$$

Now, it can be easily shown that the directional derivative of $E(\psi_1, \psi_2)$ along the trajectories of (11.24) and (11.25) is identically zero for all $t \geq 0$, which implies that, for all $t \geq 0$,

$$E(x_{1t}, x_{2t}) = E(\eta_1, \eta_2) = \eta_1(0) + \eta_2(0) + \int_{-\tau_{12}}^{0} a_{12}(\eta_2(\theta))\mathrm{d}\theta + \int_{-\tau_{21}}^{0} a_{21}(\eta_1(\theta))\mathrm{d}\theta.$$

Next, consider the functional $V : \mathcal{C}_+ \times \mathcal{C}_+ \to \mathbb{R}$ given by

$$\begin{aligned} V(\psi_1, \psi_2) = {} & 2\int_{0}^{\psi_1(0)} a_{21}(\theta)\mathrm{d}\theta + 2\int_{0}^{\psi_2(0)} a_{12}(\theta)\mathrm{d}\theta \\ & + \int_{-\tau_{12}}^{0} a_{12}^2(\psi_2(\theta))\mathrm{d}\theta + \int_{-\tau_{21}}^{0} a_{21}^2(\psi_1(\theta))\mathrm{d}\theta, \end{aligned} \tag{11.27}$$

and note that the directional derivative of $V(\psi_1, \psi_2)$ along the trajectories of (11.24) and (11.25) is given by

$$\dot{V}(x_{1t}, x_{2t}) = -[a_{21}(x_1(t)) - a_{12}(x_2(t - \tau_{12}))]^2 - [a_{12}(x_2(t)) - a_{21}(x_1(t - \tau_{21}))]^2.$$

Now, using similar arguments as in the proof of Theorem 11.3 it follows that $(x_1(t), x_2(t)) \to (\alpha^*, a_{12}^{-1}(a_{21}(\alpha^*)))$ as $t \to \infty$, where α^* is the solution to the equation

$$\begin{aligned} \alpha^* + a_{12}^{-1}(a_{21}(\alpha^*)) + (\tau_{12} + \tau_{21})a_{21}(\alpha^*) = {} & \eta_1(0) + \eta_2(0) \\ & + \int_{-\tau_{12}}^{0} a_{12}(\eta_2(\theta))\mathrm{d}\theta + \int_{-\tau_{21}}^{0} a_{21}(\eta_1(\theta))\mathrm{d}\theta, \end{aligned}$$

and $(\alpha^*, a_{12}^{-1}(a_{21}(\alpha^*)))$ is a Lyapunov stable equilibrium state. The above analysis shows that all two-dimensional nonlinear compartmental dynamical systems of the form (11.24) and (11.25) are semistable with system states reaching equilibria lying on the curve $y = a_{12}^{-1}(a_{21}(x))$ in the (x, y) plane.

To demonstrate the utility of Theorem 11.3 we consider a nonlinear two-compartment time-delay dynamical system given by

$$\dot{x}_1(t) = -\sum_{i=1}^{n_{\mathrm{d}}}[a_i(x_1(t)) + a_i(x_2(t-\tau_i))],\ x_1(\theta) = \eta_1(\theta),\ -\bar{\tau} \le \theta \le 0,\ t \ge 0, \quad (11.28)$$

$$\dot{x}_2(t) = \sum_{i=1}^{n_{\mathrm{d}}}[a_i(x_1(t-\tau_i)) - a_i(x_2(t))], \quad x_2(\theta) = \eta_2(\theta), \quad -\bar{\tau} \le \theta \le 0, \quad (11.29)$$

where $a_i : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$, $i = 1, \ldots, n_{\mathrm{d}}$, are such that for every $i = 1, \ldots, n_{\mathrm{d}}$,

$$[a_i(x_1) - a_i(x_2)](x_1 - x_2) > 0, \quad x_1 \neq x_2, \quad (11.30)$$

and $a_i(0) = 0$, $i = 1, \ldots, n_{\mathrm{d}}$. If x_1 and x_2 represent system energies, then (11.28) and (11.29) capture energy flow balance between the two compartments, and (11.30) is consistent with the second law of thermodynamics; that is, energy flows from the more energetic compartment to the less energetic compartment [104]. Furthermore, since $a_i(0) = 0$, (11.30) implies that $a_i(\cdot)$, $i = 1, 2$, is strictly increasing. Now, note that (11.28) and (11.29) can be written in the form of (11.16) with

$$f(x) = \begin{bmatrix} -\sum_{i=1}^{n_{\mathrm{d}}} a_i(x_1) \\ -\sum_{i=1}^{n_{\mathrm{d}}} a_i(x_2) \end{bmatrix}, \quad f_{\mathrm{d}i}(x) = \begin{bmatrix} a_i(x_2) \\ a_i(x_1) \end{bmatrix}, \quad i = 1, 2, \quad (11.31)$$

which implies that $f_j(x_j)$, $j = 1, 2$, are strictly decreasing. Next, with $P_i = I_n$, $i = 1, \ldots, n_{\mathrm{d}}$, (11.17) and (11.18) are trivially satisfied, and hence, it follows from Theorem 11.3 that $x_1(t) - x_2(t) \to 0$ as $t \to \infty$.

Next, we consider nonlinear compartmental time-delay dynamical systems of the form

$$\dot{x}_i(t) = -\sum_{j=1, j\neq i}^{n} a_{ji}(x_i(t)) + \sum_{j=1, j\neq i}^{n} a_{ij}(x_j(t-\tau_i)), \quad x(\theta) = \eta(\theta), \quad -\bar{\tau} \le \theta \le 0, \quad t \ge 0, \quad (11.32)$$

where $i = 1, \ldots, n$, $a_{ij} : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$, $i \neq j$, $i, j \in \{1, \ldots, n\}$, are such that $a_{ij}(0) = 0$ and $a_{ij}(\cdot)$, $i \neq j$, $i, j = 1, \ldots, n$, is strictly increasing. Note that since each transfer coefficient $a_{ij}(\cdot)$ is only a function of x_j and not x, the nonlinear compartmental system (11.32) is a *nonlinear donor controlled compartmental system* [134]. In this case, (11.32) can be written in the form given by (11.16) with $n_{\mathrm{d}} = n$,

$$f_i(x_i) = -\sum_{j=1, j\neq i}^{n} a_{ji}(x_i), \quad f_{\mathrm{d}i}(x) = \mathbf{e}_i \sum_{j=1}^{n} a_{ij}(x_j), \quad i = 1, \ldots, n. \tag{11.33}$$

Next, with $P_i = \mathbf{e}_i\mathbf{e}_i^{\mathrm{T}}$, $i = 1, \ldots, n$, so that $P = I_n$, it follows that (11.17) is trivially satisfied and (11.18) holds if and only if

$$\sum_{i=1}^{n} \left[\sum_{j=1, i\neq j}^{n} a_{ij}(x_j) \right]^2 \leq \sum_{i=1}^{n} \left[\sum_{j=1, i\neq j}^{n} a_{ji}(x_i) \right]^2, \ x \in \overline{\mathbb{R}}_+^n. \tag{11.34}$$

In the case where $n = 2$, (11.34) is trivially satisfied, and hence, it follows from Theorem 11.3 that $x_1(t) - x_2(t) \to 0$ as $t \to \infty$. In general, (11.34) does not hold for arbitrary strictly increasing functions $a_{ij}(\cdot)$. However, if $a_{ij}(\cdot) = \sigma(\cdot)$, $i \neq j$, $i, j = 1, \ldots, n$, where $\sigma : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is such that $\sigma(0) = 0$ and strictly increasing, (11.34) holds if and only if

$$\sum_{i=1}^{n} \left[\sum_{j=1, i\neq j}^{n} \sigma(x_j) \right]^2 \leq \sum_{i=1}^{n} \left[\sum_{j=1, i\neq j}^{n} \sigma(x_i) \right]^2, \quad x \in \mathbb{R}_+^n. \tag{11.35}$$

In this case, since

$$\begin{aligned} 0 \geq\ & (n-1)\sum_{i=1}^{n} \sigma^2(x_i) + (n-2)\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \sigma(x_i)\sigma(x_j) - (n-1)^2 \sum_{i=1}^{n} \sigma^2(x_i) \\ =\ & -(n-2)\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} (\sigma(x_i) - \sigma(x_j))^2, \end{aligned}$$

(11.35) holds, and hence, it follows from Theorem 11.3 that $x_i(t) - x_j(t) \to 0$ as $t \to \infty$, where $i \neq j$, $i, j = 1, \ldots, n$.

Next, we specialize Theorem 11.3 to nonlinear time-delay compartmental systems of the form

$$\dot{x}(t) = A\hat{\sigma}(x(t)) + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i}\hat{\sigma}(x(t-\tau_i)), \quad x(\theta) = \eta(\theta), \quad -\bar{\tau} \leq \theta \leq 0, \quad t \geq 0, \tag{11.36}$$

where $\hat{\sigma} : \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+^n$ is given by $\hat{\sigma}(x) = [\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_n)]^{\mathrm{T}}$, where $\sigma : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is such that $\sigma(u) = 0$ if and only if $u = 0$, and A and $A_{\mathrm{d}i}$, $i = 1, \ldots, n_{\mathrm{d}}$, are as given by (11.3).

Theorem 11.4. Consider the nonlinear time-delay system given by (11.36) where $\sigma : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is such that $\sigma(0) = 0$ and $\sigma(\cdot)$ is strictly increasing. Assume that $(A + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i})^{\mathrm{T}}\mathbf{e} = (A + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i})\mathbf{e} = 0$ and $\mathrm{rank}(A + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i}) = n - 1$. Then for every $\alpha \geq 0$, $\alpha\mathbf{e}$ is a semistable equilibrium point of (11.36). Furthermore, $x(t) \to \alpha^*\mathbf{e}$ as $t \to \infty$, where α^* satisfies

$$n\alpha^* + \sigma(\alpha^*)\sum_{i=1}^{n_{\mathrm{d}}} \tau_i \mathbf{e}^{\mathrm{T}} A_{\mathrm{d}i}\mathbf{e} = \mathbf{e}^{\mathrm{T}}\eta(0) + \sum_{i=1}^{n_{\mathrm{d}}} \int_{-\tau_i}^{0} \mathbf{e}^{\mathrm{T}} A_{\mathrm{d}i}\hat{\sigma}(\eta(\theta))\mathrm{d}\theta. \tag{11.37}$$

Proof. It follows from Lemma 11.2 that there exists Q_i, $i = 1, \ldots, n_{\mathrm{d}}$, such that (11.5) holds with Q_i given by (11.6). Now, since $A = -\sum_{i=1}^{n_{\mathrm{d}}} Q_i = -\sum_{i=1}^{n_{\mathrm{d}}} P_i^{\mathrm{D}} = -P^{-1}$, where $P = \sum_{i=1}^{n_{\mathrm{d}}} P_i$, it follows from (11.5) that, for all $x \in \overline{\mathbb{R}}_+^n$,

$$\begin{aligned} 0 &\geq 2\hat{\sigma}^{\mathrm{T}}(x)A\hat{\sigma}(x) + \hat{\sigma}^{\mathrm{T}}(x)\sum_{i=1}^{n_{\mathrm{d}}}(Q_i + A_{\mathrm{d}i}^{\mathrm{T}} Q_i^{\mathrm{D}} A_{\mathrm{d}i})\hat{\sigma}(x) \\ &= -f^{\mathrm{T}}(x)Pf(x) + \sum_{i=1}^{n_{\mathrm{d}}} f_{\mathrm{d}i}^{\mathrm{T}}(x)P_i f_{\mathrm{d}i}(x), \end{aligned}$$

where $f(x) = A\hat{\sigma}(x)$ and $f_{\mathrm{d}i}(x) = A_{\mathrm{d}i}\hat{\sigma}(x)$, $i = 1, \ldots, n_{\mathrm{d}}$, $x \in \overline{\mathbb{R}}_+^n$. Furthermore, since $P_i^{\mathrm{D}} P_i A_{\mathrm{d}i} = A_{\mathrm{d}i}$, $i = 1, \ldots, n_{\mathrm{d}}$, it follows that $P_i^{\mathrm{D}} P_i f_{\mathrm{d}i}(x) = f_{\mathrm{d}i}(x)$, $i = 1, \ldots, n_{\mathrm{d}}$, $x \in \overline{\mathbb{R}}_+^n$. Now, the result is an immediate consequence of Theorem 11.3 by noting that $\mathbf{e}^{\mathrm{T}}[f(x) + \sum_{i=1}^{n_{\mathrm{d}}} f_{\mathrm{d}i}(x)] = 0$ and $f(x) + \sum_{i=1}^{n_{\mathrm{d}}} f_{\mathrm{d}i}(x) = 0$ if and only if $x = \alpha\mathbf{e}$ for some $\alpha \geq 0$. $\square$

11.5. The Consensus Problem in Dynamical Networks

In this section, we apply the results of Sections 11.3 and 11.4 to the consensus problem in dynamical networks [164,186,187,194,240]. As discussed in Chapter 8, the consensus problem appears frequently in coordination of multiagent systems and involves finding a dynamic algorithm that enables a group of agents in a network to agree upon certain quantities of interest with directed information flow subject to possible link failures and time-delays. As in [187], we use directed graphs to represent a dynamical network and present solutions to the consensus problem for networks with *balanced* graph *topologies* (or information flow) [187] and unknown arbitrary time-delays. Specifically, let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted *directed graph* (or digraph) denoting the dynamical network (or dynamic graph) with the set of *nodes* (or vertices) $\mathcal{V} = \{1, \ldots, n\}$ denoting the agents, the set of *edges* $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denoting the direction of information flow, and a weighted *adjacency* matrix $\mathcal{A} \in \mathbb{R}^{n \times n}$ such that $\mathcal{A}_{(i,j)} = a_{ij} > 0$, $i, j = 1, \ldots, n$, if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. The *in-degree* and *out-degree* of node i are, respectively, defined as $\deg_{\mathrm{in}}(i) \triangleq \sum_{j=1}^{n} a_{ji}$ and $\deg_{\mathrm{out}}(i) \triangleq \sum_{j=1}^{n} a_{ij}$. We say that the node i of a digraph $\mathcal{G}$ is *balanced* if and only if $\deg_{\mathrm{in}}(i) = \deg_{\mathrm{out}}(i)$, and a graph $\mathcal{G}$ is called *balanced* if and only if all of its nodes are balanced, that is, $\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}$, $i = 1, \ldots, n$. Furthermore, we denote the *value* of the node $i \in \{1, \ldots, n\}$ at time t by $x_i(t) \in \mathbb{R}$. The consensus problem involves the design of a dynamic algorithm that guarantees system state equipartition, that is, $\lim_{t \to \infty} x_i(t) = \alpha \in \mathbb{R}$ for $i = 1, \ldots, n$.

The consensus problem is a dynamic graph involving the trajectories of the dynamical network characterized by the dynamical system

$$\dot{x}(t) = u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{11.38}$$

where $x(t) \triangleq [x_1(t), \ldots, x_n(t)]^{\mathrm{T}}$ is the state of the network and $u(t) \triangleq [u_1(t), \ldots, u_m(t)]^{\mathrm{T}}$ is the input to the network with components $u_i(t)$ only depending on the states of the nodes i and its neighbors. Specifically, the consensus problem deals with the design of an input $u(t)$ such that $x(t)$ converges to $\alpha \mathbf{e}$ as $t \to \infty$, where $\alpha \in \mathbb{R}$. Due to the presence of directional constraints on information flow and system time-delays, $u_i(t)$ is constrained to the feedback form $u_i(t) = f_i(x_i(t), x_{j_1}(t - \tau_{ij_1}), \ldots, x_{j_{mi}}(t - \tau_{ij_{mi}}))$, where $\tau_{ij_k} > 0$, $j_k \in \mathcal{N}_i \triangleq \{j \in \{1, \ldots, n\} : (j, i) \in \mathcal{E}\}$, are unknown constant time-delays between nodes i and j_k. For notational convenience we additionally define the parameters $\tau_{ij} \triangleq 0$ if $(j, i) \notin \mathcal{E}$.

As an example, consider the dynamical network given in Figure 11.1 where $\mathcal{V} = \{1, 2, 3\}$, $\mathcal{E} = \{(1, 2), (2, 3), (1, 3), (3, 1)\}$, with adjacency matrix $\mathcal{A}$ such that a_{13}, a_{21}, a_{31}, and $a_{32} > 0$, and with the remaining elements being zeros. In this case, the input to the network is given

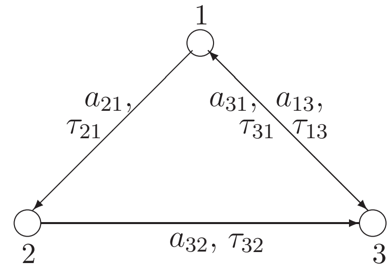

Figure 11.1: Dynamic network

by

$$
\begin{aligned}
u_1(t) &= f_1(x_1(t), x_3(t-\tau_{13})),\\
u_2(t) &= f_2(x_2(t), x_1(t-\tau_{21})),\\
u_3(t) &= f_3(x_3(t), x_2(t-\tau_{32}), x_1(t-\tau_{31})),
\end{aligned}
$$

so that for $i = 1, 2, 3$, $\dot{x}_i(t)$ is only dependent on the states (values) of the nodes that are accessible by node i and with τ_{ij} denoting the communication delay from node j to node i.

Next, we apply Theorem 11.2 and 11.4 to present linear and nonlinear solutions for the consensus problem. Specifically, first we choose

$$
f_i(x(t)) = -\sum_{j=1, i\neq j}^{n} a_{ji} x_i(t) + \sum_{j=1, i\neq j}^{n} a_{ij} x_j(t-\tau_{ij}), \quad i = 1, \ldots, n, \tag{11.39}
$$

so that the *closed-loop* system is given by

$$
\dot{x}_i(t) = -\sum_{j=1, i\neq j}^{n} a_{ji} x_i(t) + \sum_{j=1, i\neq j}^{n} a_{ij} x_j(t-\tau_{ij}), \quad x_i(\theta) = \eta_i(\theta), \quad -\bar{\tau} \le \theta \le 0, \quad t \ge 0, \tag{11.40}
$$

for all $i = 1, \ldots, n$, or, equivalently,

$$
\dot{x}(t) = Ax(t) + \sum_{l=1}^{n_{\rm d}} A_{{\rm d}l} x(t-\tau_l), \qquad x(\theta) = \eta(\theta), \quad -\bar{\tau} \le \theta \le 0, \quad t \ge 0, \tag{11.41}
$$

where $n_{\rm d} \triangleq n^2$, $A \in \mathbb{R}^{n\times n}$, and $A_{{\rm d}l} \in \mathbb{R}^{n\times n}$, $l = 1, \ldots, n_{\rm d}$, with

$$
A = \operatorname{diag}\left[-\sum_{j=2}^{n} a_{j1}, \ldots, -\sum_{j=1}^{n-1} a_{jn}\right], \tag{11.42}
$$

$A_{\mathrm{d}((i-1)n+j)} = a_{ij}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}$, and $\tau_{((i-1)n+j)} = \tau_{ij}$, $i,j = 1,\ldots,n$. Note that if $(j,i) \notin \mathcal{E}$, then $A_{\mathrm{d}((i-1)n+j)} = 0$, which implies that the algorithm is consistent with the directional constraints.

Furthermore, it can be easily shown that $(A + A_{\mathrm{d}})^{\mathrm{T}}\mathbf{e} = 0$, where $A_{\mathrm{d}} \triangleq \sum_{l=1}^{n_{\mathrm{d}}} A_{\mathrm{d}l}$, and $\mathrm{rank}(A+A_{\mathrm{d}}) = n-1$ if and only if for every pair of nodes $(i,j) \in \mathcal{V}$ there exists a *path* from node i to node j [85]. Here, we assume that the adjacency matrix $\mathcal{A}$ is chosen such that $(A + A_{\mathrm{d}})\mathbf{e} = 0$ so that the linear time-delay closed-loop dynamical system (11.41) satisfies all the conditions of Theorem 11.2. Hence, it follows from Theorem 11.2 that the dynamical network given by (11.41) solves the consensus problem, that is, $\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} x_j(t) = \alpha^*$, $i,j = 1,\ldots,n$, $i \neq j$, where α^* is given by (11.8). Alternatively, it follows from Theorem 11.4 that the nonlinear dynamical network given by

$$\dot{x}(t) = A\hat{\sigma}(x(t)) + \sum_{i=1}^{n_{\mathrm{d}}} A_{\mathrm{d}i}\hat{\sigma}(x(t-\tau_i)), \quad x(\theta) = \eta(\theta), \quad -\bar{\tau} \le \theta \le 0, \quad t \ge 0, \tag{11.43}$$

also solves the nonlinear consensus problem where $\sigma(\cdot)$ and $\hat{\sigma}(\cdot)$ satisfy the conditions in Theorem 11.4. In this case, $\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} x_j(t) = \alpha^*$, $i,j = 1,\ldots,n$, $i \neq j$, where α^* is a solution to (11.37). Note that if $\sigma(\theta) = \theta$, (11.43) specializes to (11.41). Although both (11.41) and (11.43) solve the same network consensus problem, the nonlinear function $\sigma(\cdot)$ within $\hat{\sigma}(\cdot)$ may be used to enhance the performance of the dynamic algorithm or satisfy other constraints. For example, choosing $\sigma(\theta) = \tanh(\theta)$ we can constrain bandwidth information from one agent to another.

To illustrate the two algorithms given by (11.41) and (11.43) consider the dynamical network given by the graph shown in Figure 11.2 [187] where a_{ij} and τ_{ij} denote the weight and the time-delay for each edge shown. Here, we choose $a_{(i,j)} = 1$ if $(i,j) \in \mathcal{E}$ so that $(A + A_{\mathrm{d}})\mathbf{e} = 0$. In addition, it can be easily shown that $\mathrm{rank}(A + A_{\mathrm{d}}) = n - 1 = 9$. With $x_0 = [1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10]^{\mathrm{T}}$, Figures 11.3 and 11.4 demonstrate the agreement between all nodes for the algorithms given by (11.41) and (11.43), respectively, with $\sigma(\theta) = \tanh(\theta)$ in (11.43). Finally, Figures 11.5 and 11.6 show the control input versus time for both linear and

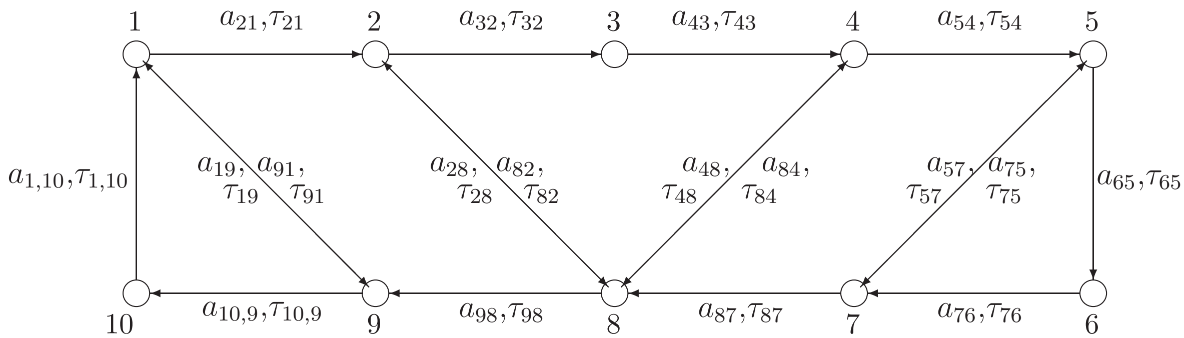

Figure 11.2: Balanced dynamic network

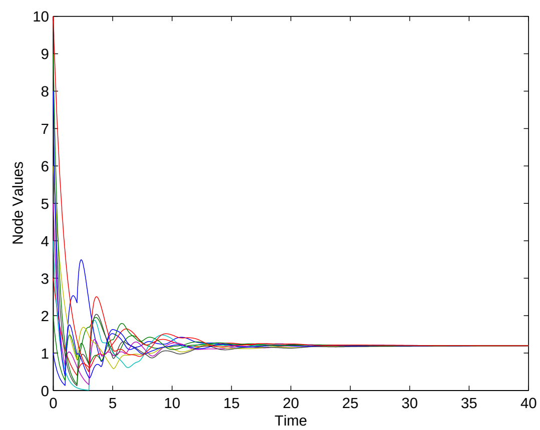

Figure 11.3: Linear consensus algorithm

nonlinear consensus algorithms. Note that the maximum amplitude of the linear consensus algorithm is about six times that of the nonlinear consensus algorithm and, as expected, the settling time of the nonlinear algorithm is longer than that of the linear algorithm.

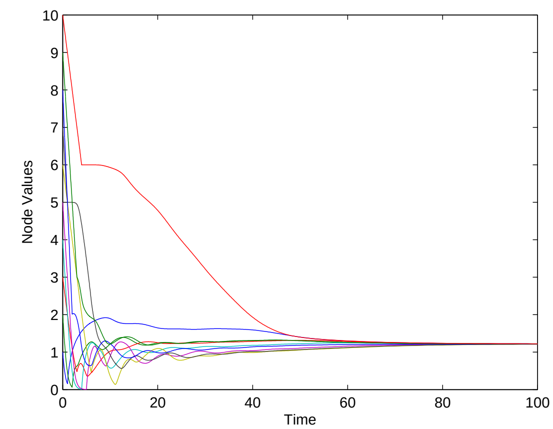

Figure 11.4: Nonlinear consensus algorithm

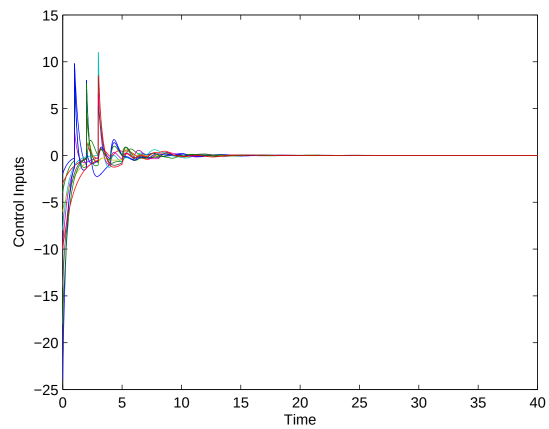

Figure 11.5: Linear consensus algorithm

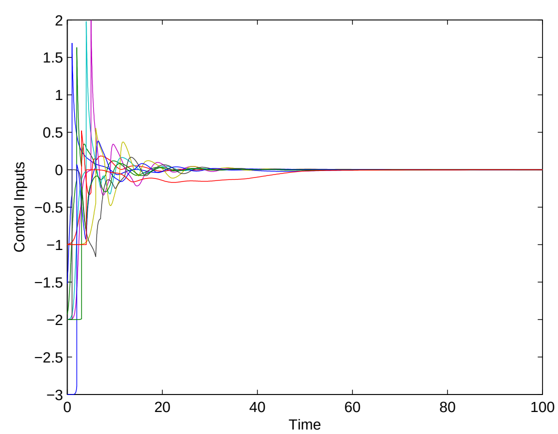

Figure 11.6: Nonlinear consensus algorithm

Chapter 12

Semistability, Differential Inclusions, and Consensus Protocols for Dynamical Networks with Switching Topologies

12.1. Introduction

Since communication links among multiagent systems are often unreliable due to multipath effects and exogenous disturbances, the information exchange topologies in network systems are often dynamic. In particular, link failures or creations in network multiagent systems result in switchings of the communication topology. This is the case, for example, if information between agents is exchanged by means of line-of-sight sensors that experience periodic communication dropouts due to agent motion. Variation in network topology introduces control input discontinuities, which in turn give rise to discontinuous dynamical systems. In addition, the communication topology may be time-varying. In this case, the vector field defining the dynamical system is a discontinuous function of the state and time, and hence, system stability can be analyzed using nonsmooth Lyapunov theory involving concepts such as weak and strong stability notions, differential inclusions, and generalized gradients of locally Lipschitz functions and proximal subdifferentials of lower semicontinuous functions [59].

In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In particular, it is important to develop information consensus protocols for networks of dynamic agents wherein a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus is the existence of a continuum of equilibria representing a state of equipartitioning or *consensus*. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well. For such systems possessing a continuum of equilibria, semistability [31, 32], and not asymptotic stability, is the relevant notion of stability.

To address agreement problems in switching networks with time-dependent and state-dependent topologies, in this chapter we extend the theory of semistability to discontinuous time-invariant and time-varying dynamical systems. In particular, we develop necessary and sufficient conditions to guarantee weak and strong invariance of Fillipov solutions under the assumption that the discontinuous system vector field is uniformly bounded. Moreover, we present Lyapunov-based tests for (strong) semistability, weak semistability, as well as uniform semistability for autonomous and nonautonomous differential inclusions. In addition, we develop sufficient conditions for finite-time semistability of autonomous discontinuous dynamical systems. Achieving agreement in finite time allows the dynamical network to use exact information in addressing other system tasks. It is important to note that our results are different from the results in the literature [56, 67] since the Lipschitz conditions in [56, 67] are not valid for autonomous and nonautonomous differential inclusions considered in the chapter.

12.2. Mathematical Preliminaries

The notation used in this chapter is fairly standard. Specifically, we write $\langle\cdot,\cdot\rangle$ for the inner product in a Hilbert space, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball *centered* at α with *radius* ε, $\operatorname{dist}(p, \mathcal{M})$ for the distance from a point p to the set $\mathcal{M}$, that is, $\operatorname{dist}(p, \mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \|p - x\|$, $x(t) \to \mathcal{M}$ as $t \to \infty$ to denote that $x(t)$ approaches the set $\mathcal{M}$, that is, for each $\varepsilon > 0$ there exists $T > 0$ such that $\operatorname{dist}(x(t), \mathcal{M}) < \varepsilon$ for all $t > T$, and $x(t) \rightrightarrows \mathcal{M}$ as $t \to \infty$ to denote $x(t)$ approaches the set $\mathcal{M}$ uniformly in the initial time $t_0 \in \mathbb{R}$.

Consider the differential equation given by

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{12.1}$$

where $f : \mathbb{R}^q \to \mathbb{R}^q$ is Lebesgue measurable and locally essentially bounded [75, 76], that is, bounded on a bounded neighborhood of every point, excluding sets of measure zero. We assume that the equilibrium set $f^{-1}(0) \triangleq \{x \in \mathbb{R}^q : f(x) = 0\}$ is closed. An absolutely continuous function $x : [0, \tau] \to \mathbb{R}^q$ is said to be a *Filippov solution* [75, 76] of (12.1) on the interval $[0, \tau]$ with initial condition $x(0) = x_0$, if $x(t)$ satisfies

$$\dot{x}(t) \in \mathcal{K}[f](x(t)), \quad \text{a. a. } t \in [0, \tau], \tag{12.2}$$

where the *Filippov set-valued map* $\mathcal{K}[f] : \mathbb{R}^q \to \mathcal{B}(\mathbb{R}^q)$ is defined by

$$\mathcal{K}[f](x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \overline{\text{co}}\,\{f(\mathcal{B}_\delta(x)\backslash\mathcal{S})\}, \quad x \in \mathbb{R}^q, \tag{12.3}$$

where $\mathcal{B}(\mathbb{R}^q)$ denotes the collection of all subsets of $\mathbb{R}^q$, $\mu(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^q$, and "$\overline{\text{co}}$" denotes the convex closure. Note that $\mathcal{K}[f] : \mathbb{R}^q \to \mathcal{B}(\mathbb{R}^q)$ is a map that assigns sets to points. Dynamical systems of the form given by (12.2) are called *differential inclusions* in the literature [9] and for each state $x \in \mathbb{R}^q$, they specify a *set* of possible evolutions rather than a single one. It follows from 1) of Theorem 1 of [193] that there exists a set $\mathcal{N}_f \subset \mathbb{R}^q$ of measure zero such that for every set $\mathcal{W} \subset \mathbb{R}^q$ of measure zero,

$$\mathcal{K}[f](x) = \overline{\text{co}}\left\{\lim_{i\to\infty} f(x_i) : x_i \to x,\, x_i \notin \mathcal{N}_f \cup \mathcal{W}\right\}. \tag{12.4}$$

Since the Filippov set-valued map given by (12.3) is upper semicontinuous with nonempty, convex, and compact values, and is also locally bounded, it follows that Filippov solutions to (12.1) exist [76]. Recall that the solution $t \mapsto x(t)$ to (12.1) is a *maximal solution* if it cannot be extended forward in time. We say that a set $\mathcal{M}$ is *weakly invariant* (resp., *strongly invariant*) with respect to (12.1) if for every $x_0 \in \mathcal{M}$, $\mathcal{M}$ contains a maximal solution (resp., all maximal solutions) of (12.1) [12, 213]. Here we assume that Filippov solutions to (12.1) exist on $[0, \infty)$.

To develop Lyapunov theory for nonsmooth dynamical systems of the form given by (12.1), we need to introduce the notion of generalized derivatives and gradients. Here we focus on Clarke generalized derivatives and gradients [55].

Definition 12.1 [12,55]. Let $V : \mathbb{R}^q \to \mathbb{R}$ be a locally Lipschitz continuous function. The *Clarke upper generalized derivative* of $V(x)$ at x in the direction of v is defined by

$$V^o(x, v) \triangleq \limsup_{y \to x, h \to 0^+} \frac{V(y + hv) - V(y)}{h}. \tag{12.5}$$

The *Clarke generalized gradient* $\partial V : \mathbb{R}^q \to \mathcal{B}(\mathbb{R}^q)$ of $V(x)$ at x is the set

$$\partial V(x) \triangleq \mathrm{co}\left\{ \lim_{i \to \infty} \nabla V(x_i) : x_i \to x,\, x_i \notin \mathcal{N} \cup \mathcal{S} \right\}, \tag{12.6}$$

where "co" denotes the convex hull, ∇ denotes the nabla operator, $\mathcal{N}$ is a set of measure zero points where ∇V does not exist, and $\mathcal{S}$ is an arbitrary set of measure zero in $\mathbb{R}^q$.

Note that (12.5) always exists. Furthermore, note that it follows from Theorem 2.5.1 of [55] that (12.6) is well defined and consists of all convex combinations of all the possible limits of the gradient at neighboring points where V is differentiable. In order to state the main results of this chapter, we need some additional notation and definitions. Given a locally Lipschitz continuous function $V : \mathbb{R}^q \to \mathbb{R}$, the *set-valued Lie derivative* $\mathcal{L}_f V : \mathbb{R}^q \to \mathcal{B}(\mathbb{R})$ of V with respect to (12.1) [12,60] is defined as

$$\mathcal{L}_f V(x) \triangleq \left\{ a \in \mathbb{R} : \text{there exists } v \in \mathcal{K}[f](x) \text{ such that } p^{\mathrm{T}} v = a \text{ for all } p \in \partial V(x) \right\}. \tag{12.7}$$

If $V(x)$ is continuously differentiable at x, then $\mathcal{L}_f V(x) = \{\nabla V(x) \cdot v, v \in \mathcal{K}[f](x)\}$. We use $\max \mathcal{L}_f V(x)$ to denote the largest nonempty element of $\mathcal{L}_f V(x)$.

Recall that a function $V : \mathbb{R}^q \to \mathbb{R}$ is *regular* at $x \in \mathbb{R}^q$ [55] if, for all $v \in \mathbb{R}^q$, the usual right directional derivative $V'_+(x, v) \triangleq \lim_{h \to 0^+} \frac{1}{h}[V(x + hv) - V(x)]$ exists and $V'_+(x, v) = V^o(x, v)$. V is called *regular* on $\mathbb{R}^q$ if it is regular at every $x \in \mathbb{R}^q$. The next definition introduces the notion of semistability for discontinuous dynamical systems. Recall that an equilibrium point $x_{\mathrm{e}} \in f^{-1}(0)$ of (12.1) is an equilibrium point of the differential inclusion if and only if $0 \in \mathcal{K}[f](x_{\mathrm{e}})$.

Definition 12.2. Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to the differential inclusion (12.2). An equilibrium point $z \in \mathcal{D}$ of (12.2) is *Lyapunov stable* with respect to $\mathcal{D}$ if for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for every initial condition $x_0 \in \mathcal{B}_\delta(z)$ and every Filippov solution $x(t)$ with the initial condition $x(0) = x_0$, $x(t) \in \mathcal{B}_\varepsilon(z)$ for all $t \geq 0$. An equilibrium point $z \in \mathcal{D}$ of (12.2) is *semistable* with respect to $\mathcal{D}$ if z is Lyapunov stable and there exists an open subset $\mathcal{D}_0$ of $\mathcal{D}$ containing z such that for all initial conditions in $\mathcal{D}_0$, the Filippov solutions of (12.2) converge to a Lyapunov stable equilibrium point. The system (12.2) is *semistable* with respect to $\mathcal{D}$ if every equilibrium point in $f^{-1}(0)$ is semistable with respect to $\mathcal{D}$. Finally, (12.2) is said to be globally semistable if (12.2) is semistable and $\mathcal{D} = \mathbb{R}^q$.

Next, we introduce the definition of finite-time semistability of (12.2).

Definition 12.3. Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to the differential inclusion (12.2). An equilibrium point $x_{\rm e} \in f^{-1}(0)$ of (12.1) is said to be *finite-time-semistable* if there exist an open neighborhood $\mathcal{U} \subseteq \mathcal{D}$ of $x_{\rm e}$ and a function $T : \mathcal{U} \backslash f^{-1}(0) \to (0, \infty)$, called the *settling-time function*, such that the following statements hold:

i) For every $x \in \mathcal{U} \backslash f^{-1}(0)$ and every Filippov solution $\psi(t)$ of (12.2) with $\psi(0) = x$, $\psi(t) \in \mathcal{U} \backslash f^{-1}(0)$ for all $t \in [0, T(x))$, and $\lim_{t \to T(x)} \psi(t)$ exists and is contained in $\mathcal{U} \cap f^{-1}(0)$.

ii) $x_{\rm e}$ is semistable.

An equilibrium point $x_{\rm e} \in f^{-1}(0)$ of (12.1) is said to be *globally finite-time-semistable* if it is finite-time-semistable with $\mathcal{D} = \mathcal{U} = \mathbb{R}^n$. The system (12.2) is said to be *finite-time-semistable* if every equilibrium point in $f^{-1}(0)$ is finite-time-semistable. Finally, (12.2) is said to be *globally finite-time-semistable* if every equilibrium point in $f^{-1}(0)$ is globally finite-time-semistable.

Given an absolutely continuous curve $\gamma : [0, \infty) \to \mathbb{R}^q$, *the positive limit set of* γ is the set $\Omega(\gamma)$ of points $y \in \mathbb{R}^q$ for which there exists an increasing sequence $\{t_i\}_{i=1}^{\infty}$ satisfying $\lim_{i\to\infty} \gamma(t_i) = y$. Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to the differential inclusion (12.2). We denote the positive limit set of a Filippov solution $\psi(\cdot)$ of (12.2) by $\Omega(\psi)$.

12.3. Semistability Theory for Differential Inclusions

In this section, we develop Lyapunov-based semistability theory for discontinuous dynamical systems of the form given by (12.1). The following proposition is needed for the main results of this section.

Proposition 12.1. Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to (12.1) and let $x \in \mathcal{D}$. If $z \in \Omega(\psi) \cap \mathcal{D}$ is a Lyapunov stable equilibrium point with respect to $\mathcal{D}$, then $z = \lim_{t\to\infty} \psi(t)$ with $\psi(0) = x$ and $\Omega(\psi) = \{z\}$.

Proof. Suppose $z \in \Omega(\psi)$ is Lyapunov stable and let $\varepsilon > 0$. Since z is Lyapunov stable, there exists $\delta = \delta(\varepsilon) > 0$ such that $\psi(t) \in \mathcal{B}_\varepsilon(z)$ for all $x \in \mathcal{B}_\delta(z)$ and $t \geq 0$. Now, since $z \in \Omega(\psi)$, it follows that there exists a divergent sequence $\{t_i\}_{i=1}^{\infty}$ in $[0, \infty)$ such that $\lim_{i\to\infty} \psi(t_i) = z$, and hence, there exists $k \geq 1$ such that $\psi(t_k) \in \mathcal{B}_\delta(z)$. We claim that $\psi(t) \in \mathcal{B}_\varepsilon(z)$ for all $t \geq t_k$. Suppose, *ad absurdum*, $\psi(t) \notin \mathcal{B}_\varepsilon(z)$ for some $t \geq t_k$. Then by continuity of $\psi(\cdot)$, for every $i \geq k$, there exists $\tau_i > t_i$ such that $\psi(\tau_i) \notin \mathcal{B}_\varepsilon(z)$. Namely, there exists a divergent sequence $\{\tau_i\}_{i=1}^{\infty}$ in $[0, \infty)$ such that $\psi(\tau_i) \notin \mathcal{B}_\varepsilon(z)$ for all $\tau_i > t_k$. This contradicts the definition of Lyapunov stability of z. Since ε is arbitrary, it follows that $z = \lim_{t\to\infty} \psi(t)$. Thus, $\lim_{n\to\infty} \psi(t_n) = z$ for every divergence sequence $\{t_n\}_{n=1}^{\infty}$, and hence, $\Omega(\psi) = \{z\}$. $\square$

Next, we present sufficient conditions for semistability of (12.1). Here, we adopt the convention $\max \emptyset = -\infty$.

Theorem 12.1. Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to (12.1) and let $V : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz continuous and regular on $\mathcal{D}$. Assume that for each $x \in \mathcal{D}$ and each Filippov solution $\psi(\cdot)$, $\psi(t)$ is bounded for all $t \geq 0$ with $\psi(0) = x$. Furthermore, assume that $\max \mathcal{L}_f V(x) \leq 0$ or $\mathcal{L}_f V(x) = \emptyset$ for all $x \in \mathcal{D}$. Finally, define

$$\mathcal{Z} \triangleq \{x \in \mathbb{R}^q : 0 \in \mathcal{L}_f V(x)\}. \tag{12.8}$$

If every point in the largest weakly invariant subset $\mathcal{M}$ of $\overline{\mathcal{Z}} \cap \mathcal{D}$ is a Lyapunov stable equilibrium point with respect to $\mathcal{D}$, then (12.1) is semistable with respect to $\mathcal{D}$.

Proof. Let $x \in \mathcal{D}$, $\psi(\cdot)$ be a Filippov solution to (12.1) with $\psi(0) = x$, and $\Omega(\psi)$ be the positive limit set of ψ. First, we show that $\Omega(\psi) \subseteq \overline{\mathcal{Z}}$. Since $\max \mathcal{L}_f V(x) \leq 0$ or $\mathcal{L}_f V(x) = \emptyset$ for all $x \in \mathcal{D}$, it follows from Lemma 1 of [12] that $\frac{\mathrm{d}}{\mathrm{d}t} V(\psi(t))$ exists almost everywhere $t \geq 0$ and $\frac{\mathrm{d}}{\mathrm{d}t} V(\psi(t)) \in \mathcal{L}_f V(\psi(t))$ almost everywhere $t \geq 0$. Now, by assumption, $V(\psi(t)) - V(\psi(\tau)) = \int_\tau^t \frac{\mathrm{d}}{\mathrm{d}t} V(\psi(s)) \mathrm{d}s \leq 0$, $t \geq \tau$, and hence, $V(\psi(t)) \leq V(\psi(\tau))$, $t \geq \tau$, which implies that $V(\psi(t))$ is a nonincreasing function of t.

Next, since $V(\cdot)$ is locally Lipschitz continuous on $\mathcal{D}$ and $\psi(t)$, $t \geq 0$, is bounded, it follows that

$$|V(\psi(t)) - V(\psi(0))| \leq L\|\psi(t) - \psi(0)\| \leq L\|\psi(t)\| + L\|\psi(0)\| \leq L\gamma + L\|x\|, \quad t \geq 0,$$

where L is a Lipschitz constant of $V(\cdot)$ and $\gamma > 0$ is a constant such that $\|\psi(t)\| < \gamma$ for almost all $t \geq 0$, which implies that $V(\psi(t))$ is bounded for almost all $t \geq 0$. Hence, $\gamma_x \triangleq \lim_{t\to\infty} V(\psi(t))$ exists. Now, for all $p \in \Omega(\psi)$, there exists an increasing unbounded sequence $\{t_n\}_{n=1}^\infty$ in $[0, \infty)$ such that $\psi(t_n) \to p$ as $n \to \infty$. Since $V(x)$, $x \in \mathcal{D}$, is continuous, it follows that $V(p) = V(\lim_{n\to\infty} \psi(t_n)) = \lim_{n\to\infty} V(\psi(t_n)) = \gamma_{x_0}$, and hence, $V(p) = \gamma_x$ for $p \in \Omega(\psi)$.

Let $y \in \Omega(\psi)$. If y is an isolated point, then there exists a Filippov solution $\hat{\psi}(\cdot)$ of (12.1) lying in $\Omega(\psi)$ such that $\hat{\psi}(t) = y$ for all $t \geq 0$. Thus, $\frac{\mathrm{d}}{\mathrm{d}t} V(\hat{\psi}(t)) = 0$, and hence, it follows from Lemma 1 of [12] that $0 \in \mathcal{L}_f V(\hat{\psi}(t))$. Hence, $\hat{\psi}(t) \in \mathcal{Z}$, that is, $y \in \mathcal{Z}$. Alternatively, if y is not an isolated point, let $\hat{\psi}(\cdot)$ be a Filippov solution of (12.1) lying in $\Omega(\psi)$ such that $\hat{\psi}(0) = y$. Since $V(\cdot)$ is continuous on $\mathcal{D}$, it follows that there exists $\delta > 0$ such that $V(z) = y$ for all $z \in \mathcal{B}_\delta(y) \cap \Omega(\psi)$. By continuity of $\hat{\psi}(\cdot)$, there exists $\hat{t} > 0$ such that $V(\hat{\psi}(t)) = y$ for all $t \in [0, \hat{t}]$. Now, it follows that $\frac{\mathrm{d}}{\mathrm{d}t}V(\hat{\psi}(t)) = 0$ for all $t \in [0, \hat{t}]$. Hence, it follows from Lemma 1 of [12] that $0 \in \mathcal{L}_f V(\hat{\psi}(t))$ for all $t \in [0, \hat{t}]$, that is, $\hat{\psi}(t) \in \mathcal{Z}$ for all $t \in [0, \hat{t}]$. Let $\{\tau_i\}_{i=1}^{\infty}$ be a positive sequence such that $\lim_{i\to\infty} \tau_i = 0$ and $\hat{\psi}(\tau_i) \in \mathcal{Z}$ for all $i \geq 1$. Since $\hat{\psi}$ is continuous, it follows that $\lim_{i\to\infty} \hat{\psi}(\tau_i) = \hat{\psi}(0) = y \in \overline{\mathcal{Z}}$. Hence, $\Omega(\psi) \subseteq \overline{\mathcal{Z}}$.

Next, since $\Omega(\psi)$ is weakly invariant, it follows that $\Omega(\psi) \subseteq \mathcal{M}$. Moreover, since every point in $\mathcal{M}$ is a Lyapunov stable equilibrium point of (12.1), it follows from Proposition 12.1 that $\Omega(\psi)$ contains a single point for every $x \in \mathcal{D}$ and $\lim_{t\to\infty} \psi(t)$ exists for every $\psi(0) = x$. Finally, since $\lim_{t\to\infty} \psi(t) \in \mathcal{M}$ is Lyapunov stable for every $x \in \mathcal{D}$, it follows from Definition 12.2 that (12.1) is semistable with respect to $\mathcal{D}$. $\square$

The following corollary to Theorem 12.1 provides sufficient conditions for *finite-time* semistability of (12.1).

Corollary 12.1. Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to (12.1) and let $V : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz continuous and regular on $\mathcal{D}$. Assume that $\max \mathcal{L}_f V(x) < 0$ or $\mathcal{L}_f V(x) = \emptyset$ for all $x \in \mathcal{D}\backslash f^{-1}(0)$. If (12.1) is Lyapunov stable with respect to $\mathcal{D}$, then (12.1) is semistable with respect to $\mathcal{D}$. If, in addition, $\max \mathcal{L}_f V(x) \leq -\varepsilon < 0$ or $\mathcal{L}_f V(x) = \emptyset$ for all $x \in \mathcal{D}\backslash f^{-1}(0)$, then (12.1) is finite-time-semistable with respect to $\mathcal{D}$.

Proof. It follows from Theorem 12.1 and $\max \mathcal{L}_f V(x) < 0$ almost everywhere that every equilibrium point of (12.1) in $\mathcal{M}$ is semistable. Note that for every $x \in f^{-1}(0)$, $\mathcal{L}_f V(x) = \{0\}$. Furthermore, since $\max \mathcal{L}_f V(x) < 0$ or $\mathcal{L}_f V(x) = \emptyset$ for all $x \in \mathcal{D}\backslash f^{-1}(0)$, it follows that $\mathcal{M} = f^{-1}(0)$, and hence, by definition, (12.1) is semistable with respect to $\mathcal{D}$. If, in addition, $\max \mathcal{L}_f V(x) \leq -\varepsilon < 0$ or $\mathcal{L}_f V(x) = \emptyset$ for all $x \in \mathcal{D}\backslash f^{-1}(0)$, then it follows from Proposition 2.8 of [60] that $\mathcal{Z}$ given by (12.8) is attained in finite time, and hence, $f^{-1}(0)$ is reached in finite time. Thus, it follows from Definition 12.3 that (12.1) is finite-time-semistable. $\square$

Example 12.1. Consider the nonlinear switched dynamical system given by

$$\dot{x}_1(t) = f_{\sigma(t)}(x_2(t)) - g_{\sigma(t)}(x_1(t)), \quad x_1(0) = x_{10}, \quad t \geq 0, \quad \sigma(t) \in \mathcal{S}, \tag{12.9}$$

$$\dot{x}_2(t) = g_{\sigma(t)}(x_1(t)) - f_{\sigma(t)}(x_2(t)), \quad x_2(0) = x_{20}, \tag{12.10}$$

where $x_1, x_2 \in \mathbb{R}$, $\sigma : [0, \infty) \to \mathcal{S}$ is a piecewise constant switching signal, $\mathcal{S}$ is a finite index set denoting the set of switching signals, for every $\sigma \in \mathcal{S}$, $f_\sigma(\cdot)$ and $g_\sigma(\cdot)$ are Lipschitz continuous, $f_\sigma(x_2) - g_\sigma(x_1) = 0$ if and only if $x_1 = x_2$, and $(x_1 - x_2)(f_\sigma(x_2) - g_\sigma(x_1)) \leq 0$, $x_1, x_2 \in \mathbb{R}$. Note that $f^{-1}(0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha \in \mathbb{R}\}$. To show that (12.9) and (12.10) is semistable, consider the Lyapunov function candidate $V(x_1 - \alpha, x_2 - \alpha) = \frac{1}{2}(x_1 - \alpha)^2 + \frac{1}{2}(x_2 - \alpha)^2$, where $\alpha \in \mathbb{R}$. Now, it follows that

$$\begin{aligned} \dot{V}(x_1 - \alpha, x_2 - \alpha) &= (x_1 - \alpha)[f_\sigma(x_2) - g_\sigma(x_1)] + (x_2 - \alpha)[g_\sigma(x_1) - f_\sigma(x_2)] \\ &= x_1[f_\sigma(x_2) - g_\sigma(x_1)] + x_2[g_\sigma(x_1) - f_\sigma(x_2)] \\ &= (x_1 - x_2)[f_\sigma(x_2) - g_\sigma(x_1)] \\ &\leq 0, \quad (x_1, x_2) \in \mathbb{R} \times \mathbb{R}, \end{aligned} \tag{12.11}$$

which, by Theorem 1 of [12], implies that $x_1 = x_2 = \alpha$ is Lyapunov stable for all $\alpha \in \mathbb{R}$.

Next, we rewrite (12.9) and (12.10) in the form of the differential inclusion (12.2) where $x \triangleq [x_1, x_2]^{\mathrm{T}} \in \mathbb{R}^2$ and $f(x) \triangleq [f_\sigma(x_2) - g_\sigma(x_1), g_\sigma(x_1) - f_\sigma(x_2)]^{\mathrm{T}}$. Let v_x be an arbitrary element of $\mathcal{K}[f](x)$ and recall that the Clarke upper generalized derivative of $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ along a vector $v_x \in \mathcal{K}[f](x)$ is given by $V^o(x, v_x) = x^{\mathrm{T}} v_x$. Note that the set $\mathcal{D}_{\mathrm{c}} \triangleq \{x \in \mathbb{R}^2 : V(x) \leq c\}$, where $c > 0$, is a compact set. Next, consider $\max V^o(x, v_x) \triangleq \max_{v_x \in \mathcal{K}[f]}\{x^{\mathrm{T}} v_x\}$. It follows from Theorem 1 of [193] and (12.11) that $x^{\mathrm{T}}\mathcal{K}[f](x) = \mathcal{K}[x^{\mathrm{T}} f](x) = \mathcal{K}[(x_1 - x_2)(f_\sigma(x_2) - g_\sigma(x_1))](x)$, and hence, by definition of a differential inclusion, it follows that

$\max V^o(x, v_x) = \max \overline{\mathrm{co}}\{(x_1 - x_2)(f_\sigma(x_2) - g_\sigma(x_1))\}$. Note that since, by (12.11), $(x_1 - x_2)(f_\sigma(x_2) - g_\sigma(x_1)) \leq 0$, $x \in \mathbb{R}^2$, it follows that $\max V^o(x, v_x)$ cannot be positive, and hence, the largest value $\max V^o(x, v_x)$ can achieve is zero.

Finally, let $\mathcal{R} \triangleq \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - x_2)(f_\sigma(x_2) - g_\sigma(x_1)) = 0\} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha \in \mathbb{R}\}$. Since $\mathcal{R}$ consists of equilibrium points, it follows that $\mathcal{M} = \mathcal{R}$. Note that $\max \mathcal{L}_f V(x) \leq \max V^o(x, v_x)$ for each $x \in \mathbb{R}^2$ [12]. Hence, it follows from Theorem 12.1 that $x_1 = x_2 = \alpha$ is semistable for all $\alpha \in \mathbb{R}$. $\triangle$

Example 12.2. Consider the discontinuous dynamical system given by

$$\dot{x}_1(t) = \mathrm{sign}(x_2(t) - x_1(t)), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{12.12}$$

$$\dot{x}_2(t) = \mathrm{sign}(x_1(t) - x_2(t)), \quad x_2(0) = x_{20}, \tag{12.13}$$

where $x_1, x_2 \in \mathbb{R}$, $\mathrm{sign}(x) \triangleq x/|x|$ for $x \neq 0$, and $\mathrm{sign}(0) \triangleq 0$. Let $f(x_1, x_2) \triangleq [\mathrm{sign}(x_2 - x_1), \mathrm{sign}(x_1 - x_2)]^{\mathrm{T}}$. Consider $V(x_1, x_2) = \frac{1}{2}(x_1 - \alpha)^2 + \frac{1}{2}(x_2 - \alpha)^2$, where $\alpha \in \mathbb{R}$. Since $V(x_1, x_2)$ is differentiable at $x = (x_1, x_2)$, it follows that $\mathcal{L}_f V(x_1, x_2) = [x_1 - \alpha, x_2 - \alpha]\mathcal{K}[f](x_1, x_2)$. Now, it follows from Theorem 1 of [193] that

$$\begin{aligned}
[x_1 - \alpha, x_2 - \alpha]\mathcal{K}[f](x) &= \mathcal{K}[[x_1 - \alpha, x_2 - \alpha]f](x) \\
&= \mathcal{K}[-(x_1 - x_2)\mathrm{sign}(x_1 - x_2)](x) \\
&= -(x_1 - x_2)\mathcal{K}[\mathrm{sign}(x_1 - x_2)](x) \\
&= -(x_1 - x_2)\mathrm{SGN}(x_1 - x_2) \\
&= -|x_1 - x_2|, \quad (x_1, x_2) \in \mathbb{R}^2,
\end{aligned} \tag{12.14}$$

where $\mathrm{SGN}(\cdot)$ is defined by ([193, 220])

$$\mathrm{SGN}(x) \triangleq \begin{cases} -1, & x < 0, \\ [-1, 1], & x = 0, \\ 1, & x > 0. \end{cases} \tag{12.15}$$

Hence, $\max \mathcal{L}_f V(x_1, x_2) \leq 0$ for all $(x_1, x_2) \in \mathbb{R}^2$. Now, it follows from Theorem 2 of [12] that $x_1 = x_2 = \alpha$ is Lyapunov stable. Finally, note that $0 \in \mathcal{L}_f V(x_1, x_2)$ if and only if

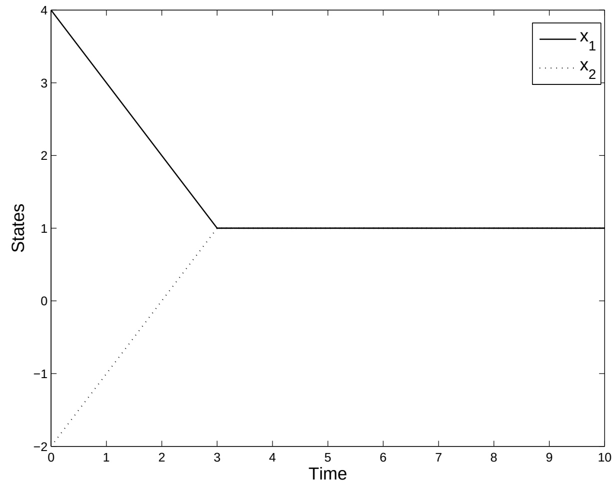

Figure 12.1: Solutions for Example 12.2

$x_1 = x_2$, and hence, $\mathcal{Z} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$. Since the largest weakly invariant subset $\mathcal{M}$ of $\mathcal{Z}$ is given by $\mathcal{M} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha \in \mathbb{R}\}$, it follows from Theorem 12.1 that (12.12) and (12.13) is semistable.

Finally, we show that (12.12) and (12.13) is finite-time-semistable. To see this, consider the nonnegative function $U(x_1, x_2) = |x_1 - x_2|$. Note that

$$\partial U(x_1, x_2) = \begin{cases} \{\text{sign}(x_1 - x_2)\} \times \{\text{sign}(x_2 - x_1)\}, & x_1 \neq x_2, \\ [-1, 1] \times [-1, 1], & x_1 = x_2. \end{cases} \tag{12.16}$$

Hence, it follows that

$$\mathcal{L}_f U(x_1, x_2) = \begin{cases} \{-2\}, & x_1 \neq x_2, \\ \{0\}, & x_1 = x_2, \end{cases} \tag{12.17}$$

which implies that $\max \mathcal{L}_f U(x_1, x_2) = -2 < 0$ for all $(x_1, x_2) \in \mathbb{R}^2 \backslash \mathcal{Z}$. Now, it follows from Corollary 12.1 that (12.12) and (12.13) is globally finite-time-semistable. Figure 12.1 shows the solutions of (12.12) and (12.13) for $x_{10} = 4$ and $x_{20} = -2$. $\triangle$

Note that in Theorem 12.1 and Corollary 12.1 Lyapunov stability is needed for semistability and finite-time semistability. However, finding the corresponding Lyapunov function can be a difficult task. To overcome this drawback, we generalize the nontangency-based approach of [32] to discontinuous dynamical systems in order to guarantee semistability and finite-time semistability by testing a condition on the vector field f, which avoids proving Lyapunov stability. Before we state this result, we need some new notation and definitions.

A set $\mathcal{E} \subseteq \mathbb{R}^q$ is *connected* if and only if every pair of open sets $\mathcal{U}_i \subseteq \mathbb{R}^q$, $i = 1, 2$, satisfying $\mathcal{E} \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$ and $\mathcal{U}_i \cap \mathcal{E} \neq \emptyset$, $i = 1, 2$, has a nonempty intersection. A *connected component* of the set $\mathcal{E} \subseteq \mathbb{R}^q$ is a connected subset of $\mathcal{E}$ that is not properly contained in any connected subset of $\mathcal{E}$. Given a set $\mathcal{E} \subseteq \mathbb{R}^q$, let $\mathrm{co}\,\mathcal{E}$ denote the union of the convex hulls of the connected components of $\mathcal{E}$, and let $\mathrm{coco}\,\mathcal{E}$ denote the cone generated by $\mathrm{co}\,\mathcal{E}$ [32].

Definition 12.4. Given $x \in \mathbb{R}^q$, the *direction cone* $\mathcal{F}_x$ of f at x relative to $\mathbb{R}^q$ is the intersection of the closed cones generated by the sets of the form $\bigcap_{\mu(\mathcal{S})=0} \mathrm{co}\{f(\mathcal{U}\backslash\mathcal{S})\}$, where $\mathcal{U} \subseteq \mathbb{R}^q$ is an open neighborhood of x. Let $z \in \mathcal{E} \subseteq \mathbb{R}^q$. A vector $v \in \mathbb{R}^q$ is *tangent* to $\mathcal{E}$ at $z \in \mathcal{E}$ if and only if there exist a sequence $\{z_i\}_{i=1}^{\infty}$ in $\mathcal{E}$ converging to z and a sequence $\{h_i\}_{i=1}^{\infty}$ of positive real numbers converging to zero such that $\lim_{i\to\infty} \frac{1}{h_i}(z_i - z) = v$. The *tangent cone* to $\mathcal{E}$ at z is the closed cone $T_z\mathcal{E}$ of all vectors tangent to $\mathcal{E}$ at z. Finally, the vector field f is *nontangent* to the set $\mathcal{E}$ at the point $z \in \mathcal{E}$ if and only if $T_z\mathcal{E} \cap \mathcal{F}_z \subseteq \{0\}$.

Definition 12.5. Given a point $x \in \mathbb{R}^q$ and a bounded open neighborhood $\mathcal{U} \subset \mathbb{R}^q$ of x, the *restricted prolongation under all Filippov solutions* of x with respect to $\mathcal{U}$ is the set $\mathcal{R}_x^{\mathcal{U}} \subseteq \overline{\mathcal{U}}$ of all subsequential limits of sequences of the form $\{\psi_i(t_i)\}$, where $\{t_i\}_{i=1}^{\infty}$ is a sequence in $[0, \infty)$, $\psi_i(\cdot)$ is a Filippov solution to (12.1) with $\psi_i(0) = x_i$, $i = 1, 2, \ldots$, and $\{x_i\}_{i=1}^{\infty}$ is a sequence in $\mathcal{U}$ converging to x such that the set $\{z \in \mathbb{R}^q : z = \psi_i(t), t \in [0, t_i], \psi_i(0) = x_i\}$ is contained in $\overline{\mathcal{U}}$ for every $i = 1, 2, \ldots$.

For the next result, we say a set $\mathcal{N} \subset \mathbb{R}^q$ is *weakly negatively invariant* if for every $x \in \mathcal{N}$, there exist $z \in \mathcal{N}$ and a Filippov solution $\psi(\cdot)$ to (12.1) with $\psi(0) = z$ such that $\psi(t) = x$ and $\psi(\tau) \in \mathcal{N}$ for all $\tau \in [0, t]$, where $t > 0$.

Lemma 12.1. Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set of (12.1). Furthermore, let $x \in \mathcal{D}$ and let $\mathcal{U} \subseteq \mathcal{D}$ be a bounded open neighborhood of x. Then $\mathcal{R}_x^{\mathcal{U}}$ is connected. Moreover, if x is an equilibrium point of (12.1), then $\mathcal{R}_x^{\mathcal{U}}$ is weakly negatively invariant.

Proof. The proof is similar to the proof of Proposition 6.1 of [32] and, hence, is omitted.

□

The following two lemmas and proposition are needed for the main result of this section.

Lemma 12.2. Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to (12.1) and let $V : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz continuous and regular on $\mathcal{D}$. Assume that $V(x) \geq 0$, $x \in \mathcal{D}$, $V(z) = 0$ for $z \in f^{-1}(0)$, and $\max \mathcal{L}_f V(x) \leq 0$ or $\mathcal{L}_f V(x) = \emptyset$ for all $x \in \mathcal{D}$. For every $z \in f^{-1}(0)$, let $\mathcal{N}_z$ denote the largest weakly negatively invariant connected subset of $\overline{\mathcal{Z}} \cap \mathcal{D}$ containing z, where $\mathcal{Z}$ is given by (12.8). Then for a bounded open neighborhood $\mathcal{V} \subset \mathcal{D}$ of z, $\mathcal{R}_z^{\mathcal{V}} \subseteq \mathcal{N}_z$.

Proof. Let $x \in f^{-1}(0)$ and let $\mathcal{V} \subset \mathcal{D}$ be a bounded open neighborhood of x. Consider $z \in \mathcal{R}_x^{\mathcal{V}}$. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{V}$ converging to x and $\{t_i\}_{i=1}^{\infty}$ a sequence in $[0, \infty)$ such that the sequence $\{\psi_i(t_i)\}_{i=1}^{\infty}$ converges to z and, for every i, $\psi_i(\tau) \in \overline{\mathcal{V}} \subset \mathcal{D}$ for every $\tau \in [0, t_i]$, where $\psi_i(\cdot)$ is a Filippov solution to (12.1) with $\psi_i(0) = x_i$. Since $\max \mathcal{L}_f V(y) \leq 0$ or $\mathcal{L}_f V(y) = \emptyset$ for all $y \in \mathcal{D}$, it follows from Lemma 1 of [12] that $\frac{\mathrm{d}}{\mathrm{d}t} V(\psi(t))$ exists almost everywhere $t \geq 0$ and $\frac{\mathrm{d}}{\mathrm{d}t} V(\psi(t)) \in \mathcal{L}_f V(\psi(t))$ for almost all $t \in [0, \tau]$, where $\psi(\cdot)$ is a Filippov solution to (12.1) with $\psi(0) = y$. Now, by assumption, $V(\psi(\tau)) - V(y) = \int_0^{\tau} \frac{\mathrm{d}}{\mathrm{d}t} V(\psi(s))\mathrm{d}s \leq 0$, $\tau \geq 0$, and hence, $V(\psi(\tau)) \leq V(y)$ for $y \in \mathcal{D}$ and $\tau \geq 0$.

Next, note that $V(z) = \lim_{i \to \infty} V(\psi_i(t_i)) \leq \lim_{i \to \infty} V(x_i) = V(x)$, and hence, $V(z) \leq V(x)$. Since $V(z) \geq 0$ and $V(x) = 0$ by assumption, it follows that $V(z) = V(x) = 0$. Hence, $\mathcal{R}_x^{\mathcal{V}} \subseteq V^{-1}(0) \cap \overline{\mathcal{V}} \subset V^{-1}(0)$. Now, it follows from Lemma 12.1 that $\mathcal{R}_x^{\mathcal{V}}$ is weakly negatively invariant and connected, and $x \in \mathcal{R}_x^{\mathcal{V}}$. Hence, $\mathcal{R}_x^{\mathcal{V}} \subseteq \mathcal{M}_x$, where $\mathcal{M}_x$ denotes the largest, weakly, negatively invariant connected subset of $V^{-1}(0)$.

Finally, we show that $\mathcal{M}_x \subseteq \mathcal{N}_x$. Let $z \in \mathcal{M}_x$ and let $t > 0$. By weak negative invariance, there exists $w \in \mathcal{M}_x$ such that $\psi(t) = z$ and $\psi(\tau) \in \mathcal{M}_x \subseteq V^{-1}(0)$ for all $\tau \in [0, t]$, where $\psi(\cdot)$ is a Filippov solution to (12.1) with $\psi(0) = w$. Thus, $V(\psi(\tau)) = V(x) = 0$ for every $\tau \in [0, t]$, and hence, $0 \in \mathcal{L}_f V(\psi(\tau))$ for every $\tau \in [0, t]$. Let $\{t_i\}_{i=1}^{\infty}$ be a sequence in $[0, t]$ converging to t. By the continuity of ψ, it follows that $\{\psi(t_i)\}_{i=1}^{\infty}$ is a sequence in $\mathcal{Z}$ that converges to z. Thus, $z \in \overline{\mathcal{Z}}$, and hence, $\mathcal{M}_x \subseteq \overline{\mathcal{Z}}$. Since $\mathcal{M}_x$ is weakly negatively invariant, connected, contains x, and is contained in $\mathcal{U}$, it follows that $\mathcal{M}_x \subseteq \mathcal{N}_x$. Hence, $\mathcal{R}_x^{\mathcal{V}} \subseteq \mathcal{M}_x \subseteq \mathcal{N}_x$. □

Lemma 12.3. Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set of (12.1). Furthermore, let $x \in \mathcal{D}$ and let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{D}$ converging to x. Let $\mathcal{I}_i \subseteq [0, \infty)$, $i = 1, 2, \ldots$, be intervals containing 0, and let $\mathcal{B} \subseteq \mathcal{D}$ be the set of all subsequential limits contained in $\mathcal{D}$ of sequences of the form $\{\psi_i(\tau_i)\}_{i=1}^{\infty}$, where, for each i, $\tau_i \in \mathcal{I}_i$ and $\psi_i : \mathcal{I}_i \to \mathcal{D}$ is a Filippov solution of (12.1) satisfying $\psi_i(0) = x_i$. Then $\mathcal{B} = \{x\}$ if and only if f is nontangent to $\mathcal{B}$ at x.

Proof. First, we note that $x \in \mathcal{B}$ since $x = \lim_{i\to\infty} \psi_i(0)$. Necessity now follows by noting that if $\mathcal{B} = \{x\}$, then $T_x\mathcal{B} = \{0\}$ and, hence, $T_x\mathcal{B} \cap \mathcal{F}_x \subseteq \{0\}$.

To prove sufficiency, let $\{\mathcal{U}_k\}_{k=0}^{\infty}$ be a nested sequence of open neighborhoods of x in $\mathcal{D}$, contained in $\mathcal{U}$, and such that $\overline{\mathcal{U}_{k+1}} \subset \mathcal{U}_k$ and $x_k \in \mathcal{U}_k$ for every $k = 1, 2, \ldots$, $\bigcap_k \mathcal{U}_k = \{x\}$ and $z_0 \notin \mathcal{U}_1$. Since $z_0 \in \mathcal{B}$, there exists a sequence $\{\tau_i\}_{i=1}^{\infty}$ such that $\tau_i \in \mathcal{I}_i$ for every i, and $\lim_{i\to\infty} \psi_i(\tau_i) = z_0 \notin \mathcal{U}_1$. The continuity of Filippov solutions implies that, for every k, there exists a sequence $\{h_j^k\}_{j=k}^{\infty}$ in $[0, \infty)$ such that, for every $j \geq k$, $h_j^k \in \mathcal{I}_j$, $h_j^k \leq \tau_j$, $\psi_j(\tau) \in \mathcal{U}_k$ for every $\tau \in [0, h_j^k)$, and $\psi_j(h_j^k) \in \partial\mathcal{U}_k$. For each k, let $z_k \in \partial\mathcal{U}_k$ be a subsequential limit of the relatively bounded sequence $\{\psi_j(h_j^k)\}_{j=k}^{\infty}$. Then, for every k, it follows that $z_k \in \mathcal{B}$, $z_k \neq x$ and $\lim_{k\to\infty} z_k = x$. Now, consider a subsequential limit v of the bounded sequence

$\{\|z_k - x\|^{-1}(z_k - x)\}$. Clearly, $v \in T_x\mathcal{B}$. Also $\|v\| = 1$ so that $v \neq 0$. We claim that $v \in \mathcal{F}_x$.

Let $\mathcal{V} \subseteq \mathcal{D}$ be an open neighborhood of x and consider $\varepsilon > 0$. By construction, there exists k such that $\|v - \|z_k - x\|^{-1}(z_k - x)\| < \varepsilon/3$. Moreover, since $\bigcap_i \mathcal{U}_i = \{x\}$, we can assume that $\mathcal{U}_k \subseteq \mathcal{V}$. Since z_k belongs to the boundary of a relatively open neighborhood of x, $\delta \triangleq \|z_k - x\| > 0$. Since $z_k = \lim_{i\to\infty} \psi_i(h_i^k)$ and $x = \lim_{i\to\infty} x_i$, there exists i such that $x_i \in \mathcal{V}$, $\|x - x_i\| < \varepsilon\delta/3$ and $\|z_k - \psi_i(h_i^k)\| < \varepsilon\delta/3$. Let $\mathcal{S} \subset \mathcal{D}$ be a zero measure set. Then, $\mathcal{K}[f](\psi_i(\tau)) \subseteq \mathrm{co}\{f(\mathcal{V}\backslash\mathcal{S})\}$ for all $\tau \in [0, h_i^k]$. Therefore, it follows from Theorem I.6.13 of [235, p. 145] that $w \triangleq \psi_i(h_i^k) - x_i = \int_0^{h_i^k} \dot{\psi}_i(\tau)d\tau$ is contained in the convex cone generated by $\mathrm{co}\{f(\mathcal{V}\backslash\mathcal{S})\}$. Since $\mathcal{S}$ was chosen to be an arbitrary zero-measure set, it follows that $w \in \bigcap_{\mu(\mathcal{S})=0} \mathrm{co}\{f(\mathcal{V}\backslash\mathcal{S})\}$.

Now,

$$\begin{aligned}\left\|v - \delta^{-1}w\right\| &= \left\|v - \delta^{-1}(z_k - x) - \delta^{-1}(\psi(h_i^k, x_i) - z_k) - \delta^{-1}(x - x_i)\right\| \\ &\leq \left\|v - \|z_k - x\|^{-1}(z_k - x)\right\| + \delta^{-1}\|\psi(h_i^k, x_i) - z_k\| + \delta^{-1}\|x - x_i\| \\ &< \varepsilon.\end{aligned}$$

We have thus shown that, for every $\varepsilon > 0$ there exists $w \in \bigcap_{\mu(\mathcal{S})=0} \mathrm{co}\{f(\mathcal{V}\backslash\mathcal{S})\}$ and $\delta > 0$ such that $w \neq 0$ and $\|v - \delta^{-1}w\| < \varepsilon$. It follows that v is contained in the closed cone generated by $\bigcap_{\mu(\mathcal{S})=0} \mathrm{co}\{f(\mathcal{V}\backslash\mathcal{S})\}$. Since $\mathcal{V}$ was chosen to be an arbitrary open neighborhood of x, it follows that v is contained in $\mathcal{F}_x$. Thus, if $\mathcal{B} \neq \{x\}$, then there exists $v \in \mathbb{R}^q$ such that $v \neq 0$ and $v \in T_x\mathcal{B} \cap \mathcal{F}_x$, that is, f is not nontangent to $\mathcal{B}$ at x. Sufficiency now follows.

□

Proposition 12.2. Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set of (12.1). Furthermore, let $x \in \mathcal{D}$ and let $\mathcal{U} \subseteq \mathcal{D}$ be a bounded open neighborhood of x. If the vector field f of (12.1) is nontangent to $\mathcal{R}_x^{\mathcal{U}}$ at x, then the point x is a Lyapunov stable equilibrium of (12.1).

Proof. Since f is nontangent to $\mathcal{R}_x^{\mathcal{U}}$ at x, by definition, it follows that $T_x\mathcal{R}_x^{\mathcal{U}} \cap \mathcal{F}_x \subseteq \{0\}$. Let $z \in \mathcal{R}_x^{\mathcal{U}}$. Then there exist a sequence $\{x_i\}_{i=1}^{\infty}$ converging to x and a sequence $\{t_i\}_{i=1}^{\infty}$ in $[0, \infty)$ such that $\mathcal{Q}_i \triangleq \{y \in \mathbb{R}^q : y = \psi_i(t), t \in [0, t_i], \psi_i(0) = x_i\}$ is contained in $\overline{\mathcal{U}}$ for every $i = 1, 2, \ldots$ and $\lim_{i\to\infty} \psi_i(t_i) = z$.

First, suppose that the sequence $\{t_i\}_{i=1}^{\infty}$ converges to 0. Then it follows from Theorem 11 of [75] that there exists a Filippov solution $\hat{\psi}(\cdot)$ to (12.1) with $\hat{\psi}(0) = x$ such that $\lim_{i\to\infty} \psi_i(t_i) = \hat{\psi}(0) = x$. Next, suppose the sequence $\{t_i\}_{i=1}^{\infty}$ does not converge to 0. Then there exists a subsequence $\{t_{i_k}\}_{k=1}^{\infty}$ of the sequence $\{t_i\}_{i=1}^{\infty}$ such that $\liminf_{k\to\infty} t_{i_k} > 0$. Let $\mathcal{I}_k \triangleq [0, t_{i_k}]$ for each k and let $\mathcal{B} \subseteq \overline{\mathcal{U}}$ denote the set of all subsequential limits of sequences of the form $\{\psi_{i_k}(\tau_k)\}_{k=1}^{\infty}$, where $\tau_k \in \mathcal{I}_k$ for every k. By construction, $z \in \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{R}_x^{\mathcal{U}}$. Hence, $T_x\mathcal{B} \cap \mathcal{F}_x \subseteq T_x\mathcal{R}_x^{\mathcal{U}} \cap \mathcal{F}_x \subseteq \{0\}$, that is, f is nontangent to $\mathcal{B}$ at x. Now, it follows from Lemma 12.3 that $\mathcal{B} = \{x\}$. Hence, $z = x$. Since $z \in \mathcal{R}_x^{\mathcal{U}}$ is arbitrary, it follows that $\mathcal{R}_x^{\mathcal{U}} = \{x\}$.

Suppose, *ad absurdum*, that x is not a Lyapunov stable equilibrium. Then there exist a bounded open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of x, a sequence $\{x_i\}_{i=1}^{\infty}$ in $\mathcal{V}$ converging to x, and a sequence $\{t_i\}_{i=1}^{\infty}$ in $[0, \infty)$ such that $\psi_i(t_i) \in \partial\mathcal{V}$ for every i. Without loss of generality, we can assume that the sequence $\{t_i\}_{i=1}^{\infty}$ is chosen such that, for every i, $\psi_i(h) \in \mathcal{V}$ for all $h \in [0, t_i)$. Now, every subsequential limit of the bounded sequence $\{\psi_i(t_i)\}_{i=1}^{\infty}$ is distinct from x by construction and is contained in $\mathcal{R}_x^{\mathcal{U}}$ by definition, which implies that $\mathcal{R}_x^{\mathcal{U}} \backslash \{x\} \neq \emptyset$. This contradicts the assumption. Hence, x is Lyapunov stable. □

The following theorem gives sufficient conditions for semistability using nontangency of the vector field f.

Theorem 12.2. Let $\mathcal{D} \subseteq \mathbb{R}^q$ be a strongly invariant set with respect to (12.1) and let $V : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz continuous and regular on $\mathcal{D}$. Assume that $V(x) \geq 0$, $x \in \mathcal{D}$, $V(z) = 0$ for $z \in f^{-1}(0)$, and $\max \mathcal{L}_f V(x) \leq 0$ or $\mathcal{L}_f V(x) = \emptyset$ for all $x \in \mathcal{D}$. Furthermore, for every $z \in f^{-1}(0)$, let $\mathcal{N}_z$ denote the largest weakly negatively invariant connected subset of $\overline{\mathcal{Z}} \cap \mathcal{D}$ containing z, where $\mathcal{Z}$ is given by (12.8). If f is nontangent to $\mathcal{N}_z$ at the point $z \in f^{-1}(0)$, then (12.1) is semistable with respect to $\mathcal{D}$.

Proof. Let $\mathcal{V} \subset \mathcal{D}$ be a bounded open neighborhood of $x \in f^{-1}(0)$. Since f is nontangent to $\mathcal{N}_x$ at the point $x \in f^{-1}(0) \cap \mathcal{V}$, it follows that $T_x\mathcal{N}_x \cap \mathcal{F}_x \subseteq \{0\}$. Next, we show that f is nontangent to $\mathcal{R}_x^{\mathcal{V}}$ at the point x. It follows from Lemma 12.2 that $\mathcal{R}_x^{\mathcal{V}} \subseteq \mathcal{N}_x$. Hence, $T_x\mathcal{R}_x^{\mathcal{V}} \cap \mathcal{F}_x \subseteq T_x\mathcal{N}_x \cap \mathcal{F}_x \subseteq \{0\}$, that is, $T_x\mathcal{R}_x^{\mathcal{V}} \cap \mathcal{F}_x \subseteq \{0\}$. By definition, f is nontangent to $\mathcal{R}_x^{\mathcal{V}}$ at the point x. Now, it follows from Proposition 12.2 that x is a Lyapunov stable equilibrium. Since x was chosen arbitrarily, it follows that (12.1) is Lyapunov stable.

By Lyapunov stability of x and local compactness of $\mathcal{D}$, it follows that there exists a strongly invariant neighborhood $\mathcal{U} \subset \mathcal{V}$ of x that is open and bounded, and such that $\overline{\mathcal{U}} \subset \mathcal{V}$. For every $z \in \mathcal{U}$, every Filippov solution $\psi(\cdot)$ to (12.1) with $\psi(0) = z$ is bounded in $\mathcal{D}$. Hence, it follows from [76, p. 129] and Theorem 3 of [12] that $\Omega(\psi) \subseteq \overline{\mathcal{U}}$ is nonempty and contained in $\overline{\mathcal{Z}}$. The invariance of $\Omega(\psi)$ implies that $\Omega(\psi)$ is contained in the largest weakly invariant subset $\mathcal{N}$ of $\overline{\mathcal{Z}} \cap \mathcal{D}$. Since every weakly invariant set is also negatively weakly invariant, it follows that $\Omega(\psi) \subseteq \mathcal{N}$ for every $z \in \mathcal{U}$. Let $z \in \mathcal{U}$ and $w \in \Omega(\psi)$. Since $\Omega(\psi)$ is connected and contained in $\mathcal{N}$, it follows that $\Omega(\psi) \subseteq \mathcal{N}_w$. Hence, $T_w\Omega(\psi) \cap \mathcal{F}_w \subseteq T_w\mathcal{N}_w \cap \mathcal{F}_w \subseteq \{0\}$. Now, it follows from Proposition 12.1 that $\lim_{t\to\infty} \psi(t)$ exists. Since $z \in \mathcal{U}$ was chosen arbitrarily, it follows that every Filippov solution in $\mathcal{U}$ converges to a limit. The strong invariance of $\mathcal{U}$ implies that the limit of every Filippov solution in $\mathcal{U}$ is contained in $\overline{\mathcal{U}}$. Since every equilibrium in $\overline{\mathcal{U}} \subset \mathcal{V}$ is Lyapunov stable, it follows from Theorem 12.1 that x is semistable. Finally, since x was chosen arbitrarily, it follows that (12.1) is semistable. □

Example 12.3. Consider the discontinuous dynamical system given by

$$\dot{x}_1(t) = \operatorname{sign}(x_3(t) - x_4(t)), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{12.18}$$

$$\dot{x}_2(t) = \operatorname{sign}(x_4(t) - x_3(t)), \quad x_2(0) = x_{20}, \tag{12.19}$$

$$\dot{x}_3(t) = \operatorname{sign}(x_4(t) - x_3(t)) + \operatorname{sign}(x_2(t) - x_1(t)), \quad x_3(0) = x_{30}, \tag{12.20}$$

$$\dot{x}_4(t) = \operatorname{sign}(x_3(t) - x_4(t)) + \operatorname{sign}(x_1(t) - x_2(t)), \quad x_4(0) = x_{40}, \tag{12.21}$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ denote the vector field of (12.18)–(12.21) and $x \triangleq [x_1, x_2, x_3, x_4] \in \mathbb{R}^4$. Consider the function $V(x) = |x_1 - x_2| + |x_3 - x_4|$. Note that

$$\partial V(x) = \begin{cases} \{\operatorname{sign}(x_1 - x_2)\} \times \{\operatorname{sign}(x_2 - x_1)\} & \\ \times\{\operatorname{sign}(x_3 - x_4)\} \times \{\operatorname{sign}(x_4 - x_3)\}, & x_1 \neq x_2,\ x_3 \neq x_4, \\ [-1,1] \times [-1,1] \times \{\operatorname{sign}(x_3 - x_4)\} \times \{\operatorname{sign}(x_4 - x_3)\}, & x_1 = x_2,\ x_3 \neq x_4, \\ \{\operatorname{sign}(x_1 - x_2)\} \times \{\operatorname{sign}(x_2 - x_1)\} \times [-1,1] \times [-1,1], & x_1 \neq x_2,\ x_3 = x_4, \\ \overline{\operatorname{co}}\{(1,1), (-1,1), (-1,-1), (1,-1)\}, & x_1 = x_2,\ x_3 = x_4. \end{cases}$$

Hence,

$$\mathcal{L}_f V(x) = \begin{cases} \{-2\}, & x_1 \neq x_2,\ x_3 \neq x_4, \\ \emptyset, & x_1 = x_2,\ x_3 \neq x_4, \\ \emptyset, & x_1 \neq x_2,\ x_3 = x_4, \\ \{0\}, & x_1 = x_2,\ x_3 = x_4, \end{cases} \tag{12.22}$$

which implies that $\max \mathcal{L}_f V(x) \leq 0$ for $x \in \mathbb{R}^4$ and $\mathcal{Z} = \{x \in \mathbb{R}^4 : x_1 = x_2, x_3 = x_4\}$. Let $\mathcal{N}$ denote the largest weakly, negatively invariant subset contained in $\mathcal{Z}$. On $\mathcal{N}$, it follows from (12.18)–(12.21) that $\dot{x}_1 = \dot{x}_2 = 0$ and $\dot{x}_3 = \dot{x}_4 = 0$. Hence, $\mathcal{N} = \{x \in \mathbb{R}^4 : x_1 = x_2 = a, x_3 = x_4 = b\}$, $a, b \in \mathbb{R}$, which implies that $\mathcal{N}$ is the set of equilibrium points.

Next, we show that f for (12.18)–(12.21) is nontangent to $\mathcal{N}$ at the point $z \in \mathcal{N}$. To see this, note that the tangent cone $T_z\mathcal{N}$ to the equilibrium set $\mathcal{N}$ is orthogonal to the vectors $\mathbf{u}_1 \triangleq [1, -1, 0, 0]^{\mathrm{T}}$ and $\mathbf{u}_2 \triangleq [0, 0, 1, -1]^{\mathrm{T}}$. On the other hand, since $f(z) \in \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ for all $z \in \mathbb{R}^4$, it follows that $f(\mathcal{V}) \subseteq \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ for every subset $\mathcal{V} \subseteq \mathbb{R}^4$. Consequently, the direction cone $\mathcal{F}_z$ of f at $z \in \mathcal{N}$ relative to $\mathbb{R}^4$ satisfies $\mathcal{F}_z \subseteq \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Hence, $T_z\mathcal{N} \cap \mathcal{F}_z = \{0\}$, which implies that the vector field f is nontangent to the set of equilibria $\mathcal{N}$ at the point $z \in \mathcal{N}$. Note that for every $z \in \mathcal{N}$, the set $\mathcal{N}_z$ required by Theorem 12.2 is contained in $\mathcal{N}$. Since nontangency to $\mathcal{N}$ implies nontangency to $\mathcal{N}_z$ at the point $z \in \mathcal{N}$, it follows from Theorem 12.2 that the system (12.18)–(12.21) is semistable.

Finally, note that $\max \mathcal{L}_f V(x) \leq -2 < 0$ or $\mathcal{L}_f V(x) = \emptyset$ for all $x \in \mathbb{R}^4 \backslash \mathcal{Z}$, it follows from Corollary 12.1 that (12.18)–(12.21) is globally finite-time-semistable. Figure 12.2 shows the solutions of (12.18)–(12.21) for $x_{10} = 4$, $x_{20} = -2$, $x_{30} = 1$, and $x_{40} = -3$. $\triangle$

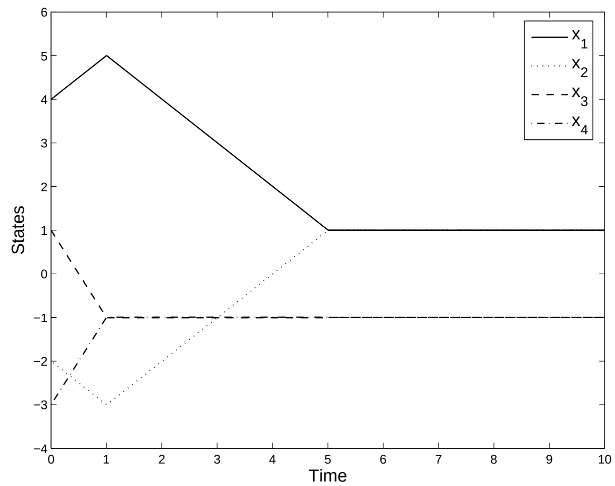

Figure 12.2: Solutions for Example 12.3

12.4. Time-Varying Discontinuous Dynamical Systems

In this and the next section, we consider time-varying differential equations given by

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{12.23}$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^q$, and $f : \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ is Lebesgue measurable and locally essentially bounded [75,76]. We assume that the equilibrium set $\mathcal{E} \triangleq \{x \in \mathbb{R}^q : f(t,x) = 0 \text{ for all } t \in \mathbb{R}\}$ is closed. An absolutely continuous function $x : [t_0, \tau] \to \mathbb{R}^q$ is said to be a *Filippov solution* [75,76] of (12.23) on the interval $[t_0, \tau]$ with initial condition $x(t_0) = x_0$, if $x(t)$ satisfies

$$\dot{x}(t) \in \mathcal{K}[f](t, x(t)), \quad \text{a.a.} \quad t \in [t_0, \tau], \tag{12.24}$$

where the Filippov set-valued map $\mathcal{K}[f] : [0, \infty) \times \mathbb{R}^q \to \mathcal{B}(\mathbb{R}^q)$ is defined by

$$\mathcal{K}[f](t,x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(\mathcal{S})=0} \overline{\text{co}}\, \{f(t, \mathcal{B}_\delta(x) \backslash \mathcal{S})\}, \quad (t,x) \in [t_0, \infty) \times \mathbb{R}^q. \tag{12.25}$$

Note that it follows from [59] that there exists a set $\mathcal{N}_f \subset \mathbb{R}^q$ of measure zero such that

$$\mathcal{K}[f](t,x) = \overline{\text{co}} \left\{ \lim_{i \to \infty} f(t, x_i) : x_i \to x,\, x_i \notin \mathcal{N}_f \cup \mathcal{W} \right\}, \tag{12.26}$$

where $\mathcal{W} \subset \mathbb{R}^q$ is an arbitrary set of measure zero. Since the Filippov set-valued map given by (12.25) is upper semicontinuous with nonempty, convex, and compact values, and is also locally bounded, it follows that Filippov solutions to (12.23) exist [76].

Let $\mathcal{S}$ be a given closed subset of $\mathbb{R}^q$. Then the pair $(\mathcal{S}, \mathcal{K}[f](t,x))$ is called *weakly invariant* (resp., *strongly invariant*) if for all initial conditions (t_0, x_0) with $x_0 \in \mathcal{S}$, $\mathcal{S}$ contains a Fillippov solution (resp., all Filippov solutions) $x(\cdot)$ of (12.2) on $[t_0, \infty)$ satisfying $x(t_0) = x_0$. Recall that an equilibrium point $x_{\rm e} \in \mathcal{E}$ of (12.23) is an equilibrium point of (12.24) if and only if $0 \in \mathcal{K}[f](t, x_{\rm e})$ for all $t \in [0, \infty)$. An equilibrium point $x_{\rm e} \in \mathcal{E}$ of (12.23) is *Lyapunov stable* if for every $t_0 \in \mathbb{R}$ and every $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that for every $\|x_0 - x_{\rm e}\| \leq \delta$, the Fillippov solutions $x(t)$, $t \geq t_0$, with the initial condition $x(t_0) = x_0$ satisfy $\|x(t) - x_{\rm e}\| < \varepsilon$ for all $t \geq t_0$. An equilibrium point $x_{\rm e} \in \mathcal{E}$ of (12.23) is *uniformly Lyapunov stable* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for every $\|x_0 - x_{\rm e}\| \leq \delta$, the Fillippov solutions $x(t)$, $t \geq t_0$, with the initial condition $x(t_0) = x_0$ satisfy $\|x(t) - x_{\rm e}\| < \varepsilon$ for all $t \geq t_0$ and for all $t_0 \in \mathbb{R}$. The following definitions are needed.

Definition 12.6. *i*) An equilibrium point $x_{\rm e} \in \mathcal{E}$ of (12.23) is *weakly semistable* (resp., *semistable*) if for every $t_0 \in \mathbb{R}$, $x_{\rm e}$ is Lyapunov stable and there exists $\delta = \delta(t_0) > 0$ such that for every $\|x_0 - x_{\rm e}\| \leq \delta$, a Fillippov solution (resp., every Filippov solution) $x(t)$, $t \geq t_0$, with the initial condition $x(t_0) = x_0$ satisfies $\lim_{t\to\infty} x(t) = z$ and $z \in \mathcal{E}$ is a Lyapunov stable equilibrium point. The system (12.23) is *weakly semistable* (resp., *semistable*) if all the equilibrium points of (12.23) are weakly semistable (resp., semistable).

ii) An equilibrium point $x_{\rm e} \in \mathcal{E}$ of (12.23) is *uniformly weakly semistable* (resp., *uniformly semistable*) if $x_{\rm e}$ is uniformly Lyapunov stable and there exists $\delta > 0$ such that for every $\|x_0 - x_{\rm e}\| \leq \delta$, a Filippov solution (resp., every Filippov solution) $x(t)$, $t \geq t_0$, with the initial condition $x(t_0) = x_0$ satisfies $\lim_{t\to\infty} x(t) = z$ uniformly in $t_0 \in \mathbb{R}$, that is, for every $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that $\|x(t)\| < \varepsilon$ for every $t \geq t_0 + T(\varepsilon)$ and every $x_0 \in \mathbb{R}^q$, and $z \in \mathcal{E}$ is a uniformly Lyapunov stable equilibrium point. The system (12.23) is *uniformly weakly semistable* (resp., *uniformly semistable*) if all the equilibrium points of (12.23) are uniformly weakly semistable (resp., uniformly semistable).

Definition 12.7 [56]. Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^q$. Given $u \notin \mathcal{S}$, let $x \in \mathcal{S}$ be such that $\|x-u\| = \inf_{s\in\mathcal{S}} \|s-u\|$. Then x is called a *projection* of u onto $\mathcal{S}$. The set of all such projections is denoted by $\mathrm{proj}(u,\mathcal{S})$. The vector $u-x$ (and all its nonnegative multiples) defines a *proximal normal direction* to $\mathcal{S}$ at x. The set of all vectors constructed in this way (for fixed x, by varying u) is called the *proximal normal cone* to $\mathcal{S}$ at x, and is denoted by $\mathcal{N}_{\mathcal{S}}^{P}(x)$.

Definition 12.8 [76]. The *contingent set* denoted by $\mathrm{Cont}(t_0,x_0)$ is the set of all limit points of the seuqences $\frac{x_i(t_i)-x_0}{t_i-t_0}$ as $t_i \to t_0$, where $x_i(\cdot)$ is a Filippov solution to (12.23) on $[t_0,t_i]$ satisfying $x_i(t_0)=x_0$, $i=1,2,\ldots$.

12.5. Lyapunov-Based Semistability Analysis for Time-Varying Discontinuous Dynamical Systems

In this section, we develop Lyapunov-based semistability theory for time-varying discontinuous dynamical systems of the form given by (12.23). The following lemmas are needed for the main results of this section.

Lemma 12.4. Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^q$. Assume that there exists $M>0$ such that for every $(t,x)\in\mathbb{R}^{q+1}$ and almost every $v\in\mathcal{K}[f](t,x)$, $\|v\|\leq M$. If $(\mathcal{S},\mathcal{K}[f](t,x))$ is weakly invariant, then $\mathcal{K}[f](t,x)\cap\mathrm{Cont}(t,x)\neq\emptyset$ for every $x\in\mathcal{S}$ and $t\geq t_0$.

Proof. Since $(\mathcal{S},\mathcal{K}[f](t,x))$ is weakly invariant, it follows that for every $x_0\in\mathcal{S}$ there exists a Filippov solution $x(\cdot)$ to (12.23) on $[t_0,\infty)$ such that $x(t)\in\mathcal{S}$ for all $t\geq t_0$, where $x(t_0)=x_0$. Hence, for a sequence $\{t_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} t_n = t_0$, it follows that there exist Filippov solutions $x_n(\cdot)$ to (12.23) on $[t_0,t_n]$ such that $x_n(t_n)\in\mathcal{S}$ with $x_n(t_0)=x_0$. Since $\|v\|\leq M$ for every $v\in\mathcal{K}[f](t,x)$, it follows that $\|\dot{x}_n(t)\|\leq M$ almost everywhere $t\geq t_0$, where $\dot{x}_n(t)\in\mathcal{K}[f](t,x_n(t))$. Note that $x_n(t_n)-x_0=\int_{t_0}^{t_n}\dot{x}_n(t)\mathrm{d}t$. Then it follows that $\|x_n(t_n)-x_0\|\leq M(t_n-t_0)$ for all $n=1,2,\ldots$. Hence, we can take a subsequence $\{t_{n_i}\}_{i=1}^{\infty}$ satisfying $\frac{x_{n_i}(t_{n_i})-x_0}{t_{n_i}-t_0}\to\nu$ as $n_i\to\infty$ for some ν. Note that $\nu\in\mathrm{Cont}(t_0,x_0)$ by definition. Next, we show that $\nu \in \mathcal{K}[f](t_0, x_0)$.

For a given $\delta > 0$ and all sufficiently large n_i, it follows that the set $\{x_{n_i}(t) : t_0 \leq t \leq t_{n_i}\}$ is contained in $\mathcal{B}_\delta(x_0)$. Furthermore, for a given $\varepsilon > 0$ and sufficiently small δ, it follows from Theorem 1 of [76, p. 87] that for $x \in \mathcal{B}_\delta(x_0)$ and $|t-t_0| < \sigma$, $\sigma > 0$, $\mathcal{K}[f](t, x) \subset \mathcal{K}[f](t_0, x_0) + \varepsilon\mathcal{B}$, where $A + \varepsilon\mathcal{B} \triangleq \{y : y \in \mathcal{B}_\varepsilon(x), x \in A\}$. Hence, for sufficiently large n_i, it follows from Theorem 1 of [76, p. 70] that $\frac{x_{n_i}(t_{n_i})-x_0}{t_{n_i}-t_0} \in \mathrm{Cont}(t_0, x_0) \subset \mathcal{K}[f](t, x) \subset \mathcal{K}[f](t_0, x_0) + \varepsilon\mathcal{B}$, which implies that $\nu \in \mathcal{K}[f](t_0, x_0) + \varepsilon\overline{\mathcal{B}}$, where $A + \varepsilon\overline{\mathcal{B}} \triangleq \{y : y \in \overline{\mathcal{B}}_\varepsilon(x), x \in A\}$. Since ε was chosen arbitrarily, it follows that $\nu \in \mathcal{K}[f](t_0, x_0)$. □

Lemma 12.5. Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^q$ and consider $(t, x) \in [t_0, t_0+a] \times \overline{\mathcal{B}}_b(x_0)$ for (12.24). Assume that for every $(t, z) \in [t_0, t_0 + d] \times \overline{\mathcal{B}}_b(x_0)$ there exists $w \in \mathrm{proj}(z, \mathcal{S})$ such that $\langle f(t, z), z - w\rangle \leq 0$, where $d = \min\{a, \frac{b}{m}\}$ and $m = \sup_{(t,x)\in[t_0,t_0+a]\times\overline{\mathcal{B}}_b(x_0)} \|\mathcal{K}[f](t, x)\|$. Then $\mathrm{dist}(x(t), \mathcal{S}) \leq \mathrm{dist}(x(t_0), \mathcal{S})$ for every $t \in [t_0, t_0 + d]$, where $x(\cdot)$ is a Filippov solution of (12.24) on $[t_0, t_0 + d]$ with $x(t_0) = x_0$.

Proof. First, it follows from Lemma 15 of [76, p. 66] that $m < \infty$. For $k = 1, 2, \ldots$, let $h_k = d/k$ and $t_{ki} = t_0 + ih_k$, $i = 0, 1, \ldots, k$. Next, construct an approximate solution $x_k(t)$ to (12.23) as follows: Let $x_k(t_{k0}) = x_0$. If for some $i \geq 0$ the value $x_k(t_{ki}) = x_{ki}$ is defined and $\|x_{ki} - x_0\| \leq m(t_{ki} - t_0)$, then define $x_k(t)$, $t_{ki} < t \leq t_{k,i+1}$, by $x_k(t) \triangleq x_{ki} + \int_{t_{ki}}^t f(s, x_{ki})\mathrm{d}s$. Hence, $x_k(t)$ is constructed successively on intervals $[t_{ki}, t_{k,i+1}]$, $i = 0, 1, \ldots, k - 1$. Furthermore, it follows that $\|x_k(t) - x_0\| \leq m(t - t_0)$, $t_{ki} < t \leq t_{k,i+1}$. Since $\dot{x}_k(t) = f(t, x_{ki}) \in \mathcal{K}[f](t, x_{ki})$, it follows that $\|\dot{x}_k(t)\| \leq m$ for almost all $t \geq t_0$. Hence, the functions $\{x_k(t)\}_{k=1}^\infty$ are uniformly bounded and equicontinuous. By the Arzelà-Ascoli theorem [49, p. 180] and Lemma 1 of [76, p. 76], there exists a subsequence of $x_k(t)$ uniformly converging to $x(t)$, where $x(\cdot)$ is a Filippov solution of (12.24) with $x(t_0) = x_0$.

Next, it follows that for each $i = 0, 1, \ldots, k$, there exists a point $w_{ki} \in \operatorname{proj}(x_{ki}, \mathcal{S})$ such that $\langle f(t, x_{ki}), x_{ki} - w_{ki}\rangle \leq 0$, $t_{ki} < t \leq t_{k,i+1}$. Hence,

$$\begin{aligned}(\operatorname{dist}(x_{k1}, \mathcal{S}))^2 \leq\ & \|x_{k1} - w_{k0}\|^2 \\ =\ & \|x_{k1} - x_{k0}\|^2 + \|x_{k0} - w_{k0}\|^2 + 2\langle x_{k1} - x_{k0}, x_{k0} - w_{k0}\rangle \\ \leq\ & m^2(t_{k1} - t_0)^2 + (\operatorname{dist}(x_0, \mathcal{S}))^2 + 2\int_{t_0}^{t_1} \langle f(t, x_0), x_0 - w_{k0}\rangle \mathrm{d}t \\ \leq\ & m^2(t_{k1} - t_0)^2 + (\operatorname{dist}(x_0, \mathcal{S}))^2. \end{aligned} \tag{12.27}$$

Similarly, $(\operatorname{dist}(x_{ki}, \mathcal{S}))^2 \leq (\operatorname{dist}(x_{k,i-1}, \mathcal{S}))^2 + m^2(t_{ki} - t_{k,i-1})^2$. Thus,

$$\begin{aligned}(\operatorname{dist}(x_{ki}, \mathcal{S}))^2 \leq\ & (\operatorname{dist}(x_0, \mathcal{S}))^2 + m^2 \sum_{r=1}^{i} (t_{kr} - t_{k,r-1})^2 \\ \leq\ & (\operatorname{dist}(x_0, \mathcal{S}))^2 + m^2 h_k d. \end{aligned} \tag{12.28}$$

Let $\{x_{n_k}(t)\}_{k=1}^{\infty}$ be a subsequence of $x_k(t)$ uniformly converging to $x(t)$. Note that $h_{n_k} \to 0$ as $n_k \to \infty$. Hence, taking the limit on both sides of (12.28) yields $\operatorname{dist}(x(t), \mathcal{S}) \leq \operatorname{dist}(x(t_0), \mathcal{S})$ for every $t \in [t_0, t_0 + d]$. $\square$

Next, we present necessary and sufficient conditions for characterizing weak invariance. It is important to note that our results are different from the results in [56, 67] since the Lipschitz conditions in [56, 67] do not hold for the nonautonomous differential inclusion discussed in this section; see Examples 12.4 and 12.5 below. A similar observation holds for Proposition 12.6 below.

Proposition 12.3. Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^q$. Assume that there exists $M > 0$ such that for every $(t, x) \in \mathbb{R}^{q+1}$ and almost every $v \in \mathcal{K}[f](t, x)$, $\|v\| \leq M$. Then $(\mathcal{S}, \mathcal{K}[f](t, x))$ is weakly invariant if and only if, for every $\zeta \in \mathcal{N}_{\mathcal{S}}^{P}(x)$,

$$\min_{v \in \mathcal{K}[f](t,x)} \langle \zeta, v\rangle \leq 0, \quad t \in \mathbb{R}, \quad x \in \mathcal{S}. \tag{12.29}$$

Proof. (Necessity.) Define the function f_{P} as follows. For every $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, choose any $w = w(x) \in \operatorname{proj}(x, \mathcal{S})$ and let $v \in \mathcal{K}[f](t, w)$ minimize the function $v \mapsto \langle v, x - w\rangle$ over

$\mathcal{K}[f](t,w)$. Set $f_{\mathrm{P}}(t,x)=v$, $x\in\mathbb{R}^n$, $t\in\mathbb{R}$. Since $x-w\in\mathcal{N}_{\mathcal{S}}^{P}(w)$, it follows from (12.29) that $\langle f_{\mathrm{P}}(t,x), x-w\rangle\le 0$. Note that $\|f_{\mathrm{P}}(t,x)\|=\|v\|\le M$, $x\in\mathbb{R}^n$, $t\in\mathbb{R}$. Hence, by taking $t_0=0$, $a=1$, and $b=M$ in Lemma 12.5, it follows that the Filippov solutions $x(\cdot)$ to $\dot{x}(t)=f_{\mathrm{P}}(t,x(t))$ with $x(0)=x_0$ on $[0,1]$ satisfy $\mathrm{dist}(x(t),\mathcal{S})\le\mathrm{dist}(x_0,\mathcal{S})$, which implies that if $x_0\in\mathcal{S}$, then $x(t)\in\mathcal{S}$ for all $t\in[0,1]$. We can extend $x(\cdot)$ to $[0,\infty)$ by considering the interval $[n,n+1]$ successively for $n=1,2,\ldots$.

To complete the proof, we need to show that $x(\cdot)$ is a Filippov solution to (12.24). Define the Filippov set-valued map $\mathcal{K}_{\mathcal{S}}[f](t,x)$ by $\mathcal{K}_{\mathcal{S}}[f](t,x)\triangleq\mathrm{co}\{\mathcal{K}[f](t,w): w\in\mathrm{proj}(x,\mathcal{S})\}$. We claim that $\mathcal{K}_{\mathcal{S}}[f](t,x)=\mathcal{K}[f](t,x)$ for $x\in\mathcal{S}$. To see this, note that if $x\in\mathcal{S}$, then $w=x\in\mathcal{S}$. Hence, it follows from the definition of differential inclusions that $\mathcal{K}_{\mathcal{S}}[f](t,x)=\mathrm{co}\{\mathcal{K}[f](t,x): x\in\mathcal{S}\}=\mathcal{K}[f](t,x)$. Next, since $f_{\mathrm{P}}\in\mathcal{K}_{\mathcal{S}}[f]$, it follows that $\mathcal{K}[f_{\mathrm{P}}]\subseteq\mathcal{K}_{\mathcal{S}}[f]$. By definition, the Filippov solution $x(\cdot)$ of $\dot{x}(t)=f_{\mathrm{P}}(t,x(t))$ satisfies $\dot{x}(t)\in\mathcal{K}_{\mathcal{S}}[f](t,x(t))$ almost everywhere on $[0,1]$ with $x(0)=x_0$. Since $x(t)\in\mathcal{S}$ on $[0,1]$ and $\mathcal{K}_{\mathcal{S}}[f](t,x)=\mathcal{K}[f](t,x)$ for $x\in\mathcal{S}$, it follows that $x(\cdot)$ is a Filippov solution to (12.24).

(Sufficiency.) Suppose $(\mathcal{S},\mathcal{K}[f])$ is weakly invariant. Then it follows from Lemma 12.4 that $\mathcal{K}[f](t,x)\cap\mathrm{Cont}(t,x)\neq\emptyset$ for every $x\in\mathcal{S}$ and $t\ge t_0$. Next, we show that $\mathrm{Cont}(t,x)\subseteq\mathcal{H}_{\mathcal{S}}(x)\triangleq\{\eta\in\mathbb{R}^n:\langle\zeta,\eta\rangle\le 0,\zeta\in\mathcal{N}_{\mathcal{S}}^{P}(x)\}$ for $x\in\mathcal{S}$. To see this, choose $\nu\in\mathrm{Cont}(t_0,x_0)$. Then it follows that $\nu=\lim_{i\to\infty}\frac{x_i(t_i)-x_0}{t_i-t_0}$, where $t_i\to t_0$ as $i\to\infty$ and $t_i>t_0$. Let $\zeta\in\mathcal{N}_{\mathcal{S}}^{P}(x_0)$. Then $\langle\zeta,x_i(t_i)-x_0\rangle=\langle w-x_0,x_i(t_i)-x_0\rangle$, where $w\in\mathrm{proj}(x_0,\mathcal{S})$. Since $\|w-x_0\|\le\|x_i(t_i)-x_0\|$, it follows from the Cauchy-Schwarz inequality that $\langle w-x_0,x_i(t_i)-x_0\rangle\le\|w-x_0\|\cdot\|x_i(t_i)-x_0\|\le\|x_i(t_i)-x_0\|^2$. Hence, $\langle\zeta,\frac{x_i(t_i)-x_0}{t_i-t_0}\rangle\le\|x_i(t_i)-x_0\|\cdot\|\frac{x_i(t_i)-x_0}{t_i-t_0}\|$. Finally, note that since $\lim_{i\to\infty}x_i(t_i)=x_0$, it follows that $\langle\zeta,\nu\rangle=\langle\zeta,\lim_{i\to\infty}\frac{x_i(t_i)-x_0}{t_i-t_0}\rangle=\lim_{i\to\infty}\langle\zeta,\frac{x_i(t_i)-x_0}{t_i-t_0}\rangle\le\lim_{i\to\infty}\|x_i(t_i)-x_0\|\cdot\|\frac{x_i(t_i)-x_0}{t_i-t_0}\|=0\cdot\|\nu\|=0$, which implies that $\nu\in\mathcal{H}_{\mathcal{S}}(x_0)$. This shows that for every $\zeta\in\mathcal{N}_{\mathcal{S}}^{P}(x)$, $\langle\zeta,v\rangle\le 0$, where $v\in\mathcal{K}[f](t,x)\cap\mathrm{Cont}(t,x)$, which implies (12.29) holds. $\square$

The following propositions are needed for the main results of this section. For the first proposition recall that the *epigraph* of a function $f : \mathcal{X} \to \mathbb{R}$ is defined by the α-sublevel set $\mathrm{Ep}(f) \triangleq \{(x, \alpha) \in \mathcal{X} \times \mathbb{R} : f(x) \leq \alpha\}$ [208, p. 23].

Proposition 12.4. Assume that there exists $M > 0$ such that for every $(t, x) \in \mathbb{R}^{q+1}$ and almost every $v \in \mathcal{K}[f](t, x)$, $\|v\| \leq M$. Furthermore, assume that there exist a continuously differentiable function $V(\cdot)$ and a continuous function $W(\cdot)$ such that the following statements hold:

i) $\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|)$, $x \in \mathbb{R}^q$, where $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}_\infty$ functions.

ii) $\min_{v \in \mathcal{K}[f](t,x)} \langle \nabla V(x), v \rangle \leq -W(x)$ for all $x \in \mathbb{R}^q$ and $t \in \mathbb{R}$, where $W(x) \geq 0$ for all $x \in \mathbb{R}^q$.

Then $(V^{-1}([0, c]), \mathcal{K}[f](t, x))$ is weakly invariant and, for every $x_0 \in \mathbb{R}^q$, there exists a Filippov solution $x(\cdot)$ to (12.23) on $[t_0, \infty)$ with $x(t_0) = x_0$ such that $x(t) \to W^{-1}(0)$ as $t \to \infty$, where $c > 0$.

Proof. Since $V(\cdot)$ is continuously differentiable, it follows from Proposition 2 of [9, p. 32] that $\{\nabla V(x)\} = \partial V(x)$, $x \in \mathbb{R}^q$. Thus, it follows from *ii)* that $\min_{v \in \mathcal{K}[f](t,x)} \langle p, v \rangle \leq 0$, $p \in \partial V(x)$, $x \in \mathbb{R}^q$. Consider the epigraph of $V(\cdot)$ defined by $\mathrm{Ep}(V) \triangleq \{(x, z) \in \mathbb{R}^q \times \mathbb{R} : V(x) \leq z\}$. Note that $\mathrm{Ep}(V)$ is closed. Let $(\zeta, \lambda) \in \mathbb{R}^q \times \mathbb{R}$ belong to $\mathcal{N}^P_{\mathrm{Ep}(V)}(x, z)$ for some $(x, z) \in \mathrm{Ep}(V)$. We show that for $(\zeta, \lambda) \in \mathcal{N}^P_{\mathrm{Ep}(V)}(x, z)$, there exists $v \in \mathcal{K}[f](t, x)$ such that $\langle \zeta, v \rangle \leq 0$.

First, we show that $\lambda \leq 0$. Let y be in the domain of V and $(y^*, 0) \in \mathcal{N}^P_{\mathrm{Ep}(V)}(y, V(y))$ with $y^* \neq 0$. Without loss of generality, assume that $\|y^*\| = 1$. Then there exists $(x, V(y)) \notin \mathrm{Ep}(V)$ such that $\|(x, V(y)) - (y, V(y))\| = \inf_{(s, V(s)) \in \mathrm{Ep}(V)} \|(x, V(s)) - (s, V(s))\|$ and $(x - y)/\|x - y\| = y^*$, where $(y, V(y)) \in \mathrm{Ep}(V)$. By Proposition 2.1 of [203] we can assume, without loss of generality, that $(y^*, 0) \in \partial \mathrm{dist}((x, V(y)), \mathrm{Ep}(V))$. Note that for every $(\hat{x}, V(\hat{y}))$, it follows from the definition of an epigraph that $\mathrm{dist}((\hat{x}, V(\hat{y})), \mathrm{Ep}(V)) \leq \mathrm{dist}((\hat{x}, V(\hat{y}) - t), \mathrm{Ep}(V))$ for every $t > 0$. Suppose that there exists $(\hat{x}, V(\hat{y}))$ arbitrarily close to $(x, V(y))$ and $t > 0$ arbitrarily small so that $\mathrm{dist}((\hat{x}, V(\hat{y})), \mathrm{Ep}(V)) < \mathrm{dist}((\hat{x}, V(\hat{y}) - t), \mathrm{Ep}(V))$. Then it follows from Theorem 1.4 of [203] that there exists $(\zeta, \lambda) \in \partial\mathrm{dist}((\bar{x}, V(\bar{y})), \mathrm{Ep}(V))$, where $(\bar{x}, V(\bar{y}))$ is arbitrarily close to $(x, V(y))$ such that $\langle(\zeta, \lambda), (\hat{x}, V(\hat{y}) - t) - (\hat{x}, V(\hat{y}))\rangle > 0$, which implies that $\lambda < 0$. For the case where $\mathrm{dist}((\hat{x}, V(\hat{y})), \mathrm{Ep}(V)) = \mathrm{dist}((\hat{x}, V(\hat{y}) - t), \mathrm{Ep}(V))$, $t > 0$, it follows that $\langle(\zeta, \lambda), (\hat{x}, V(\hat{y}) - t) - (\hat{x}, V(\hat{y}))\rangle = 0$, which implies that $\lambda = 0$. Hence, $\lambda \leq 0$.

If $\lambda < 0$, then $(\zeta/(-\lambda), -1) \in \mathcal{N}^P_{\mathrm{Ep}(V)}(x, z)$, which implies that $-\zeta/\lambda \in \partial V(x)$. Now, it follows from *ii*) that there exists $v \in \mathcal{K}[f](t, x)$ such that $\langle(-\zeta/\lambda), v\rangle \leq 0$, and hence, $\langle\zeta, v\rangle \leq 0$. Alternatively, if $\lambda = 0$, then $(\zeta, 0) \in \mathcal{N}^P_{\mathrm{Ep}(V)}(x, V(x))$. Now, it follows from Theorem 2.4 of [203] that there exist sequences $\{(\zeta_i, -\varepsilon_i)\}_{i=1}^{\infty}$, with $\varepsilon_i > 0$, and $\{x_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty}(\zeta_i, -\varepsilon_i) = (\zeta, 0)$, $(\zeta_i, -\varepsilon_i) \in \mathcal{N}^P_{\mathrm{Ep}(V)}(x_i, V(x_i))$, and $\lim_{i\to\infty} x_i = x$. Using the above result for the case where $\lambda < 0$, it follows that there exists $v_i \in \mathcal{K}[f](t, x_i)$ such that $\langle\zeta_i, v_i\rangle \leq 0$. By assumption, the sequence $\{v_i\}_{i=1}^{\infty}$ is uniformly bounded. Hence, there exists a subsequence $\{n_i\}_{i=1}^{\infty}$ such that $\{v_{n_i}\}_{i=1}^{\infty}$ converges to the limit v. Furthermore, $v \in \mathcal{K}[f](t, x)$ since $\mathcal{K}[f]$ is upper semicontinuous. Thus, $\langle\zeta, v\rangle \leq 0$.

Since for $(\zeta, \lambda) \in \mathcal{N}^P_{\mathrm{Ep}(V)}(x, z)$, there exists $v \in \mathcal{K}[f](t, x)$ such that $\langle\zeta, v\rangle \leq 0$, it follows from Proposition 12.3 that the pair $(\mathrm{Ep}(V), \mathcal{K}[f] \times \{0\})$ is weakly invariant, and hence, for every $x_0 \in \mathbb{R}^q$, there exists a Filippov solution $x(\cdot)$ to (12.23) on $[t_0, \infty)$ with $x(t_0) = x_0$ such that $V(x(t)) \leq V(x_0)$ for all $t \geq t_0$, which implies that $(V^{-1}([0, c]), \mathcal{K}[f])$ is weakly invariant.

To show the second assertion, define a function $U : \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}$ by $U(x, y) \triangleq V(x) + y$ and a set-valued map $\mathcal{F}(t, x, y) \triangleq \mathcal{K}[f](t, x) \times \{y : y = W(x)\}$. We claim that for every $\alpha \in \mathbb{R}^q$, there exists a Filippov solution $z = (x, y)$ to the differential inclusion $\dot{z} \in \mathcal{F}(t, z)$ almost everywhere on $[t_0, \infty)$ with $x(t_0) = \alpha$ and $y(t_0) = 0$ such that $U(x(t), y(t)) \leq U(\alpha, 0)$ for all $t \geq t_0$. Let $(\zeta, \eta) \in \partial U(x, y)$. Then $\zeta \in \partial V(x)$ and $\eta = 1$. Since $\langle v, \zeta\rangle \leq -W(x)$ for some $v \in \mathcal{K}[f](t,x)$, it follows that $\langle v, \zeta \rangle + W(x) \leq 0$, or, equivalently, $\langle (v, W(x)), (\zeta, 1) \rangle \leq 0$. Using similar arguments as above, it can be shown that the pair $(\mathrm{Ep}(U), \mathcal{F} \times \{0\})$ is weakly invariant, which implies that for every $\alpha \in \mathbb{R}^q$, there exists a Filippov solution (x, y) to $\dot{z} \in \mathcal{F}(t, z)$ almost everywhere on $[t_0, \infty)$ with $x(t_0) = \alpha$ and $y(t_0) = 0$ such that $U(x(t), y(t)) \leq U(\alpha, 0)$ for all $t \geq t_0$. Note that $U(x(t), y(t)) \leq U(\alpha, 0)$ for $t \geq t_0$ implies that $V(x(t)) + \int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau \leq V(\alpha)$, where $x(\cdot)$ is a Filippov solution to (12.23). Hence, $V(x(t))$ and $\int_{t_0}^{t} W(x(\tau))\mathrm{d}\tau$ are bounded for almost all $t \geq t_0$. Furthermore, note that $\dot{x}(t)$ is uniformly bounded for almost all $t \geq t_0$. Now, using similar arguments as in the proof of Theorem 8.4 of [141], it can be shown that $x(t) \to W^{-1}(0)$ as $t \to \infty$. □

Proposition 12.5. Consider the time-varying discontinuous dynamical system (12.23). Assume that every point in $\mathcal{E}$ is Lyapunov stable. Furthermore, assume that, for a given $x_0 \in \mathbb{R}^q$, there exists a Filillpov solution to (12.23) satisfying $x(t) \to \mathcal{E}$ as $t \to \infty$. Then $x(t) \to z$ as $t \to \infty$, where $z \in \mathcal{E}$. Alternatively, assume that every point in $\mathcal{E}$ is uniformly Lyapunov stable and, for given $x_0 \in \mathbb{R}^q$, there exists a Filillpov solution to (12.23) satisfying $x(t) \rightrightarrows \mathcal{E}$ as $t \to \infty$. Then $x(t) \rightrightarrows z$ as $t \to \infty$, where $z \in \mathcal{E}$.

Proof. The proof is similar to the proof of Proposition 12.1 and, hence, is omitted. □

Next, we present sufficient conditions for weak semistability and uniform weak semistability for (12.23).

Theorem 12.3. Assume that there exists $M > 0$ such that for almost every $v \in \mathcal{K}[f](t,x)$, $\|v\| \leq M$. Furthermore, assume that there exist a continuously differentiable function $V(\cdot)$ and a continuous function $W(\cdot)$ such that *i*) and *ii*) of Proposition 12.4 hold, and $\mathcal{E} \subseteq W^{-1}(0)$. If every point in $W^{-1}(0)$ is a Lyapunov stable equilibrium of (12.23), then (12.23) is weakly semistable. Alternatively, if every point in $W^{-1}(0)$ is a uniformly Lyapunov stable equilibrium of (12.23), then (12.23) is uniformly weakly semistable.

Proof. It follows from Proposition 12.4 that there exists a Filippov solution $x(\cdot)$ to (12.23) such that $x(t) \to W^{-1}(0)$ as $t \to \infty$. Since every point in $W^{-1}(0)$ is a Lyapunov stable equilibrium of (12.23), it follows that $W^{-1}(0) \subseteq \mathcal{E}$. Furthermore, since, by assumption, $\mathcal{E} \subseteq W^{-1}(0)$, it follows that $W^{-1}(0) = \mathcal{E}$. Hence, $x(t) \to \mathcal{E}$ as $t \to \infty$ and every point in $\mathcal{E}$ is Lyapunov stable. Now, it follows from Proposition 12.5 that $x(t) \to z$ as $t \to \infty$, where $z \in \mathcal{E}$. By definition, (12.23) is weakly semistable. To show the second assertion, note that since $\dot{x}(t)$ is uniformly bounded, it follows using similar arguments as in the proof of Proposition 12.4 that $x(t) \rightrightarrows W^{-1}(0)$ as $t \to \infty$. Now, using similar arguments as above, it can be shown that (12.23) is uniformly weakly semistable. $\square$

Remark 12.1. If all the conditions in Theorem 12.3 are satisfied and (12.23) has a unique Filippov solution, then it follows from Theorem 12.3 that (12.23) is semistable. Sufficient conditions for guaranteeing uniqueness of Filippov solutions can be found in [59, 76].

Example 12.4. Consider the time-varying discontinuous dynamical system given by

$$\dot{x}_1(t) = \frac{1+2t^2}{1+t^2}\mathrm{sign}(x_2(t) - x_1(t)), \quad x_1(t_0) = x_{10}, \quad t \geq t_0, \tag{12.30}$$

$$\dot{x}_2(t) = \frac{1+2t^2}{1+t^2}\mathrm{sign}(x_1(t) - x_2(t)), \quad x_2(t_0) = x_{20}, \tag{12.31}$$

where $x_1, x_2 \in \mathbb{R}$. Note that, for $x = [x_1, x_2]^{\mathrm{T}}$,

$$\mathcal{K}[f](t,x) = \begin{cases} \{\frac{1+2t^2}{1+t^2}\} \times \{-\frac{1+2t^2}{1+t^2}\}, & x_2 > x_1, \\ \left[-\frac{1+2t^2}{1+t^2}, \frac{1+2t^2}{1+t^2}\right] \times \left[-\frac{1+2t^2}{1+t^2}, \frac{1+2t^2}{1+t^2}\right], & x_1 = x_2, \\ \{-\frac{1+2t^2}{1+t^2}\} \times \{\frac{1+2t^2}{1+t^2}\}, & x_1 > x_2, \end{cases} \quad t \geq t_0. \tag{12.32}$$

Clearly, $\|v\| \leq 2\sqrt{2}$ for almost all $v \in \mathcal{K}[f](t,x)$. Next, consider $V(x_1, x_2) = \frac{1}{2}(x_1 - \alpha)^2 + \frac{1}{2}(x_2 - \alpha)^2$, where $\alpha \in \mathbb{R}$. Then it follows from the time-dependent version of Theorem 1 of [193] that

$$\begin{aligned} [x_1 - \alpha, x_2 - \alpha]^{\mathrm{T}}\mathcal{K}[f](t,x) &= \mathcal{K}[[x_1 - \alpha, x_2 - \alpha]^{\mathrm{T}} f](t,x) \\ &= \mathcal{K}\left[-\frac{1+2t^2}{1+t^2}(x_1 - x_2)\mathrm{sign}(x_1 - x_2)\right](t,x) \end{aligned}$$

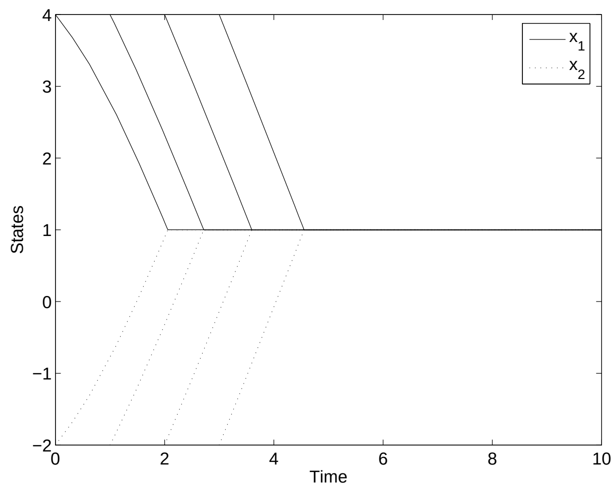

Figure 12.3: State trajectories versus time for Example 12.4

$$
\begin{aligned}
&= -\frac{1+2t^2}{1+t^2}(x_1 - x_2)\mathcal{K}[\mathrm{sign}(x_1 - x_2)](x) \\
&= -\frac{1+2t^2}{1+t^2}(x_1 - x_2)\mathrm{SGN}(x_1 - x_2) \\
&= -\frac{1+2t^2}{1+t^2}|x_1 - x_2|, \quad t \in \mathbb{R}, \quad (x_1, x_2) \in \mathbb{R}^2, \quad (12.33)
\end{aligned}
$$

which further implies that $\langle \nabla V(x_1, x_2), v\rangle \leq -|x_1 - x_2|$ for every $v \in \mathcal{K}[f](t, x)$. Now, it follows from Theorem 1 of [76, p. 153] that $x_1 = x_2 = \alpha$ is Lyapunov stable. In fact, it can be shown that $x_1 = x_2 = \alpha$ is uniformly Lyapunov stable. Next, let $W(x_1, x_2) = |x_1 - x_2|$ and note that $W^{-1}(0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\} = \mathcal{E}$. Now, it follows from Theorem 12.3 that (12.30) and (12.31) is weakly semistable. Moreover, it can be shown that (12.30) and (12.31) is uniformly weakly semistable. Figure 12.3 shows the solutions of (12.30) and (12.31) for $x_{10} = 4$, $x_{20} = -2$, and $t_0 = 0, 1, 2, 3$. △

The next proposition characterizes strong invariance of (12.23).

Proposition 12.6. Consider the time-varying discontinuous dynamical system (12.23). Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^q$ and assume that there exists $M > 0$ such that for every $(t, x) \in \mathbb{R}^{q+1}$, $\|f(t, x)\| \leq M$ for almost all $t \in \mathbb{R}$ and $x \in \mathbb{R}^q$. Then $(\mathcal{S}, \mathcal{K}[f](t, x))$ is strongly invariant if and only if, for every $\zeta \in \mathcal{N}_{\mathcal{S}}^{P}(x)$ and $x \in \mathcal{S}$,

$$\max_{v \in \mathcal{K}[f](t,x)} \langle \zeta, v \rangle \leq 0, \quad t \in \mathbb{R}, \quad x \in \mathcal{S}. \tag{12.34}$$

Proof. First, note that it follows from $\|f(t,x)\| \leq M$ for almost all $t \in \mathbb{R}$ and $x \in \mathbb{R}^q$, and (12.26) that for almost every $v \in \mathcal{K}[f](t,x)$, $\|v\| \leq M$. To show necessity, let $x_0 \in \mathcal{S}$ and define the Filippov set-valued function G by

$$G(t,x) \triangleq \{v \in \mathcal{K}[f](t,x) : \langle \zeta, v \rangle \leq 0, \zeta \in \mathcal{N}_{\mathcal{S}}^{P}(x)\}, \quad (t,x) \in [t_0, \infty) \times \mathcal{S}. \tag{12.35}$$

Note that the pair $(\mathcal{S}, G)$ is weakly invariant. Then it follows that there exists a Filippov solution $y(\cdot)$ to the differential inclusion given by

$$\dot{y}(t) \in G(t, y(t)), \quad y(t_0) = x_0, \quad \text{a.a.} \quad t \geq t_0, \tag{12.36}$$

such that $y(t) \in \mathcal{S}$ for all $t \geq t_0$. Note that $G(t,x) = \mathcal{K}[f](t,x)$ provided that (12.34) holds and $y(t_0) = x_0$. Now, it follows from Theorem 1 of [76, p. 87] that for $\varepsilon > 0$, $\|x(t) - y(t)\| \leq \varepsilon$ for all $t \in [t_0, \tau]$, where $x(\cdot)$ denotes any Filippov solution of (12.23) with $x(t_0) = x_0$. If $\mathrm{dist}(y(t), \partial\mathcal{S}) > 0$ for all $t \geq t_0$, then by taking $\varepsilon < \mathrm{dist}(y(t), \partial\mathcal{S})$ it follows that $x(t) \in \mathcal{S}$ for all $t \geq t_0$. Alternatively, consider the case where $\mathrm{dist}(y(t), \partial\mathcal{S}) = 0$. In this case, we claim that $x(t) \in \mathcal{S}$ for all $t \geq t_0$. To see this, suppose, *ad absurdum*, that there exists a time instant t^* such that $x(t^*) \in \partial\mathcal{S}$ and $x(t) \notin \mathcal{S}$ for $t^* < t \leq t^* + \delta$. Then it follows that $\langle \dot{x}(t^*), \zeta^* \rangle > 0$ for $\zeta^* \in \mathcal{N}_{\mathcal{S}}^{P}(x(t^*))$. Note that $\dot{x}(t^*) \in \mathcal{K}[f](t^*, x(t^*))$. Hence, $\langle v^*, \zeta^* \rangle > 0$ for some $v^* \in \mathcal{K}[f](t^*, x(t^*))$, which contradicts (12.34). Thus, for $\mathrm{dist}(y(t), \partial\mathcal{S}) = 0$, $x(t) \in \mathcal{S}$ for all $t \geq t_0$. Thus, $(\mathcal{S}, \mathcal{K}[f](t,x))$ is strongly invariant.

To show sufficiency, consider any $\tilde{x} \in \mathcal{S}$. Let $\tilde{v} \in \mathcal{K}[f](t, \tilde{x})$ be given. Define the set-valued function $\mathcal{F}(t,x) \triangleq \{g(t,x)\}$, where $g(t,x)$ is such that $\|g(t,x) - \tilde{v}\| = \inf_{\mu \in \mathcal{K}[f](t,x)} \|\mu - \tilde{v}\|$ for some fixed $t \in \mathbb{R}$. Note that $g(t, \tilde{x}) = \tilde{v}$. Next, since $(\mathcal{S}, \mathcal{F})$ is strongly invariant, it follows that $(\mathcal{S}, \mathcal{F})$ is weakly invariant, and hence, by Theorem 12.3, $\langle \tilde{\zeta}, \tilde{v} \rangle \leq 0$ for any $\tilde{\zeta} \in \mathcal{N}_{\mathcal{S}}^{P}(\tilde{x})$. Since $\tilde{v}$ is arbitrary in $\mathcal{K}[f](t, \tilde{x})$, it follows that (12.34) holds. $\square$

Finally, we present sufficient conditions for semistability and uniform semistability for (12.23).

Theorem 12.4. Assume that there exists $M > 0$ such that for almost every $(t, x) \in \mathbb{R}^{q+1}$, $\|f(t,x)\| \leq M$. Furthermore, assume that there exist a continuously differentiable function $V(\cdot)$ and a continuous function $W(\cdot)$ such that i) of Proposition 12.4 holds, $\mathcal{E} \subseteq W^{-1}(0)$, and

$$\max_{v \in \mathcal{K}[f](t,x)} \langle \nabla V(x), v \rangle \leq -W(x) \tag{12.37}$$

for every $x \in \mathcal{S}$ and $t \in \mathbb{R}$. If every point in $W^{-1}(0)$ is a Lyapunov stable equilibrium of (12.23), then (12.23) is semistable. Alternatively, if every point in $W^{-1}(0)$ is a uniformly Lyapunov stable equilibrium of (12.23), then (12.23) is uniformly semistable.

Proof. Using similar arguments as in the proof of Proposition 12.4 and Proposition 12.6 it can be shown that every Filippov solution $x(\cdot)$ of (12.23) satisfies $x(t) \to W^{-1}(0)$ as $t \to \infty$. Since every point in $W^{-1}(0)$ is a Lyapunov stable equilibrium of (12.23), it follows that $W^{-1}(0) \subseteq \mathcal{E}$. Since, by assumption, $\mathcal{E} \subseteq W^{-1}(0)$, it follows that $W^{-1}(0) = \mathcal{E}$. Hence, $x(t) \to \mathcal{E}$ as $t \to \infty$ and every point in $\mathcal{E}$ is Lyapunov stable. Now, it follows from Proposition 12.5 that $x(t) \to z$ as $t \to \infty$, where $z \in \mathcal{E}$. By definition, (12.23) is semistable. To prove the second assertion, note that since $\dot{x}(t)$ is uniformly bounded for almost all $t \geq t_0$, it follows using similar arguments as in the proof of Proposition 12.4 that $x(t) \rightrightarrows W^{-1}(0)$ as $t \to \infty$. Now, using similar arguments as above, it can be shown that (12.23) is uniformly semistable.

$\square$

Example 12.5. Consider the time-varying discontinuous dynamical system given by

$$\dot{x}_1(t) = (2 - \cos t)\text{sign}(x_2(t) - x_1(t)), \quad x_1(t_0) = x_{10}, \quad t \geq t_0, \tag{12.38}$$

$$\dot{x}_2(t) = (2 - \cos t)\text{sign}(x_1(t) - x_2(t)), \quad x_2(t_0) = x_{20}, \tag{12.39}$$

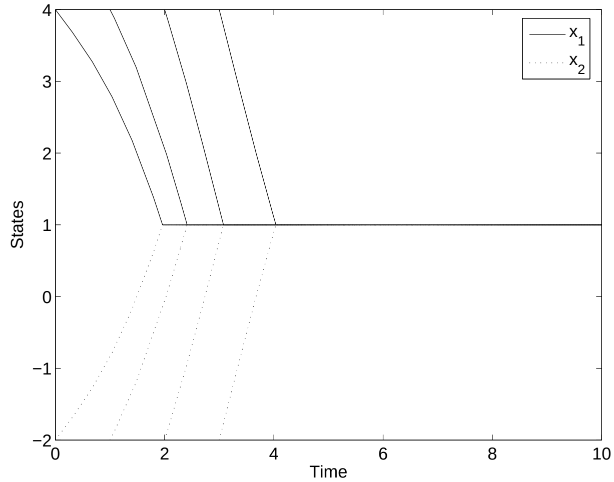

Figure 12.4: State trajectories versus time for Example 12.5

where $x_1, x_2 \in \mathbb{R}$. Clearly, $\|f(t,x)\| \leq 3\sqrt{2}$ for almost all $t \geq t_0$ and $x \in \mathbb{R}^2$. Next, consider $V(x_1,x_2) = \frac{1}{2}(x_1-\alpha)^2 + \frac{1}{2}(x_2-\alpha)^2$, where $\alpha \in \mathbb{R}$. Then it follows from the time-dependent version of Theorem 1 of [193] that

$$\begin{aligned}
[x_1-\alpha, x_2-\alpha]^{\mathrm{T}}\mathcal{K}[f](t,x) &= \mathcal{K}[[x_1-\alpha, x_2-\alpha]^{\mathrm{T}}f](t,x) \\
&= \mathcal{K}\left[-(2-\cos t)(x_1-x_2)\mathrm{sign}(x_1-x_2)\right](t,x) \\
&= -(2-\cos t)(x_1-x_2)\mathcal{K}[\mathrm{sign}(x_1-x_2)](x) \\
&= -(2-\cos t)(x_1-x_2)\mathrm{SGN}(x_1-x_2) \\
&= -(2-\cos t)|x_1-x_2|, \quad t \in \mathbb{R}, \quad (x_1,x_2) \in \mathbb{R}^2, \qquad (12.40)
\end{aligned}$$

which implies that $\langle \nabla V(x_1,x_2), v\rangle \leq -|x_1-x_2|$ for every $v \in \mathcal{K}[f](t,x)$. Now, it follows from Theorem 1 of [76, p. 153] that $x_1 = x_2 = \alpha$ is Lyapunov stable. In fact, it can be shown that $x_1 = x_2 = \alpha$ is uniformly Lyapunov stable. Next, let $W(x_1,x_2) = |x_1-x_2|$ and note that $W^{-1}(0) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 = x_2\} = \mathcal{E}$. Now, it follows from Theorem 12.4 that (12.38) and (12.39) is semistable. Moreover, it can be shown that (12.38) and (12.39) is uniformly semistable. Figure 12.4 shows the solutions of (12.38) and (12.39) for $x_{10} = 4$, $x_{20} = -2$, and $t_0 = 0, 1, 2, 3$. $\triangle$

12.6. Applications to Network Consensus with Switching Topology

Communication links among multiagent systems are often unreliable due to multipath effects and exogenous disturbances leading to dynamic information exchange topologies. In the remainder of the chapter, we use the semistability theory developed in Sections 12.3 and 12.5 to develop switched consensus protocols to achieve agreement over a network with switching topology. Specifically, consider q mobile agents with the dynamics $\mathcal{G}_i$ given by

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \tag{12.41}$$

where for each $i \in \{1, \ldots, q\}$, $x_i(t) \in \mathbb{R}$ denotes the information state and $u_i(t) \in \mathbb{R}$ denotes the information control input for all $t \geq 0$. The general consensus protocol is given by

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \phi_{ij}(x_i(t), x_j(t)), \tag{12.42}$$

where $\phi_{ij}(\cdot,\cdot)$, $i, j = 1, \ldots, q$, are Lebesgue measurable and locally essentially bounded. Note that (12.41) and (12.42) describe an interconnected network $\mathcal{G}$ with a graph topology $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, \ldots, q\}$ denotes the set of *nodes* (or vertices) involving a finite nonempty set denoting the agents, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of *edges* involving a set of ordered pairs denoting the direction of information flow, and $\mathcal{A}$ denotes an *adjacency matrix* such that $\mathcal{A}_{(i,j)} = 1$, $i, j = 1, \ldots, q$, if $(j, i) \in \mathcal{E}$, and 0 otherwise. For further details, see [126]. Furthermore, note that it follows from (12.41) and (12.42) that information states are updated using a distributed nonlinear controller involving neighbor-to-neighbor interaction between agents. The following assumptions are needed for the main results of this section.

Assumption 1: For the *connectivity matrix*[6] $\mathcal{C} \in \mathbb{R}^{q\times q}$ associated with the multiagent dynamical system $\mathcal{G}$ defined by

$$\mathcal{C}_{(i,j)} \triangleq \begin{cases} 0, & \text{if } \phi_{ij}(x_i, x_j) \equiv 0, \\ 1, & \text{otherwise,} \end{cases} \quad i \neq j, \quad i, j = 1, \ldots, q, \tag{12.43}$$

[6] The negative of the connectivity matrix, that is, $-\mathcal{C}$, is known as the Laplacian of the directed graph $\mathfrak{G}$ in the literature.

and $\mathcal{C}_{(i,i)} \triangleq -\sum_{k=1,\, k\neq i}^{q} \mathcal{C}_{(i,k)}$, $i = 1,\ldots,q$, $\operatorname{rank}\mathcal{C} = q-1$, and for $\mathcal{C}_{(i,j)} = 1$, $i \neq j$, $\phi_{ij}(x_i, x_j) = 0$ if and only if $x_i = x_j$.

Assumption 2: For $i, j = 1,\ldots,q$, $(x_i - x_j)\phi_{ij}(x_i, x_j) \leq 0$, $x_i, x_j \in \mathbb{R}$.

For details concerning Assumptions 1 and 2 and their connection to system thermodynamics see [104, 125]. The following proposition is needed.

Proposition 12.7 [125]. Consider the multiagent dynamical system (12.41) and (12.42) and assume that Assumptions 1 and 2 hold. Then $f_i(x) = 0$ for all $i = 1,\ldots,q$ if and only if $x_1 = \cdots = x_q$. Furthermore, $\alpha\mathbf{e}$, $\alpha \in \mathbb{R}$, $\mathbf{e} \triangleq [1,\ldots,1] \in \mathbb{R}^q$, is an equilibrium state of (12.41) and (12.42).

To address the network consensus problem with a switching topology, consider the switched controller $\mathcal{G}_{\mathrm{si}}$ given by

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \phi_{ij}^{\sigma(t)}(x_i(t), x_j(t)), \tag{12.44}$$

where $\sigma : [0,\infty) \to \mathcal{S}$ is a piecewise constant switching signal, $\mathcal{S}$ is a finite index set, and $\phi_{ij}^{\sigma} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and locally essentially bounded and satisfies Assumptions 1 and 2 for every $\sigma \in \mathcal{S}$. Furthermore, we assume that $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$ in Assumption 1, where $\mathcal{C} = \mathcal{C}(t)$, $t \geq 0$.

Theorem 12.5. Consider the closed-loop system $\tilde{\mathcal{G}}$ given by the multiagent dynamical system (12.41) and the switched controller (12.44). Assume that Assumptions 1 and 2 hold for every $\sigma \in \mathcal{S}$. Furthermore, assume that $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$, where $\mathcal{C} = \mathcal{C}(t)$, $t \geq 0$, in Assumption 1. Then for every $\alpha \in \mathbb{R}$, $x_1 = \cdots = x_q = \alpha$ is a semistable state of $\tilde{\mathcal{G}}$. Furthermore, $x_i(t) \to \frac{1}{q}\sum_{i=1}^{q} x_{i0}$ and $\frac{1}{q}\sum_{i=1}^{q} x_{i0}$ is a semistable equilibrium state.

Proof. Consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}(x - \alpha\mathbf{e})^{\mathrm{T}}(x - \alpha\mathbf{e}), \tag{12.45}$$

where $x \triangleq [x_1, \ldots, x_q]^{\mathrm{T}} \in \mathbb{R}^q$ and $\alpha \in \mathbb{R}$. Then the Lyapunov derivative along the trajectories of the closed-loop system (12.41) and (12.44) is given by

$$\dot{V}(x) = (x - \alpha\mathbf{e})^{\mathrm{T}}\dot{x} = x^{\mathrm{T}}\dot{x} = \sum_{i=1}^{q} x_i \left[\sum_{j=1, j\neq i}^{q} \phi_{ij}^{\sigma}(x_i, x_j) \right] = \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} (x_i - x_j)\phi_{ij}^{\sigma}(x_i, x_j) \leq 0, \quad x \in \mathbb{R}^q, \qquad (12.46)$$

which establishes Lyapunov stability of $x \equiv \alpha\mathbf{e}$.

Next, we rewrite the closed-loop system (12.41) and (12.44) as the differential inclusion (12.2). For any $v \in \mathcal{K}[f](x)$, let $V^o(x, v) \triangleq x^{\mathrm{T}}v$ and $\max V^o(x, v) \triangleq \max_{v\in\mathcal{K}[f]}\{x^{\mathrm{T}}v\}$. Now, it follows from Theorem 1 of [193] and (12.46) that

$$x^{\mathrm{T}}\mathcal{K}[f](x) = \mathcal{K}[x^{\mathrm{T}}f](x) = \mathcal{K}\left[\sum_{i=1}^{q-1} \sum_{j=i+1}^{q} (x_i - x_j)\phi_{ij}^{\sigma}(x_i, x_j) \right](x), \quad x \in \mathbb{R}^q, \quad (12.47)$$

and hence, by definition of differential inclusions, it follows that $\max V^o(x, v) = \max \overline{\mathrm{co}}\{\sum_{i=1}^{q-1} \sum_{j=i+1}^{q}(x_i - x_j)\phi_{ij}^{\sigma}(x_i, x_j)\}$. Note that since, by (12.46), $\sum_{i=1}^{q-1} \sum_{j=i+1}^{q}(x_i - x_j)\phi_{ij}^{\sigma}(x_i, x_j) \leq 0$, $x_i \in \mathbb{R}$, it follows that $\max V^o(x, v)$ cannot be positive, and hence, the largest value $\max V^o(x, v)$ can achieve is zero.

Finally, note that $0 \in \mathcal{L}_f V(x)$ if and only if $\sum_{i=1}^{q-1} \sum_{j=i+1}^{q}(x_i - x_j)\phi_{ij}^{\sigma}(x_i, x_j) = 0$, and hence, $\mathcal{Z} \triangleq \{x \in \mathbb{R}^q : \sum_{i=1}^{q-1} \sum_{j=i+1}^{q}(x_i - x_j)\phi_{ij}^{\sigma}(x_i, x_j) = 0\}$. Now, it follows from Proposition 12.7 that $\mathcal{Z} = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$. Since $\mathcal{Z}$ consists of equilibrium points, it follows that $\mathcal{M} = \mathcal{Z}$. Hence, it follows from Theorem 12.1 that $x = \alpha\mathbf{e}$ is semistable for all $\alpha \in \mathbb{R}$. $\square$

Note that Example 12.1 serves as a special case of Theorem 12.5. Next, we extend Theorem 12.5 to the discontinuous controllers $\mathcal{G}_{\mathrm{n}i}$ of the form

$$u_i = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}\mathrm{sign}(x_j - x_i). \qquad (12.48)$$

It is important to note that the consensus protocol (12.48) is a logic-based, distributed decision-making protocol. Although a similar consensus protocol based on nonsmooth gradient flows is proposed in [58], the key difference between (12.48) and the one in [58] is that (12.48) is a *distributed* protocol while the consensus protocol in [58] is a *centralized* protocol.

In [125], the authors prove that the consensus protocol given by the form

$$u_i = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} \mathrm{sign}(x_j - x_i)|x_j - x_i|^{\alpha} \tag{12.49}$$

is a finite-time consensus protocol for $0 < \alpha < 1$. Next, we show that (12.49) is also a finite-time consensus protocol for $\alpha = 0$. Note that in this case, (12.49) reduces to (12.48). Furthermore, note that Example 12.2 is a special case of the closed-loop system given by (12.41) and (12.48).

Theorem 12.6. Consider the closed-loop system $\tilde{\mathcal{G}}$ given by the multiagent dynamical system (12.41) and the discontinuous controller (12.48). Assume that Assumptions 1 and 2 hold. Furthermore, assume that $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$ in Assumption 1. Then for every $\alpha \in \mathbb{R}$, $x_1 = \cdots = x_q = \alpha$ is a finite-time-semistable state of $\tilde{\mathcal{G}}$. Furthermore, $x_i(t) = \frac{1}{q}\sum_{i=1}^{q} x_{i0}$ for $t \geq T(x_{10}, \ldots, x_{q0})$ and $\frac{1}{q}\sum_{i=1}^{q} x_{i0}$ is a semistable equilibrium state.

Proof. Consider the Lyapunov function candidate (12.45). Since $V(x)$ is differentiable at x, it follows that $\mathcal{L}_f V(x) = (x - \alpha\mathbf{e})^{\mathrm{T}}\mathcal{K}[f](x)$. Now, it follows from Theorem 1 of [193] that

$$\begin{aligned}
(x - \alpha\mathbf{e})^{\mathrm{T}}\mathcal{K}[f](x) &= \mathcal{K}[(x - \alpha\mathbf{e})^{\mathrm{T}} f](x) \\
&= \mathcal{K}[x^{\mathrm{T}} f](x) \\
&= \mathcal{K}\left[\sum_{i=1}^{q} x_i \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} \mathrm{sign}(x_j - x_i)\right](x) \\
&= \mathcal{K}\left[-\sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} (x_i - x_j)\mathrm{sign}(x_i - x_j)\right](x) \\
&\subseteq -\sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} (x_i - x_j)\mathcal{K}[\mathrm{sign}(x_i - x_j)](x) \\
&= -\sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} (x_i - x_j)\mathrm{SGN}(x_i - x_j)
\end{aligned}$$

$$= -\sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} |x_i - x_j|, \quad x \in \mathbb{R}^q, \tag{12.50}$$

which implies that $\max \mathcal{L}_f V(x) \leq 0$ for all $x \in \mathbb{R}^q$. Hence, it follows from Theorem 2 of [12] that $x_1 = \cdots = x_q = \alpha$ is Lyapunov stable. Next, note that since

$$\begin{aligned} \mathcal{L}_f V(x) &= \mathcal{K}\left[-\sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} (x_i - x_j) \mathrm{sign}(x_i - x_j) \right](x) \\ &= \mathcal{K}\left[-\sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} |x_i - x_j| \right](x), \end{aligned}$$

it follows that $0 \in \mathcal{L}_f V(x)$ if and only if $x_1 = \cdots = x_q$, and hence, $\mathcal{Z} = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$. Since the largest weakly invariant subset $\mathcal{M}$ of $\mathcal{Z}$ is given by $\mathcal{M} = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q = \alpha, \alpha \in \mathbb{R}\}$, it follows from Theorem 12.1 that $\tilde{\mathcal{G}}$ is semistable.

Finally, we show that $\tilde{\mathcal{G}}$ is finite-time-semistable. To see this, consider the nonnegative function $U(x) = \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} \mathcal{C}_{(i,j)}|x_i - x_j|$. In this case, it follows using similar arguments as in Example 12.2 that

$$\mathcal{L}_f U(x) = \begin{cases} \left\{ -2\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} \mathcal{C}_{(i,j)} \right\}, & x_i \neq x_j,\ i,j = 1,\ldots,q,\ i \neq j, \\ \emptyset, & x_k = x_l \text{ for some } k,l \in \{1,\ldots,q\},\ k \neq l, \\ \{0\}, & x_1 = \cdots = x_q, \end{cases} \tag{12.51}$$

which implies that $\max \mathcal{L}_f U(x) \leq -2\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} \mathcal{C}_{(i,j)} < 0$ or $\mathcal{L}_f U(x) = \emptyset$ for all $x \in \mathbb{R}^q \backslash \mathcal{Z}$. Hence, it follows from Corollary 12.1 that $\tilde{\mathcal{G}}$ is globally finite-time-semistable. □

Finally, we design discontinuous *dynamic* consensus protocols for (12.41). In contrast to the static controllers addressed in [135] and [187], the proposed controller is a dynamic compensator. This controller architecture allows us to design finite-time consensus protocols via quantized feedback in a dynamical network. Specifically, consider the q mobile agents with dynamics $\mathcal{G}_i$ given by (12.41). Furthermore, consider the discontinuous dynamic compensators $\mathcal{G}_{\mathrm{c}i}$ given by

$$\dot{x}_{\mathrm{c}i}(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} \mathrm{sign}(x_{\mathrm{c}j}(t) - x_{\mathrm{c}i}(t)) + \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} \mathrm{sign}(x_i(t) - x_j(t)),$$

$$x_{ci}(0) = x_{ci0}, \quad t \geq 0, \tag{12.52}$$

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(j,i)} \text{sign}(x_{cj}(t) - x_{ci}(t)), \tag{12.53}$$

where $x_{ci}(t) \in \mathbb{R}$, $t \geq 0$. Here, we assume that Assumption 1 holds and $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$.

Theorem 12.7. Consider the closed-loop system $\tilde{\mathcal{G}}$ given by the multiagent dynamical system (12.41) and the discontinuous dynamic controller (12.52) and (12.53). Assume that Assumption 1 holds and $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$. Then for every $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, $x_1 = \cdots = x_q = \alpha$ and $x_{c1} = \cdots = x_{cq} = \beta$ is a finite-time-semistable state of $\tilde{\mathcal{G}}$. Furthermore, $x_i(t) = \frac{1}{q}\sum_{i=1}^{q} x_{i0}$ and $x_{ci}(t) = \frac{1}{q}\sum_{i=1}^{q} x_{ci0}$ for all $t \geq T(x_{10}, \ldots, x_{q0}, x_{c10}, \ldots, x_{cq0})$ and $(\frac{1}{q}\sum_{i=1}^{q} x_{i0}, \frac{1}{q}\sum_{i=1}^{q} x_{ci0})$ is a semistable equilibrium state.

Proof. Note that for every $a, b \in \mathbb{R}$, $x(t) \equiv a\mathbf{e}$ and $x_c(t) \equiv b\mathbf{e}$ are the equilibrium points for the closed-loop system $\tilde{\mathcal{G}}$. Consider the nonnegative function given by

$$V(\tilde{x}) = \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} \mathcal{C}_{(i,j)}|x_i - x_j| + \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} \mathcal{C}_{(i,j)}|x_{ci} - x_{cj}|, \tag{12.54}$$

where $\tilde{x} \triangleq [x^{\mathrm{T}}, x_c^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2q}$. In this case, it follows using similar arguments as in Example 12.3 that

$$\mathcal{L}_f V(\tilde{x}) = \begin{cases} \left\{-2\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} \mathcal{C}_{(i,j)}\right\}, & x_i \neq x_j,\ x_{ci} \neq x_{cj},\ i,j = 1,\ldots,q,\ i \neq j, \\ \emptyset, & x_k = x_l \text{ or } x_{ck} = x_{cl} \\ & \text{for some } k, l \in \{1,\ldots,q\},\ k \neq l, \\ \{0\}, & x_1 = \cdots = x_q,\ x_{c1} = \cdots = x_{cq}, \end{cases} \tag{12.55}$$

which implies that $\max \mathcal{L}_f V(\tilde{x}) \leq 0$ or $\mathcal{L}_f V(\tilde{x}) = \emptyset$ for all $\tilde{x} \in \mathbb{R}^{2q}$. Next, define $\mathcal{Z} \triangleq \{\tilde{x} \in \mathbb{R}^{2q} : x_1 = \cdots = x_q, x_{c1} = \cdots = x_{cq}\}$ and let $\mathcal{N}$ denote the largest negatively invariant set of $\mathcal{Z}$. On $\mathcal{N}$, it follows from (12.41), (12.52), and (12.53) that $\dot{x}_i = 0$ and $\dot{x}_{ci} = 0$, $i = 1, \ldots, q$. Hence, $\mathcal{N} = \{\tilde{x} \in \mathbb{R}^{2q} : x = a\mathbf{e}, x_c = b\mathbf{e}\}$, $a, b \in \mathbb{R}$, which implies that $\mathcal{N}$ is the set of equilibrium points.

Since the connectivity matrix $\mathcal{C}$ of the closed-loop system is irreducible, assume, without loss of generality, that $\mathcal{C}_{(i,i+1)} = \mathcal{C}_{(q,1)} = 1$, where $i = 1, \ldots, q-1$. Now, for $q = 2$, it was shown in Example 12.3 that the vector field f of the closed-loop system given by (12.41), (12.52), and (12.53) is nontangent to $\mathcal{N}$ at a point $\tilde{x} \in \mathcal{N}$. Next, we show that for $q \geq 3$, the vector field f of the closed-loop system given by (12.41), (12.52), and (12.53) is nontangent to $\mathcal{N}$ at a point $\tilde{x} \in \mathcal{N}$. To see this, note that the tangent cone $T_{\tilde{x}}\mathcal{N}$ to the equilibrium set $\mathcal{N}$ is orthogonal to the $2q$ vectors $\mathbf{u}_i \triangleq [0_{1\times(i-1)}, \mathcal{C}_{(i,i+1)}, -\mathcal{C}_{(i,i+1)}, 0_{1\times(2q-i-1)}]^{\mathrm{T}} \in \mathbb{R}^{2q}$, $\mathbf{u}_q \triangleq [-\mathcal{C}_{(q,1)}, 0_{1\times(q-2)}, \mathcal{C}_{(q,1)}, 0_{1\times q}]^{\mathrm{T}} \in \mathbb{R}^{2q}$, $\mathbf{v}_i \triangleq [0_{1\times(q+i-1)}, -\mathcal{C}_{(i,i+1)}, \mathcal{C}_{(i,i+1)}, 0_{1\times(q-i-1)}]^{\mathrm{T}} \in \mathbb{R}^{2q}$, and $\mathbf{v}_q \triangleq [0_{1\times q}, \mathcal{C}_{(q,1)}, 0_{1\times(q-2)}, -\mathcal{C}_{(q,1)}]^{\mathrm{T}} \in \mathbb{R}^{2q}$, $i = 1, \ldots, q-1$, $q \geq 3$. On the other hand, since $f(\tilde{x}) \in \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_q, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ for all $\tilde{x} \in \mathbb{R}^{2q}$, it follows that $f(\mathcal{V}) \subseteq \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_q, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ for every subset $\mathcal{V} \subseteq \mathbb{R}^{2q}$. Consequently, the direction cone $\mathcal{F}_{\tilde{x}}$ of f at $\tilde{x} \in \mathcal{N}$ relative to $\mathbb{R}^{2q}$ satisfies $\mathcal{F}_{\tilde{x}} \subseteq \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_q, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$. Hence, $T_{\tilde{x}}\mathcal{N} \cap \mathcal{F}_{\tilde{x}} = \{0\}$, which implies that the vector field f is nontangent to the set of equilibria $\mathcal{N}$ at the point $\tilde{x} \in \mathcal{N}$. Note that for every $z \in \mathcal{N}$, the set $\mathcal{N}_z$ required by Theorem 12.2 is contained in $\mathcal{N}$. Since nontangency to $\mathcal{N}$ implies nontangency to $\mathcal{N}_z$ at the point $z \in \mathcal{N}$, it follows from Theorem 12.2 that the closed-loop system $\tilde{\mathcal{G}}$ is semistable.

Finally, note that $\max \mathcal{L}_f V(x) \leq -2\sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} < 0$ or $\mathcal{L}_f V(\tilde{x}) = \emptyset$ for all $x \in \mathbb{R}^4 \backslash \mathcal{Z}$, and hence, it follows from Corollary 12.1 that $\tilde{\mathcal{G}}$ is globally finite-time-semistable.

□

The dynamic compensator (12.52) and (12.53) is a state feedback controller. A natural question regarding (12.41) is how to design finite-time consensus protocols for multiagent coordination via output feedback. To address this question, we consider q continuous-time integrator agents given by (12.41) with the output given by

$$y_i = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}(x_j - x_i), \quad i = 1, \ldots, q. \tag{12.56}$$

Specifically, consider the dynamic output feedback compensator given by

$$\dot{x}_{\mathrm{c}i}(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}\operatorname{sign}(x_{\mathrm{c}j}(t) - x_{\mathrm{c}i}(t)) + y_i(t), \quad x_{\mathrm{c}i}(0) = x_{\mathrm{c}i0}, \quad t \geq 0, \tag{12.57}$$

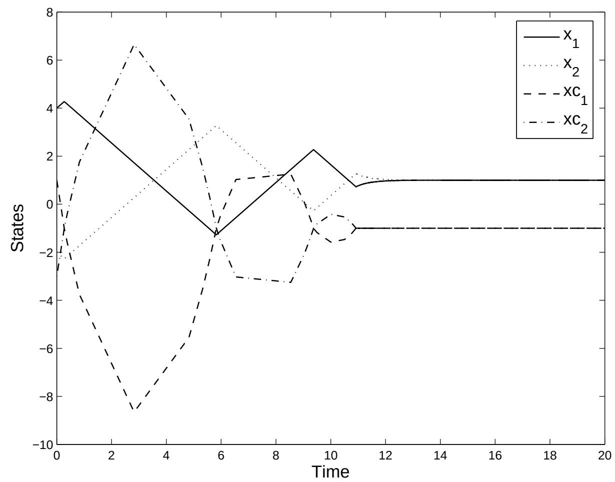

Figure 12.5: State trajectories for the case where $q = 2$ of Theorem 12.8

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(j,i)} \mathrm{sign}(x_{\mathrm{c}j}(t) - x_{\mathrm{c}i}(t)), \tag{12.58}$$

where $x_{\mathrm{c}i}(t) \in \mathbb{R}$, $t \geq 0$. Here, once again, we assume that Assumption 1 holds and $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$.

Theorem 12.8. Consider the closed-loop system $\tilde{\mathcal{G}}$ given by the multiagent dynamical system (12.41) and the nonsmooth dynamic controller (12.57) and (12.58) with (12.56). Assume that Assumption 1 holds and $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$. Then for every $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, $x_1 = \cdots = x_q = \alpha$ and $x_{\mathrm{c}1} = \cdots = x_{\mathrm{c}q} = \beta$ is a finite-time-semistable state of $\tilde{\mathcal{G}}$. Furthermore, $x_i(t) = \frac{1}{q}\sum_{i=1}^{q} x_{i0}$ and $x_{\mathrm{c}i}(t) = \frac{1}{q}\sum_{i=1}^{q} x_{\mathrm{c}i0}$ for all $t \geq T(x_{10}, \ldots, x_{q0}, x_{\mathrm{c}10}, \ldots, x_{\mathrm{c}q0})$ and $(\frac{1}{q}\sum_{i=1}^{q} x_{i0}, \frac{1}{q}\sum_{i=1}^{q} x_{\mathrm{c}i0})$ is a semistable equilibrium state.

Proof. The proof is similar to the proof of Theorem 12.7 with $V(\tilde{x}) = \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} \mathcal{C}_{(i,j)}(x_i - x_j)^2 + \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} \mathcal{C}_{(i,j)}|x_{\mathrm{c}i} - x_{\mathrm{c}j}|$. □

To illustrate Theorem 12.8, consider the case where $q = 2$. Figure 12.5 shows the states of the closed-loop system (12.41), (12.56), (12.57), and (12.58).

12.7. Discontinuous Time-Varying Consensus Protocols

In this section, we consider a discontinuous consensus protocol $\mathcal{G}$ with time-dependent and state-dependent communication links given by

$$\dot{x}_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}(x_i(t), x_j(t)) a_{ij}(t, x_i(t), x_j(t)) \mathrm{sign}(x_j(t) - x_i(t)), \quad x_i(t_0) = x_{i0},$$
$$t \geq t_0, \quad i = 1, \ldots, q, \tag{12.59}$$

where $t \geq t_0$, $x_i(t) \in \mathbb{R}$, $a_{ij} : \mathbb{R}^3 \to \mathbb{R}$ satisfies $a_{ij}(t, x_i, x_j) = a_{ji}(t, x_j, x_i)$ and $m \leq a_{ij}(t, x_i, x_j) \leq M$, $a_{ij}(t, x_i, x_j) \not\equiv 0$, $i, j = 1, \ldots, q$, $i \neq j$, $0 < m < M$ is a constant, and $\mathcal{C}_{(i,j)} : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following assumption:

Assumption 3: For the connectivity matrix $\mathcal{C}(x) \in \mathbb{R}^{q\times q}$, $x \triangleq [x_1, \ldots, x_q]^{\mathrm{T}} \in \mathbb{R}^q$, associated with $\mathcal{G}$ defined by

$$\mathcal{C}_{(i,j)}(x_i, x_j) \triangleq \begin{cases} 0, & \text{if } (j, i) \in \mathcal{E}, \\ 1, & \text{otherwise}, \end{cases} \quad i \neq j, \quad i, j = 1, \ldots, q, \tag{12.60}$$

and $\mathcal{C}_{(i,i)}(x_i, x_i) = -\sum_{k=1,\, k\neq i}^{q} \mathcal{C}_{(i,k)}(x_i, x_k)$, $i = 1, \ldots, q$, $\mathrm{rank}\,\mathcal{C}(x) = q - 1$, $x \in \mathbb{R}^q$, and $\mathcal{C}(x) = \mathcal{C}^{\mathrm{T}}(x)$, $x \in \mathbb{R}^q$.

Theorem 12.9. Consider the time-varying discontinuous consensus protocol $\mathcal{G}$ given by (12.59). Assume that Assumption 3 holds. Then $\mathcal{G}$ is uniformly semistable and $x_i(t) \rightrightarrows \frac{1}{q}\sum_{i=1}^{q} x_{i0}$ as $t \to \infty$, $i = 1, \ldots, q$.

Proof. First, note that $\|f(t, x)\| \leq M(q-1)\sqrt{q}$ for almost all $t \geq t_0$ and $x \in \mathbb{R}^q$. Next, consider the Lyapunov function candidate (12.45) and note that

$$\begin{aligned}
(x - \alpha\mathbf{e})^{\mathrm{T}} \mathcal{K}[f](t, x) &= \mathcal{K}[(x - \alpha\mathbf{e})^{\mathrm{T}} f](t, x) \\
&= \mathcal{K}[x^{\mathrm{T}} f](t, x) \\
&= \mathcal{K}\left[\sum_{i=1}^{q} x_i \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)} a_{ij} \mathrm{sign}(x_i - x_j)\right](t, x)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{K}\left[-\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}a_{ij}(x_i-x_j)\mathrm{sign}(x_i-x_j)\right](t,x)\\
&\subseteq -\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}a_{ij}(x_i-x_j)\mathcal{K}[\mathrm{sign}(x_i-x_j)](x)\\
&= -\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}a_{ij}(x_i-x_j)\mathrm{SGN}(x_i-x_j)\\
&= -\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}a_{ij}|x_i-x_j|, \quad (t,x)\in[t_0,\infty)\times\mathbb{R}^q, \qquad (12.61)
\end{aligned}
$$

which implies that $\langle\nabla V(x),v\rangle \leq -\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} m\mathcal{C}_{(i,j)}|x_i-x_j|$ for every $v\in\mathcal{K}[f](t,x)$. Now, it follows from Theorem 1 of [76, p. 153] that $x_1=\cdots=x_q=\alpha$ is Lyapunov stable. In fact, it can be shown that $x_1=\cdots=x_q=\alpha$ is uniformly Lyapunov stable. Next, let $W(x)=\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q} m\mathcal{C}_{(i,j)}|x_i-x_j|$ and note that $W^{-1}(0)=\{x\in\mathbb{R}^q : x_1=\cdots=x_q\}=\mathcal{E}$. Now, it follows from Theorem 12.4 that $\mathcal{G}$ is uniformly semistable. Finally, since $\sum_{i=1}^{q}\dot{x}_i(t)=0$, $t\geq t_0$, it follows that $x_i(t)\rightrightarrows \frac{1}{q}\sum_{i=1}^{q}x_{i0}$ as $t\to\infty$, $i=1,\ldots,q$. □

Note that Example 12.5 serves as a special case of Theorem 12.9.

Chapter 13

Semistability of Switched Linear Systems

13.1. Introduction

Building on the results of [117, 135] and Chapter 12, in this chapter we develop semistability and uniform semistability analysis results for switched linear systems. Since solutions to switched systems are a function of both the system initial conditions and the admissible switching signals, uniformity here refers to the convergence rate to a Lyapunov stable equilibrium as the switching signal ranges over a given switching set. The main results of this chapter involve sufficient conditions for semistability and uniform semistability using multiple Lyapunov functions and sufficient regularity assumptions on the class of switching signals considered. Specifically, using multiple Lyapunov functions whose derivatives are negative semidefinite, semistability of the switched linear system is established. If, in addition, the admissible switching signals have infinitely many disjoint intervals of length bounded from below and above, uniform semistability can be concluded. Finally, we note that the results of the present chapter can be viewed as an extension of asymptotic stability results for switched linear systems developed in [117, 138, 201].

13.2. Switched Dynamical Systems

In this chapter, we consider *switched linear systems* $\mathcal{G}_\sigma$ given by

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad \sigma(t) \in \mathcal{S}, \quad x(0) = x_0, \quad t \geq 0, \tag{13.1}$$

where $x(t) \in \mathbb{R}^n$, $A_{\sigma(t)} \in \mathbb{R}^{n \times n}$, $\sigma : [0, \infty) \to \mathcal{P}$ denotes a piecewise constant switching signal, and $\mathcal{S}$ denotes the set of switching signals. The switching signal σ effectively switches the right-hand side of (13.1) by selecting different vector fields from the parameterized family

$\{A_px : p \in \mathcal{P}\}$. The *switching times* of (13.1) refer to the time instants at which the switching signal σ is discontinuous. Our convention here is that $\sigma(\cdot)$ is left-continuous, that is, $\sigma(t^-) = \sigma(t)$, where $\sigma(t^-) \triangleq \lim_{h\to 0^+} \sigma(t-h)$. The pair $(x, \sigma) : [0,\infty) \times \mathcal{S} \to \mathbb{R}^n$ is a *solution* to the switched system (13.1) if $x(\cdot)$ is piecewise continuously differentiable and satisfies (13.1) for all $t \geq 0$. The set $\mathcal{S}_p[\tau, T]$, $\tau > 0$, $T \in [0, \infty]$, denotes the set of signals σ for which there is an infinite number of disjoint intervals of length no smaller than τ on which σ is constant, and consecutive intervals with this property are separated by no more than T [117] (including the initial time). Finally, a point $x_{\rm e} \in \mathbb{R}^n$ is an *equilibrium point* of (13.1) if and only if $A_{\sigma(t)}x_{\rm e} = 0$ for all $\sigma(t) \in \mathcal{S}$ and for all $t \geq 0$.

We assume that the following assumption holds for (13.1).

Assumption 1: $\bigcap_{p\in\mathcal{P}} \mathcal{N}(A_p) - \{0\} \neq \emptyset$.

Let $\mathcal{E} \triangleq \{x_{\rm e} \in \mathbb{R}^n : A_{\sigma(t)}x_{\rm e} = 0, \sigma(t) \in \mathcal{S}, t \geq 0\}$. Then $\mathcal{E} = \bigcap_{p\in\mathcal{P}} \mathcal{N}(A_p)$ and $\mathcal{E}$ contains an element other than 0. It is important to note that our results also hold for the case where $\bigcap_{p\in\mathcal{P}} \mathcal{N}(A_p) = \{0\}$. However, due to space limitations, we do not consider this case in this chapter.

Definition 13.1. *i*) An equilibrium point $x_{\rm e} \in \mathcal{E}$ of (13.1) is *Lyapunov stable* if for every switching signal $\sigma \in \mathcal{S}$ and every $\varepsilon > 0$, there exists $\delta = \delta(\sigma, \varepsilon) > 0$ such that for all $\|x_0 - x_{\rm e}\| \leq \delta$, $\|x(t) - x_{\rm e}\| < \varepsilon$ for all $t \geq 0$. An equilibrium point $x_{\rm e} \in \mathcal{E}$ of (13.1) is *uniformly Lyapunov stable* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $\|x_0 - x_{\rm e}\| \leq \delta$, $\|x(t) - x_{\rm e}\| < \varepsilon$ for all $t \geq 0$.

ii) An equilibrium point $x_{\rm e} \in \mathcal{E}$ of (13.1) is *semistable* if for every switching signal $\sigma \in \mathcal{S}$, $x_{\rm e}$ is Lyapunov stable and there exists $\delta = \delta(\sigma) > 0$ such that for all $\|x_0 - x_{\rm e}\| \leq \delta$, $\lim_{t\to\infty} x(t) = z$ and $z \in \mathcal{E}$ is a Lyapunov stable equilibrium point. An equilibrium point $x_{\rm e} \in \mathcal{E}$ of (13.1) is *uniformly semistable* if $x_{\rm e}$ is uniformly Lyapunov stable and there exists $\delta > 0$ such that for all $\|x_0 - x_{\rm e}\| \leq \delta$, $\lim_{t\to\infty} x(t) = z$ uniformly in σ and $z \in \mathcal{E}$ is a uniformly

Lyapunov stable equilibrium point.

iii) The switched system (13.1) is *semistable* if all the equilibrium points of (13.1) are semistable. The switched system (13.1) is *uniformly semistable* if all the equilibrium points of (13.1) are uniformly semistable.

Next, we present the notion of semiobservability which plays a critical role in semistability analysis of linear dynamical systems. For details, see [107].

Definition 13.2 [107]. Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$. The pair (A, C) is *semiobservable* if

$$\bigcap_{k=1}^{n} \mathcal{N}\left(CA^{k-1}\right) = \mathcal{N}(A). \tag{13.2}$$

The following lemmas and propositions are needed for the main results of this chapter.

Lemma 13.1. Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$. If the pair (A, C) is semiobservable, then

$$\mathcal{N}(A) \cap \mathcal{N}(C) = \mathcal{N}(A). \tag{13.3}$$

Proof. Note that, by definition of semiobservability, $\mathcal{N}(A) \cap \mathcal{N}(C) \subseteq \mathcal{N}(A)$. Let $x \in \mathcal{N}(A)$. Then it follows from (13.2) that $Cx = 0$, and hence, $\mathcal{N}(A) \subseteq \mathcal{N}(A) \cap \mathcal{N}(C)$. Thus, (13.3) holds. □

Lemma 13.2 [29, 107]. Consider the switched dynamical system (13.1). Assume that there exists a family $\{P_p : p \in \mathcal{P}\}$ of symmetric, nonnegative-definite matrices such that, for every $\sigma \in \mathcal{S}$,

$$0 = A_p^{\mathrm{T}} P_p + P_p A_p + R_p, \quad p \in \mathcal{P}, \tag{13.4}$$

where $R_p = C_p^{\mathrm{T}} C_p$, $C_p \in \mathbb{R}^{l\times n}$, and the pair (A_p, C_p) is semiobservable for every $p \in \mathcal{P}$ and for an appropriately defined set of symmetric, nonnegative-definite matrices $\{R_p : p \in \mathcal{P}\}$. Then the following statements hold:

i) $\mathcal{N}(P_p) \subseteq \mathcal{N}(A_p) \subseteq \mathcal{N}(R_p)$, $p \in \mathcal{P}$.

ii) $\mathcal{N}(A_p) \cap \mathcal{R}(A_p) = \{0\}$, $p \in \mathcal{P}$.

Proposition 13.1. Consider the switched dynamical system (13.1). Assume that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric, nonnegative-definite matrices such that, for every $\sigma \in \mathcal{S}$, (13.4) holds, the pair (A_p, C_p) is semiobservable for every $p \in \mathcal{P}$ and for an appropriately defined set of symmetric, nonnegative-definite matrices $\{R_p : p \in \mathcal{P}\}$, and

$$x^{\mathrm{T}}(t)(P_{\sigma(t)} + L_{\sigma(t)}^{\mathrm{T}} L_{\sigma(t)})x(t) \leq x^{\mathrm{T}}(t)(P_{\sigma(t^-)} + L_{\sigma(t^-)}^{\mathrm{T}} L_{\sigma(t^-)})x(t), \quad t \geq 0, \tag{13.5}$$

where $L_p \triangleq I_n - A_p A_p^{\mathrm{D}}$. Then (13.1) is Lyapunov stable. If, in addition, $\{A_p : p \in \mathcal{P}\}$ is a compact set, then (13.1) is uniformly Lyapunov stable.

Proof. Let $p \in \mathcal{P}$. Since, by Lemma 13.2, $\mathcal{N}(A_p) \cap \mathcal{R}(A_p) = \{0\}$, it follows from Lemma 4.14 of [19] that A_p is group invertible. Furthermore, since $L_p^2 = L_p$, L_p is the unique $n \times n$ matrix satisfying $\mathcal{N}(L_p) = \mathcal{R}(A_p)$, $\mathcal{R}(L_p) = \mathcal{N}(A_p)$, and $L_p x = x$ for all $x \in \mathcal{N}(A_p)$.

Consider the multiple nonnegative functions

$$V_p(x) = x^{\mathrm{T}} P_p x + x^{\mathrm{T}} L_p^{\mathrm{T}} L_p x, \quad p \in \mathcal{P}, \quad x \in \mathbb{R}^n, \tag{13.6}$$

where P_p satisfies (13.4). If $V_p(x) = 0$ for some $x \in \mathbb{R}^n$, then $P_p x = 0$ and $L_p x = 0$. It follows from *i)* of Lemma 13.2 that $x \in \mathcal{N}(A_p)$, while $L_p x = 0$ implies $x \in \mathcal{R}(A_p)$. Now, it follows from *ii)* of Lemma 13.2 that $x = 0$. Hence, the family of functions $V_p(\cdot)$ are positive definite. Now, for every $x_{\mathrm{e}} \in \mathcal{E}$, consider the multiple Lyapunov function candidates $V_p(x - x_{\mathrm{e}})$, $p \in \mathcal{P}$. Note that since $A_p x_{\mathrm{e}} = 0$ for all $p \in \mathcal{P}$, it follows that $x(t) - x_{\mathrm{e}}$, $t \geq 0$, is also a solution of (13.1). Now, it follows from (13.5) that

$$V_{\sigma(t)}(x(t) - x_{\mathrm{e}}) \leq V_{\sigma(t^-)}(x(t) - x_{\mathrm{e}}). \tag{13.7}$$

Next, note that

$$\dot{V}_{\sigma(t)}(x(t) - x_{\mathrm{e}}) = -(x(t) - x_{\mathrm{e}})^{\mathrm{T}} R_{\sigma(t)}(x(t) - x_{\mathrm{e}}) + 2(x(t) - x_{\mathrm{e}})^{\mathrm{T}} L_{\sigma(t)}^{\mathrm{T}} L_{\sigma(t)} A_{\sigma(t)}(x(t) - x_{\mathrm{e}})$$

$$
\begin{aligned}
&= -(x(t) - x_{\rm e})^{\rm T} R_{\sigma(t)} (x(t) - x_{\rm e}) \\
&\leq 0, \quad t \geq 0. \qquad (13.8)
\end{aligned}
$$

Now, it follows from Theorem 2.3 of [38] that (13.1) is Lyapunov stable. Finally, if $\{A_p : p \in \mathcal{P}\}$ is compact, then $\{L_p^{\rm T} L_p : p \in \mathcal{P}\}$ is compact. Hence, it follows from Theorem 3 of [117] that (13.1) is uniformly Lyapunov stable. □

Proposition 13.2. Consider the switched dynamical system (13.1). Assume that every point in $\mathcal{E}$ is Lyapunov stable. Furthermore, assume that for a given $\sigma(t) \in \mathcal{S}$ and $x_0 \in \mathbb{R}^q$, the trajectory of (13.1) satisfies $x(t) \to \mathcal{E}$ as $t \to \infty$. Then $x(t) \to z$ as $t \to \infty$, where $z \in \mathcal{E}$. Alternatively, assume that every point in $\mathcal{E}$ is uniformly Lyapunov stable and for a given $x_0 \in \mathbb{R}^q$, the trajectory of (13.1) satisfies $x(t) \to \mathcal{E}$ as $t \to \infty$ uniformly in $\sigma(t) \in \mathcal{S}$. Then $x(t) \to z$ as $t \to \infty$ uniformly in $\sigma(t) \in \mathcal{S}$, where $z \in \mathcal{E}$.

Proof. Let $x_{\rm e} \in \mathcal{E}$. Choosing x_0 sufficiently close to $x_{\rm e}$, it follows from Lyapunov stability of $x_{\rm e}$ that the trajectories of (13.1) starting sufficiently close to $x_{\rm e}$ are bounded, and hence, there exists an increasing sequence $\{t_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} x(t_i)$ exists. Next, since $x(t) \to \mathcal{E}$ as $t \to \infty$, it follows that $\lim_{i\to\infty} x(t_i) \in \mathcal{E}$. Let $z \triangleq \lim_{i\to\infty} x(t_i) \in \mathcal{E}$. We show that $\lim_{t\to\infty} x(t) = z$. Note that, by assumption, $z \in \mathcal{E}$ is a Lyapunov stable equilibrium point. Let $\varepsilon > 0$ and note that since z is Lyapunov stable, it follows that there exists $\delta > 0$ such that $x(t) \in \mathcal{B}_\varepsilon(z)$ for all $x_0 \in \mathcal{B}_\delta(z)$ and $t \geq 0$. Next, since $z = \lim_{i\to\infty} x(t_i)$, it follows that there exists $k \geq 1$ such that $x(t_k) \in \mathcal{B}_\delta(z)$. We claim that $x(t) \in \mathcal{B}_\varepsilon(z)$ for all $t \geq t_k$. Suppose, *ad absurdum*, $x(t) \notin \mathcal{B}_\varepsilon(z)$ for some $t \geq t_k$. Then by continuity of $x(\cdot)$, there exists $\tau_i > t_i$ such that $x(\tau_i) \notin \mathcal{B}_\varepsilon(z)$ for every $i \geq k$. Namely, there exists a divergent sequence $\{\tau_i\}_{i=1}^{\infty}$ such that $x(\tau_i) \notin \mathcal{B}_\varepsilon(z)$ for all $\tau_i > t_k$. This contradicts Lyapunov stability of z. Since ε is arbitrary, it follows that $z = \lim_{t\to\infty} x(t)$. The proof of the second assertion is similar and, hence, is omitted. □

Lemma 13.3. Let $A \in \mathbb{R}^{n\times n}$. Assume that there exists a symmetric, nonnegative-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$0 = A^{\mathrm{T}}P + PA + R, \tag{13.9}$$

where $R = C^{\mathrm{T}}C$, $C \in \mathbb{R}^{l\times n}$, and the pair (A, C) is semiobservable. Then $\mathrm{spec}(A) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\} \cup \{0\}$ and, if $0 \in \mathrm{spec}(A)$, then 0 is semisimple. Alternatively, assume that there exists a symmetric, positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that (13.9) holds and

$$\mathrm{rank}\begin{bmatrix} A - \jmath\omega I_n \\ C \end{bmatrix} = n \tag{13.10}$$

for every nonzero $\omega \in \mathbb{R}$. Then $\mathrm{spec}(A) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\} \cup \{0\}$ and, if $0 \in \mathrm{spec}(A)$, then 0 is semisimple.

Proof. Consider the dynamical system $\mathcal{G}$ given by $\dot{x} = Ax$. Then it follows from Theorem 2.2 of [107] that $\mathcal{G}$ is semistable. Note that $\mathcal{G}$ is semistable if and only if the matrix A is semistable. Hence, it follows from *ii*) of Definition 11.7.1 of [22] that $\mathrm{spec}(A) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\} \cup \{0\}$ and, if $0 \in \mathrm{spec}(A)$, then 0 is semisimple. The second assertion is a direct consequence of Corollary 11.8.1 of [22]. $\square$

Lemma 13.4. Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$. If $\mathrm{rank}\,A < n$ and the pair (A, C) is semiobservable, then there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that

$$S^{-1}AS = \begin{bmatrix} \hat{A}_{11} & 0_{(n-1)\times 1} \\ [0_{1\times(n-3)}, 1, 0_{1\times 1}] & 0_{1\times 1} \end{bmatrix}, \quad CS = \begin{bmatrix} \hat{C}_1 & 0_{l\times 1} \end{bmatrix}, \tag{13.11}$$

where $\hat{A}_{11} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\hat{C}_1 \in \mathbb{R}^{l\times(n-1)}$. Furthermore, if $\mathrm{rank}\,A = n-1$ and the pair (A, C) is semiobservable, then there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that

$$T^{-1}AT = \begin{bmatrix} A_{11} & 0_{(n-r-1)\times r} & 0_{(n-r-1)\times 1} \\ A_{21} & A_{22} & 0_{r\times 1} \\ A_{31} & A_{32} & 0_{1\times 1} \end{bmatrix}, \quad CT = \begin{bmatrix} C_1 & 0_{l\times(r+1)} \end{bmatrix}, \tag{13.12}$$

where the pair (A_{11}, C_1) is observable, A_{22} is asymptotically stable, $A_{11} \in \mathbb{R}^{(n-r-1)\times(n-r-1)}$, $A_{21} \in \mathbb{R}^{r\times(n-r-1)}$, $A_{22} \in \mathbb{R}^{r\times r}$, $A_{31} \in \mathbb{R}^{1\times(n-r-1)}$, $A_{32} \in \mathbb{R}^{1\times r}$, $[A_{31}, A_{32}] = [0_{1\times(n-3)}, 1, 0_{1\times 1}]$ U^{-1}, $U \in \mathbb{R}^{(n-1)\times(n-1)}$ is nonsingular, and $C_1 \in \mathbb{R}^{l\times(n-r-1)}$.

Proof. Since $\operatorname{rank} A < n$, it follows that 0 is an eigenvalue of A. Now, since the pair (A, C) is semiobservable, it follows from Lemma 13.1 that $\mathcal{N}(A) \cap \mathcal{N}(C) = \mathcal{N}(A)$, that is, $\mathcal{N}\left(\left[\begin{array}{c} A \\ C \end{array}\right]\right) = \mathcal{N}(A)$. Next, it follows from the real Jordan decomposition (Theorem 5.3.5 of [22]) that there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that

$$S^{-1}AS = \left[\begin{array}{cc} \hat{A}_{11} & 0_{(n-1)\times 1} \\ [0_{1\times(n-3)}, 1, 0_{1\times 1}] & 0_{1\times 1} \end{array}\right], \tag{13.13}$$

where $\hat{A}_{11} \in \mathbb{R}^{(n-1)\times(n-1)}$. Note that $\mathcal{N}(AS) = \mathcal{N}(S^{-1}AS)$ and $\mathcal{N}\left[\begin{array}{c} AS \\ CS \end{array}\right] = \mathcal{N}(AS)$. Hence,

$$\mathcal{N}\left(\begin{array}{c} S\left[\begin{array}{cc} \hat{A}_{11} & 0_{(n-1)\times 1} \\ [0_{1\times(n-3)}, 1, 0_{1\times 1}] & 0_{1\times 1} \end{array}\right] \\ \left[\begin{array}{cc} \hat{C}_1 & \hat{C}_2 \end{array}\right] \end{array}\right) = \mathcal{N}\left(\left[\begin{array}{cc} \hat{A}_{11} & 0_{(n-1)\times 1} \\ [0_{1\times(n-3)}, 1, 0_{1\times 1}] & 0_{1\times 1} \end{array}\right]\right), \tag{13.14}$$

where $[\hat{C}_1, \hat{C}_2] = CS$. Now, it follows from (13.14) that $\hat{C}_2 = 0_{l\times 1}$, which implies that (13.11) holds.

To show the second assertion, consider the pair $(\hat{A}_{11}, \hat{C}_1)$. Then it follows from the Kalman decomposition (Proposition 12.9.11 of [22]) that there exists an invertible matrix $U \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$U^{-1}\hat{A}_{11}U = \left[\begin{array}{cc} A_{11} & 0 \\ A_{21} & A_{22} \end{array}\right], \quad \hat{C}_1 U = \left[\begin{array}{cc} C_1 & 0 \end{array}\right]. \tag{13.15}$$

Now, with

$$T \triangleq S\left[\begin{array}{cc} U & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & 1 \end{array}\right] \tag{13.16}$$

and $[A_{31}, A_{32}] \triangleq [0_{1\times(n-3)}, 1, 0]U^{-1}$, it follows that (13.12) holds. $\square$

13.3. Semistability of Switched Linear Systems

In this section, we present several sufficient conditions for semistability of switched linear systems.

Theorem 13.1. Consider the switched dynamical system (13.1). Assume that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric, nonnegative-definite matrices such that, for every $\sigma \in \mathcal{S}$, (13.4) and (13.5) hold, and the pair (A_p, C_p) is semiobservable for every $p \in \mathcal{P}$ and for an appropriately defined compact set of matrices $\{C_p : p \in \mathcal{P}\}$. Furthermore, assume that $\{A_p : p \in \mathcal{P}\}$ is compact. Then the following statements hold:

i) If $\mathcal{S} \subset \mathcal{S}_p[\tau, T]$ for some $\tau > 0$, $0 < T < \infty$, and $\mathcal{N}(A_{\sigma(t)}) \subseteq \bigcap_{s\in[0,t]} \mathcal{N}(A_{\sigma(s)})$, $t \geq 0$, then (13.1) is uniformly semistable.

ii) If $\mathcal{S} \subset \bigcup_{\tau>0, 0<T\leq\infty} \mathcal{S}_p[\tau, T]$ and $\mathcal{N}(A_{\sigma(t)}) \subseteq \bigcap_{s\in[0,t]} \mathcal{N}(A_{\sigma(s)})$, $t \geq 0$, then (13.1) is semistable.

Proof. *i*) It follows from Proposition 13.1 that (13.1) is uniformly Lyapunov stable. To show uniform semistability, it follows from Proposition 13.2 that we need to show $x(t) \to \mathcal{E}$ as $t \to \infty$ uniformly in σ. Let $\sigma \in \mathcal{S}$, let $x(t)$, $t \geq 0$, be a solution to (13.1), and let $\mathcal{T} \triangleq \{t_1, \tau_1, t_2, \tau_2, \ldots, t_k, \tau_k\} \subset (0, t)$ be an increasing sequence of time instants in the interval $(0, t)$ such that the lengths of the intervals $[t_i, \tau_i)$ are no smaller than τ on which $\sigma = p_i$ and the intervals between these have length no larger than T, that is, $\tau_i \geq t_i + \tau$ for $i \in \{1, 2, \ldots, k\}$, $t_{i+1} \leq \tau_i + T$ for $i \in \{1, 2, \ldots, k-1\}$, $t \leq \tau_k + T$, and $t_1 \leq T$. Next, it follows from Lemma 13.3 and Assumption 1 that $\mathrm{spec}(A_p) = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\} \cup \{0\}$ and 0 is semisimple for every $p \in \mathcal{P}$. Now, it follows from Lemma 13.4 that there exists an invertible matrix $S_p \in \mathbb{R}^{n\times n}$ such that, with $[x_{\mathrm{a}}^{\mathrm{T}}, x_{\mathrm{s}}]^{\mathrm{T}} = S_p x$, (13.1) can be transformed into the form

$$\begin{bmatrix} \dot{x}_{\mathrm{a}} \\ \dot{x}_{\mathrm{s}} \end{bmatrix} = \begin{bmatrix} \hat{A}_{p11} & 0_{(n-1)\times 1} \\ [0_{1\times(n-3)}, 1, 0_{1\times 1}] & 0_{1\times 1} \end{bmatrix} \begin{bmatrix} x_{\mathrm{a}} \\ x_{\mathrm{s}} \end{bmatrix}, \quad y = \begin{bmatrix} \hat{C}_{p1} & 0_{l\times 1} \end{bmatrix} \begin{bmatrix} x_{\mathrm{a}} \\ x_{\mathrm{s}} \end{bmatrix}, \qquad (13.17)$$

where $x_{\mathrm{a}} \in \mathbb{R}^{n-1}$, $x_{\mathrm{s}} \in \mathbb{R}$, and $\hat{A}_{p11}$ is asymptotically stable. Since $\hat{A}_{p11}$ is asymptotically stable, it follows that $\|e^{\hat{A}_{p11}t}\| < 1$ for every $t > 0$ and $p \in \mathcal{P}$.

Let $\mathcal{J}$ be the set of all sequences $p_1, p_2, \ldots, p_q \in \mathcal{P}$ with length of at most $\lceil T/\tau \rceil$, where $\lceil \cdot \rceil$ is a ceiling function defined by $\lceil x \rceil \triangleq \min\{n \in \mathbb{Z} : x \leq n\}$. Define

$$\mu \triangleq \max_{\tau_1 \in [\tau, \tau+T]} \max_{\tau_2 \in [\tau, \tau+T]} \cdots \max_{\tau_q \in [\tau, \tau+T]} \max_{\mathcal{J}} \|e^{\hat{A}_{p_q 11}\tau_q} \cdots e^{\hat{A}_{p_2 11}\tau_2} e^{\hat{A}_{p_1 11}\tau_1}\|. \tag{13.18}$$

Note that $\mathcal{J}$ is a finite set and $[\tau, \tau+T]$ is compact. Hence, it follows that

$$\mu \leq \max_{\mathcal{J}} \prod_{i=1}^{q} \max_{\tau_i \in [\tau, \tau+T]} \|e^{\hat{A}_{p_i 11}\tau_i}\| < 1. \tag{13.19}$$

Next, it follows from (13.18) that

$$\|e^{\hat{A}_{\sigma(t_i)11}(t_{i+1}-t_i)}\| \leq \mu, \quad i \in \{1, 2, \ldots, k\}. \tag{13.20}$$

Let $\Phi_\sigma(t, s)$ denote the state transition matrix of $\dot{x}_{\mathrm{a}} = \hat{A}_{\sigma 11} x_{\mathrm{a}}$ and note that

$$\Phi_\sigma(t, 0) = \Phi_\sigma(t, t_k)\Phi_\sigma(t_k, t_{k-1}) \cdots \Phi_\sigma(t_1, 0), \quad t > 0. \tag{13.21}$$

If $t < T + \tau$, then $\mathcal{T} = \emptyset$. Hence, for $t \geq T + \tau$, it follows that $\Phi_\sigma(t_{i+1}, t_i) = e^{\hat{A}_{\sigma(t_i)11}(t_{i+1}-t_i)}$, $i \in \{1, 2, \ldots, k-1\}$. Hence, it follows from (13.20) and (13.21) that

$$\|\Phi_\sigma(t, 0)\| \leq \|\Phi_\sigma(t, t_k)\| \cdot \|\Phi_\sigma(t_k, t_{k-1})\| \cdots \|\Phi_\sigma(t_1, 0)\| \leq \mu^k. \tag{13.22}$$

Since $x_{\mathrm{a}}(t) = \Phi_\sigma(t, 0)x_{\mathrm{a}}(0)$ and $0 < \mu < 1$, it follows from (13.22) that $\lim_{t\to\infty} x_{\mathrm{a}}(t) = 0$. Furthermore, since $t_1 \leq T$, and μ and k are independent of the switching signal σ, it follows that $x(t) \to 0$ as $t \to \infty$ uniformly in σ.

Next, note that $\dot{x}_{\mathrm{s}}(t) = [0_{1\times(n-3)}, 1, 0]x_{\mathrm{a}}(t)$, $t \geq 0$. Hence, $x_{\mathrm{s}}(t)$ is continuously differentiable and $\lim_{t\to\infty} \dot{x}_{\mathrm{s}}(t) = 0$ uniformly in σ. Thus, for every $h > 0$,

$$|x_{\mathrm{s}}(t+h) - x_{\mathrm{s}}(t)| \leq h|\dot{x}(\xi)|, \quad t < \xi < t+h, \tag{13.23}$$

which implies that $\lim_{t\to\infty} |x_{\mathrm{s}}(t+h) - x_{\mathrm{s}}(t)| = 0$ uniformly in σ, and hence, $\lim_{t\to\infty} x_{\mathrm{s}}(t)$ exists. Let $\lim_{t\to\infty} x_{\mathrm{s}}(t) = \alpha_{\mathrm{s}} \in \mathbb{R}$. Now, since

$$x(t_i + h_i) - x(t_i) = S_{\sigma(t_i)} \begin{bmatrix} x_{\mathrm{a}}(t_i + h_i) - x_{\mathrm{a}}(t_i) \\ x_{\mathrm{s}}(t_i + h_i) - x_{\mathrm{s}}(t_i) \end{bmatrix}, \tag{13.24}$$

where $0 < h_i < t_{i+1} - t_i$, $i \in \overline{\mathbb{Z}}_+$, and $\{S_p : p \in \mathcal{P}\}$ is compact, it follows that $\lim_{i\to\infty} \|x(t_i + h_i) - x(t_i)\| = 0$. Furthermore, since for $i \in \overline{\mathbb{Z}}_+$,

$$x(t_{i+1}^-) - x(t_i) = S_{\sigma(t_{i+1}^-)} \begin{bmatrix} x_{\mathrm{a}}(t_{i+1}^-) \\ x_{\mathrm{s}}(t_{i+1}^-) \end{bmatrix} - S_{\sigma(t_i)} \begin{bmatrix} x_{\mathrm{a}}(t_i) \\ x_{\mathrm{s}}(t_i) \end{bmatrix} = S_{\sigma(t_i)} \begin{bmatrix} x_{\mathrm{a}}(t_{i+1}) - x_{\mathrm{a}}(t_i) \\ x_{\mathrm{s}}(t_{i+1}) - x_{\mathrm{s}}(t_i) \end{bmatrix},$$

it follows that $\lim_{i\to\infty} \|x(t_{i+1}) - x(t_i)\| = 0$. Hence, for every $t \geq 0$ and $h > 0$, it follows that

$$x(t+h) - x(t) = x(t+h) - x(t_{i+j}) + \sum_{k=0}^{j-1} x(t_{i+k}) - x(t_{i+k-1}) + x(t_{i-1}) - x(t),$$

where $t_{i-1} < t \leq t_i < t_{i+1} < \cdots < t_{i+j} < t + h \leq t_{i+j+1}$. Hence,

$$\|x(t+h) - x(t)\| \leq \|x(t+h) - x(t_{i+j})\| + \sum_{k=0}^{j-1} \|x(t_{i+k}) - x(t_{i+k-1})\| + \|x(t) - x(t_{i-1})\|,$$

which implies that $\lim_{t\to\infty} \|x(t+h) - x(t)\| = 0$, and hence, $\lim_{t\to\infty} x(t)$ exists. Let $\lim_{t\to\infty} x(t) = \beta \in \mathbb{R}^n$. Note that this convergence is also uniform in σ.

Define $z_\sigma \triangleq S_\sigma^{-1}[0_{1\times(n-1)}, \alpha_{\mathrm{s}}]^{\mathrm{T}}$. Then $x(t) - z_{\sigma(t)} = S_{\sigma(t)}^{-1}[x_{\mathrm{a}}^{\mathrm{T}}(t), x_{\mathrm{s}}(t) - \alpha_{\mathrm{s}}]^{\mathrm{T}}$. Since the set $\{S_p^{-1} : p \in \mathcal{P}\}$ is compact, it follows that there exists $b > 0$ such that $\|S_p^{-1}\| \leq b$ for all $p \in \mathcal{P}$. Hence,

$$\|x(t) - z_{\sigma(t)}\| \leq b \left\| \begin{bmatrix} x_{\mathrm{a}}^{\mathrm{T}}(t) \\ x_{\mathrm{s}}(t) - \alpha_{\mathrm{s}} \end{bmatrix} \right\|, \quad t \geq 0, \tag{13.25}$$

which implies that $\lim_{t\to\infty} \|\beta - z_{\sigma(t)}\| = 0$. Hence, $\lim_{t\to\infty} z_{\sigma(t)} = \beta$. Note that $z_\sigma \in \mathcal{N}(A_\sigma)$ for every $\sigma \in \mathcal{S}$. Now, it follows from $\mathcal{N}(A_{\sigma(t_i)}) \subseteq \bigcap_{l=0}^{i} \mathcal{N}(A_{\sigma(t_l)})$, $i \in \overline{\mathbb{Z}}_+$, that $\beta \in \bigcap_{i=0}^{\infty} \mathcal{N}(A_{\sigma(t_i)}) = \bigcap_{p\in\mathcal{P}} \mathcal{N}(A_p) = \mathcal{E}$. Hence, $x(t) \to \mathcal{E}$ as $t \to \infty$, uniformly in σ. Finally, it follows from Proposition 13.2 that (13.1) is uniformly semistable.

ii) It follows from Proposition 13.1 that (13.1) is Lyapunov stable. To show semistability, it follows from Lemma 13.2 that we need to show $x(t) \to \mathcal{E}$ as $t \to \infty$. Let $\sigma \in \mathcal{S}$ and let $x(t)$, $t \geq 0$, be a solution to (13.1). Then $\sigma \in \mathcal{S}_p[\tau, T]$ for some $\tau > 0$ and $T \leq \infty$. However, τ and T are not uniform over all switching signals $\sigma(\cdot)$. If $T = \infty$, then it follows that there exists a switching time instant $t_m < \infty$ such that $x(t)$ is continuously differentiable for all $t > t_m$. In this case, it follows from Lemma 13.3 that $x(t) \to \mathcal{E}$ as $t \to \infty$.

Now we consider the case where $T < \infty$. Let $\mathcal{T} \triangleq \{t_1, \tau_1, t_2, \tau_2, \ldots, t_k, \tau_k\} \subset (0, t)$ be as defined in *i*). Next, it follows from Lemma 13.4 that there exists an invertible matrix $T_p \in \mathbb{R}^{n\times n}$ such that with $[x_{\mathrm{o}}^{\mathrm{T}}, x_{\mathrm{u}}^{\mathrm{T}}, x_{\mathrm{s}}]^{\mathrm{T}} = T_p x$, (13.1) can be transformed into the form

$$\begin{bmatrix} \dot{x}_{\mathrm{o}} \\ \dot{x}_{\mathrm{u}} \\ \dot{x}_{\mathrm{s}} \end{bmatrix} = \begin{bmatrix} A_{p11} & 0_{(n-r-1)\times r} & 0_{(n-r-1)\times 1} \\ A_{p21} & A_{p22} & 0_{r\times 1} \\ A_{p31} & A_{p32} & 0_{1\times 1} \end{bmatrix} \begin{bmatrix} x_{\mathrm{o}} \\ x_{\mathrm{u}} \\ x_{\mathrm{s}} \end{bmatrix}, \quad y = \begin{bmatrix} C_{p1} & 0_{l\times(r+1)} \end{bmatrix} \begin{bmatrix} x_{\mathrm{o}} \\ x_{\mathrm{u}} \\ x_{\mathrm{s}} \end{bmatrix}, \tag{13.26}$$

where $x_{\mathrm{o}} \in \mathbb{R}^{n-r-1}$, $x_{\mathrm{u}} \in \mathbb{R}^r$, $x_{\mathrm{s}} \in \mathbb{R}$, $y \in \mathbb{R}^l$, the pair (A_{p11}, C_{p1}) is observable, and A_{p22} is asymptotically stable. Since (A_{p11}, C_{p1}) is observable, it follows from Lemma 1 of [195] that for $\lambda, \delta > 0$ there exists a matrix $K_p \in \mathbb{R}^{(n-r-1)\times l}$ such that $\|e^{(A_{p11}+K_pC_{p1})t}\| \le \delta e^{-\lambda(t-\tau)}$, $t \ge \tau$, $p \in \mathcal{P}$.

Now, consider $\dot{x}_{\mathrm{o}} = (A_{\sigma 11} + K_\sigma C_{\sigma 1})x_{\mathrm{o}} - K_\sigma y$. First, we show that $\int_0^\infty \|y(t)\|^2 \mathrm{d}t < \infty$. Note that it follows from (13.8) that $\dot{V}_{\sigma(t)}(x(t)) = -x^{\mathrm{T}}(t)C_{\sigma(t)}^{\mathrm{T}}C_{\sigma(t)}x(t) = -\|y(t)\|^2$. Hence, $\int_0^\infty \|y(t)\|^2 \mathrm{d}t \le V_{\sigma(0)}(x(0)) < \infty$. Next, note that

$$x_{\mathrm{o}}(t) = e^{(A_{p11}+K_pC_{p1})t}x_{\mathrm{o}}(\tau_k) - \int_{\tau_k}^t e^{(A_{p11}+K_pC_{p1})(t-s)}K_p y(s)\mathrm{d}s, \quad t \in [\tau_k, t_{k+1}). \tag{13.27}$$

Hence, for every $t \in [\tau_k, t_{k+1})$, it follows from the Cauchy-Schwarz inequality that

$$\|x_{\mathrm{o}}(t)\| \le \delta e^{-\lambda(t-\tau)}\|x_{\mathrm{o}}(\tau_k)\| + \alpha \left(\int_{\tau_k}^t \|y(s)\|^2 \mathrm{d}s \right)^{1/2}, \tag{13.28}$$

where $\alpha \triangleq (\int_0^\infty \|e^{(A_{\sigma 11}+K_\sigma C_{\sigma 1})s}K_\sigma\|^2 \mathrm{d}s)^{1/2} < \infty$ since $\{A_p : p \in \mathcal{P}\}$ and $\{C_p : p \in \mathcal{P}\}$ are compact. Since (13.1) is Lyapunov stable, $\|x_{\mathrm{o}}(t)\|$, $t \ge 0$, is bounded.

Next, we show that $\lim_{t\to\infty} x_{\mathrm{o}}(t) = 0$. Suppose, *ad absurdum*, $x_{\mathrm{o}}(t) \not\to 0$ as $t \to \infty$. Then $\lim_{t\to\infty} x_{\mathrm{o}}(t) = \nu \ne 0$ or $\liminf_{t\to\infty} x_{\mathrm{o}}(t) \ne \limsup_{t\to\infty} x_{\mathrm{o}}(t)$. Note that τ_k was chosen so that $\tau_k \to \infty$ as $t \to \infty$. Since $\int_0^\infty \|y(t)\|^2 \mathrm{d}t < \infty$, it follows that $\lim_{\tau_k\to\infty} \int_{t_k}^\infty \|y(t)\|^2 \mathrm{d}t = 0$. Hence, $\lim_{t\to\infty} \int_{\tau_k}^t \|y(s)\|^2 \mathrm{d}s = 0$. Thus, if $\lim_{t\to\infty} x_{\mathrm{o}}(t) = \nu \ne 0$, then by taking the limit on both sides of (13.28), it follows that $\|\nu\| \le \delta\|\nu\|$, which is a contradiction since δ is arbitrary. Next, let $a \triangleq \liminf_{t\to\infty} \|x_{\mathrm{o}}(t)\|$ and $b \triangleq \limsup_{t\to\infty} \|x_{\mathrm{o}}(t)\|$ and note that $0 \le a < b < \infty$. Choose an unbounded sequence $\{\eta_n\}_{n=1}^\infty$ with $\tau_k \le \eta_{n_k} < t_{k+1}$ so that

$\limsup_{n\to\infty} \|x_{\rm o}(\eta_n)\| = b$. By taking $t = \eta_{n_k}$ in (13.28) and $n_k \to \infty$, it follows that $b \leq \delta b$, which is a contradiction since δ is arbitrary. Thus, $\lim_{t\to\infty} x_{\rm o}(t) = 0$.

Next, since $U_p^{-1}[0, x_{\rm u}^{\rm T}]^{\rm T}$ belongs to the unobservable subspace of the pair $(\hat{A}_{p11}, \hat{C}_{p1})$, where $U_p \in \mathbb{R}^{(n-1)\times(n-1)}$ denotes the Kalman transformation matrix of the pair $(\hat{A}_{p11}, \hat{C}_{p1})$, and $\hat{A}_{p11}$ and $\hat{C}_{p1}$ are given by (13.17), it follows that $U_p^{-1}[0, x_{\rm u}^{\rm T}]^{\rm T}$ belongs to the smallest subspace $\mathcal{M}$ that is $\hat{A}_{p11}$-invariant[7] for all $p \in \mathcal{P}$ and contains the unobservable subspaces of all pairs $(\hat{A}_{p11}, \hat{C}_{p1})$, $p \in \mathcal{P}$. Since $\hat{A}_{p11}$ is a full rank matrix, it follows that $\mathcal{M} = \{0\}$. Hence, $\lim_{t\to\infty} x_{\rm u}(t) = 0$.

Note that $\begin{bmatrix} A_{p11} & 0_{(n-r-1)\times r} \\ A_{p21} & A_{p22} \end{bmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}$ is a full rank matrix and $[A_{p31}, A_{p32}] \in \mathbb{R}^{1\times(n-1)}$. Then it follows that there exists $g_p \in \mathbb{R}^{1\times(n-1)}$ such that

$$[A_{p31}, A_{p32}] = g_p \begin{bmatrix} A_{p11} & 0_{(n-r-1)\times r} \\ A_{p21} & A_{p22} \end{bmatrix}. \tag{13.29}$$

Hence,

$$\dot{x}_{\rm s} = [A_{p31}, A_{p32}] \begin{bmatrix} x_{\rm o} \\ x_{\rm u} \end{bmatrix} = g_p \begin{bmatrix} A_{p11} & 0_{(n-r-1)\times r} \\ A_{p21} & A_{p22} \end{bmatrix} \begin{bmatrix} x_{\rm o} \\ x_{\rm u} \end{bmatrix} = g_p \begin{bmatrix} \dot{x}_{\rm o} \\ \dot{x}_{\rm u} \end{bmatrix}. \tag{13.30}$$

Now, it follows that

$$x_{\rm s}(t_i + h_i) - x_{\rm s}(t_i) = g_{\sigma(t_i)} \begin{bmatrix} x_{\rm o}(t_i + h_i) - x_{\rm o}(t_i) \\ x_{\rm u}(t_i + h_i) - x_{\rm u}(t_i) \end{bmatrix}, \quad 0 < h_i \leq t_{i+1} - t_i, \quad i \in \overline{\mathbb{Z}}_+,$$

which implies that $\lim_{i\to\infty} |x_{\rm s}(t_i + h_i) - x_{\rm s}(t_i)| = 0$. Using similar arguments as in the proof of i), it follows that $\lim_{t\to\infty} |x(t+h) - x(t)| = 0$ for $h > 0$, and hence, $\lim_{t\to\infty} x_{\rm s}(t)$ exists. The rest of the proof is similar to the proof of i). $\square$

Next, we present a stronger result for ensuring semistability for the switched linear system (13.1).

Theorem 13.2. Consider the switched dynamical system (13.1). Assume that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric, positive-definite matrices such that, for

[7] Given a matrix $A \in \mathbb{R}^{n\times n}$, a subspace $\mathcal{M}$ of $\mathbb{R}^n$ is *A-invariant* if and only if the state of $\dot{x} = Ax$ starting at time τ is such that $x(\tau) \in \mathcal{M}$, then $x(t) \in \mathcal{M}$ for all $t \geq \tau$.

every $\sigma \in \mathcal{S}$, (13.4) holds and

$$x^{\mathrm{T}}(t)P_{\sigma(t)}x(t) \le x^{\mathrm{T}}(t)P_{\sigma(t^-)}x(t), \quad t \ge 0, \tag{13.31}$$

for every $p \in \mathcal{P}$ and for an appropriately defined compact set of matrices $\{C_p : p \in \mathcal{P}\}$. Assume that $\{A_p : p \in \mathcal{P}\}$ is compact and $\operatorname{rank} A_p < n$ for every $p \in \mathcal{P}$. Furthermore, assume that there exists an invertible matrix $S_p \in \mathbb{R}^{n\times n}$, $p \in \mathcal{P}$, such that (13.1) can be transformed into (13.17). If $\mathcal{S} \subset \bigcup_{\tau>0,0<T<\infty} \mathcal{S}_p[\tau, T]$ and $\mathcal{N}(A_{\sigma(t)}) \subseteq \bigcap_{s\in[0,t]} \mathcal{N}(A_{\sigma(s)})$, $t \ge 0$, then (13.1) is semistable.

Proof. The proof is similar to the proof of *ii*) of Theorem 13.1 and, hence, is omitted. □

The next result uses the geometric (rank) condition given in Lemma 13.3 to develop a sufficient condition for semistability.

Theorem 13.3. Consider the switched dynamical system (13.1). Assume that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric, positive-definite matrices such that, for every $\sigma \in \mathcal{S}$, (13.4) and (13.31) hold, and $\operatorname{rank} \begin{bmatrix} A_p - \jmath\omega I_n \\ C_p \end{bmatrix} = n$ for every nonzero $\omega \in \mathbb{R}$ and every $p \in \mathcal{P}$, and for an appropriately defined compact set of matrices $\{C_p : p \in \mathcal{P}\}$. Furthermore, assume that $\{A_p : p \in \mathcal{P}\}$ is compact. Then the following statements hold:

i) If $\mathcal{S} \subset \mathcal{S}_p[\tau, T]$ for some $\tau > 0$, $0 < T < \infty$, and $\mathcal{N}(A_{\sigma(t)}) \subseteq \bigcap_{s\in[0,t]} \mathcal{N}(A_{\sigma(s)})$, $t \ge 0$, then (13.1) is uniformly semistable.

ii) If $\mathcal{S} \subset \bigcup_{\tau>0,0<T<\infty} \mathcal{S}_p[\tau, T]$ and $\mathcal{N}(A_{\sigma(t)}) \subseteq \bigcap_{s\in[0,t]} \mathcal{N}(A_{\sigma(s)})$, $t \ge 0$, then (13.1) is semistable.

Proof. The proofs of Lyapunov stability and uniform Lyapunov stability are similar to the proof of Proposition 13.1 by considering the family of Lyapunov functions $V_p(x) = x^{\mathrm{T}}P_p x$. Next, it follows from Lemma 13.3 and Assumption 1 that $\operatorname{spec}(A_p) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda <$

$0\} \cup \{0\}$ and 0 is semisimple for every $p \in \mathcal{P}$. Now, the proofs of $i)$ and $ii)$ are similar to the proofs of $i)$ and $ii)$ of Theorem 13.1, respectively. $\square$

Finally, we develop sufficient conditions for semistability of switched linear systems involving conditions less restrictive than those assumed in Theorems 13.1 and 13.3.

Theorem 13.4. Consider the switched dynamical system (13.1). Assume that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric, positive-definite matrices such that, for every $p \in \mathcal{P}$ and $\sigma \in \mathcal{S}$, (13.4) and (13.31) hold, and there exists an infinite sequence of nonempty, bounded, nonoverlapping time-intervals $[t_{i_j}, t_{i_j+k_j})$, $i \in \mathbb{Z}_+$, $j \in \mathbb{Z}_+$, where t_k denotes switching time instant, such that the switching times t_k satisfy $t_{k+1} - t_k \geq \tau > 0$, $k \in \overline{\mathbb{Z}}_+$, $t_0 \triangleq 0$, with the property that across each such interval, rank $\begin{bmatrix} A_{\sigma(t_{i_j+\ell})} - \jmath\omega_\ell I_n \\ C_{\sigma(t_{i_j+\ell})} \end{bmatrix} = n$ for all nonzero $\omega_\ell \in \mathbb{R}$ and every $\ell = 0, 1, \ldots, k_j - 1$, and an appropriately defined compact set of matrices $\{C_p : p \in \mathcal{P}\}$. Furthermore, assume that $\{A_p : p \in \mathcal{P}\}$ is compact. If $\mathcal{N}(A_{\sigma(t_i)}) \subseteq \bigcap_{l=0}^{i} \mathcal{N}(A_{\sigma(t_l)})$, $i \in \overline{\mathbb{Z}}_+$, then (13.1) is semistable.

Proof. The proof of Lyapunov stability is similar to the proof of Proposition 13.1 by considering the family of Lyapunov functions $V_p(x) = x^{\mathrm{T}} P_p x$. Since A_σ is Lyapunov stable for $\sigma \in \mathcal{S}$, it follows from $i)$ of Definition 11.7.1 of [22] that $\mathrm{spec}(A_\sigma) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \leq 0\}$ and, if $\lambda \in \mathrm{spec}(A_\sigma)$ and $\mathrm{Re}\,\lambda = 0$, then λ is semisimple. Since, by assumption, $\bigcap_{p \in \mathcal{P}} \mathcal{N}(A_p) - \{0\} \neq \emptyset$, it follows that there exists $z \in \mathbb{R}^n$, $z \neq 0$, such that $A_{\sigma(t)} z = 0$ for all $t \geq 0$, which further implies that 0 is a common eigenvalue of $A_{\sigma(t)}$ for all $t \geq 0$. Hence, $0 \in \mathrm{spec}(A_\sigma)$ and 0 is semisimple. Then, using similar arguments as in the proof of Lemma 13.4, it follows that for every $\sigma \in \mathcal{S}$ there exists an invertible matrix $S_\sigma \in \mathbb{R}^{n \times n}$ such that the matrix A_σ can be transformed into the form

$$S_\sigma^{-1} A_\sigma S_\sigma = \begin{bmatrix} \hat{A}_{\sigma 11} & 0_{(n-1)\times 1} \\ [0_{1\times(n-3)}, 1, 0_{1\times 1}] & 0_{1\times 1} \end{bmatrix}, \tag{13.32}$$

where $\hat{A}_{\sigma 11} \in \mathbb{R}^{(n-1)\times(n-1)}$ is Lyapunov stable. Furthermore, since

$$\operatorname{rank}\begin{bmatrix} A_{\sigma(t_{i_j+\ell})} - \jmath\omega_\ell I_n \\ C_{\sigma(t_{i_j+\ell})} \end{bmatrix} = n$$

for all nonzero $\omega_\ell \in \mathbb{R}$ and every $\ell = 0, 1, \ldots, k_j - 1$, it follows from Lemma 13.3 that $\hat{A}_{\sigma(t_{i_j+\ell})11} \in \mathbb{R}^{(n-1)\times(n-1)}$, $\ell = 0, 1, \ldots, k_j - 1$, is asymptotically stable. Since $\hat{A}_{\sigma 11}$ is Lyapunov stable, it follows from Proposition 11.2.3 of [22] that

$$\|e^{\hat{A}_{\sigma(t_i)11}(t_{i+1}-t_i)}\| \le 1, \quad i \in \overline{\mathbb{Z}}_+. \tag{13.33}$$

Moreover, since $\hat{A}_{\sigma(t_{i_j+\ell})11} \in \mathbb{R}$ is asymptotically stable, it follows that $\|e^{\hat{A}_{\sigma(t_{i_j+\ell})11}t}\| < 1$ for every $t > 0$ and $\ell = 0, 1, \ldots, k_j - 1$.

Consider the switched dynamical system given by

$$\begin{bmatrix} \dot{x}_{\mathrm{a}}(t) \\ \dot{x}_{\mathrm{s}}(t) \end{bmatrix} = \begin{bmatrix} \hat{A}_{\sigma(t)11} & 0_{(n-1)\times 1} \\ [0_{1\times(n-3)}, 1, 0_{1\times 1}] & 0_{1\times 1} \end{bmatrix} \begin{bmatrix} x_{\mathrm{a}}(t) \\ x_{\mathrm{s}}(t) \end{bmatrix}, \quad \begin{bmatrix} x_{\mathrm{a}}(0) \\ x_{\mathrm{s}}(0) \end{bmatrix} = S_{\sigma(0)}x(0), \quad t \ge 0. \tag{13.34}$$

Clearly, $[x_{\mathrm{a}}^{\mathrm{T}}(t), x_{\mathrm{s}}(t)]^{\mathrm{T}} = S_{\sigma(t)}x(t)$, where $x(t)$ denotes the solution of (13.1). By assumption there exists a finite upper bound T on the lengths of the intervals $[t_{i_j}, t_{i_j+k_j})$ across which

$$\operatorname{rank}\begin{bmatrix} A_{\sigma(t_{i_j+\ell})} - \jmath\omega_\ell I_n \\ C_{\sigma(t_{i_j+\ell})} \end{bmatrix} = n$$

for all nonzero $\omega_\ell \in \mathbb{R}$ and every $\ell = 0, 1, \ldots, k_j - 1$. Since $t_{i+1} - t_i \ge \tau$, $i \ge 0$, it follows that $k_j \le \lceil T/\tau \rceil$, $j \ge 1$.

Let $\mathcal{J}$ be the set of all sequences $p_1, p_2, \ldots, p_q \in \mathcal{P}$ with length of at most $\lceil T/\tau \rceil$ for which

$$\operatorname{rank}\begin{bmatrix} A_{\sigma(t_{i_j+\ell})} - \jmath\omega_\ell I_n \\ C_{\sigma(t_{i_j+\ell})} \end{bmatrix} = n$$

for all nonzero $\omega_\ell \in \mathbb{R}$ and every $\ell = 0, 1, \ldots, k_j - 1$, and define

$$\mu \triangleq \max_{\tau_1\in[\tau,T]} \max_{\tau_2\in[\tau,T]} \cdots \max_{\tau_q\in[\tau,T]} \max_{\mathcal{J}} \|e^{\hat{A}_{p_q 11}} \cdots e^{\hat{A}_{p_2 11}\tau_2} e^{\hat{A}_{p_1 11}\tau_1}\|. \tag{13.35}$$

Note that since $\mathcal{J}$ is a finite set and $[\tau, T]$ is compact, it follows that

$$\mu \leq \max_{\mathcal{J}} \prod_{i=1}^{q} \max_{\tau_i \in [\tau, T]} \|e^{\hat{A}_{p_i 11}\tau_i}\| < 1. \tag{13.36}$$

Next, it follows from (13.35) that

$$\|e^{\hat{A}_{\sigma(t_{i_j+k_j-1})}(t_{i_j+k_j}-t_{i_j+k_j-1})} \cdots e^{\hat{A}_{\sigma(t_{i_j+1})}(t_{i_j+2}-t_{i_j+1})} e^{\hat{A}_{\sigma(t_{i_j})}(t_{i_j+1}-t_{i_j})}\| \leq \mu, \quad j \geq 1. \tag{13.37}$$

Furthermore, note that

$$\begin{aligned} & e^{\hat{A}_{\sigma(t_{i_{j+1}-1})}(t_{i_{j+1}}-t_{i_{j+1}-1})} \cdots e^{\hat{A}_{\sigma(t_{i_j+1})}(t_{i_j+2}-t_{i_j+1})} e^{\hat{A}_{\sigma(t_{i_j})}(t_{i_j+1}-t_{i_j})} \\ &\quad = \left(e^{\hat{A}_{\sigma(t_{i_{j+1}-1})}(t_{i_{j+1}}-t_{i_{j+1}-1})} \cdots e^{\hat{A}_{\sigma(t_{i_j+k_j})}(t_{i_j+k_j+1}-t_{i_j+k_j})} \right) \\ &\qquad \cdot \left(e^{\hat{A}_{\sigma(t_{i_j+k_j-1})}(t_{i_j+k_j}-t_{i_j+k_j-1})} \cdots e^{\hat{A}_{\sigma(t_{i_j+1})}(t_{i_j+2}-t_{i_j+1})} e^{\hat{A}_{\sigma(t_{i_j})}(t_{i_j+1}-t_{i_j})} \right). \end{aligned} \tag{13.38}$$

Then it follows from (13.33) and (13.37) that

$$\|e^{\hat{A}_{\sigma(t_{i_{j+1}-1})}(t_{i_{j+1}}-t_{i_{j+1}-1})} \cdots e^{\hat{A}_{\sigma(t_{i_j+1})}(t_{i_j+2}-t_{i_j+1})} e^{\hat{A}_{\sigma(t_{i_j})}(t_{i_j+1}-t_{i_j})}\| \leq \mu, \quad j \geq 1. \tag{13.39}$$

Now, it follows from (13.39) that

$$\|x_{\mathrm{a}}(t_{i_{j+1}})\| \leq \mu \|x_{\mathrm{a}}(t_{i_j})\|, \quad j \geq 1. \tag{13.40}$$

Hence, $\|x_{\mathrm{a}}(t_{i_j})\| \leq \mu^{j-1}\|x_{\mathrm{a}}(t_{i_1})\|$, which implies that $\lim_{t\to\infty} x_{\mathrm{a}}(t) = 0$. Furthermore, note that $\dot{x}_{\mathrm{s}}(t) = [0_{1\times(n-3)}, 1, 0]x_{\mathrm{a}}(t)$, $t \geq 0$. Hence, $x_{\mathrm{s}}(\cdot)$ is continuously differentiable and $\lim_{t\to\infty} \dot{x}_{\mathrm{s}}(t) = 0$. Thus, for every $h > 0$,

$$|x_{\mathrm{s}}(t+h) - x_{\mathrm{s}}(t)| \leq h|\dot{x}(\xi)|, \quad t < \xi < t+h, \tag{13.41}$$

which implies that $\lim_{t\to\infty} |x_{\mathrm{s}}(t+h) - x_{\mathrm{s}}(t)| = 0$, and hence, $\lim_{t\to\infty} x_{\mathrm{s}}(t)$ exists. Let $\lim_{t\to\infty} x_{\mathrm{s}}(t) = \alpha_{\mathrm{s}} \in \mathbb{R}$.

Next, since

$$x(t_i + h_i) - x(t_i) = S_{\sigma(t_i)} \begin{bmatrix} x_{\mathrm{a}}(t_i+h_i) - x_{\mathrm{a}}(t_i) \\ x_{\mathrm{s}}(t_i+h_i) - x_{\mathrm{s}}(t_i) \end{bmatrix}, \tag{13.42}$$

where $0 < h_i < t_{i+1} - t_i$, $i \in \overline{\mathbb{Z}}_+$, and $\{S_p : p \in \mathcal{P}\}$ is compact, it follows that $\lim_{i\to\infty} \|x(t_i + h_i) - x(t_i)\| = 0$. Furthermore, since

$$\begin{aligned} x(t_{i+1}^-) - x(t_i) &= S_{\sigma(t_{i+1}^-)} \begin{bmatrix} x_{\mathrm{a}}(t_{i+1}^-) \\ x_{\mathrm{s}}(t_{i+1}^-) \end{bmatrix} - S_{\sigma(t_i)} \begin{bmatrix} x_{\mathrm{a}}(t_i) \\ x_{\mathrm{s}}(t_i) \end{bmatrix} \\ &= S_{\sigma(t_i)} \begin{bmatrix} x_{\mathrm{a}}(t_{i+1}) - x_{\mathrm{a}}(t_i) \\ x_{\mathrm{s}}(t_{i+1}) - x_{\mathrm{s}}(t_i) \end{bmatrix}, \end{aligned} \tag{13.43}$$

$i \in \overline{\mathbb{Z}}_+$, it follows that $\lim_{i\to\infty} \|x(t_{i+1}) - x(t_i)\| = 0$. Hence, for every $t \geq 0$ and $h > 0$, it follows that

$$x(t+h) - x(t) = x(t+h) - x(t_{i+j}) + \sum_{k=0}^{j-1} x(t_{i+k}) - x(t_{i+k-1}) + x(t_{i-1}) - x(t), \tag{13.44}$$

where $t_{i-1} < t \leq t_i < t_{i+1} < \cdots < t_{i+j} < t + h \leq t_{i+j+1}$. Hence,

$$\|x(t+h) - x(t)\| \leq \|x(t+h) - x(t_{i+j})\| + \sum_{k=0}^{j-1} \|x(t_{i+k}) - x(t_{i+k-1})\| + \|x(t) - x(t_{i-1})\|,$$

which implies that $\lim_{t\to\infty} \|x(t+h) - x(t)\| = 0$, and hence, $\lim_{t\to\infty} x(t)$ exists. Let $\lim_{t\to\infty} x(t) = \beta \in \mathbb{R}^n$.

Define $z_\sigma \triangleq S_\sigma^{-1}[0_{1\times(n-1)}, \alpha_{\mathrm{s}}]^{\mathrm{T}}$. Then $x(t) - z_{\sigma(t)} = S_{\sigma(t)}^{-1}[x_{\mathrm{a}}^{\mathrm{T}}(t), x_{\mathrm{s}}(t) - \alpha_{\mathrm{s}}]^{\mathrm{T}}$. Since the set $\{S_p^{-1} : p \in \mathcal{P}\}$ is compact, it follows that there exists $b > 0$ such that $\|S_p^{-1}\| \leq b$ for all $p \in \mathcal{P}$. Hence,

$$\|x(t) - z_{\sigma(t)}\| \leq b \left\| \begin{bmatrix} x_{\mathrm{a}}^{\mathrm{T}}(t) \\ x_{\mathrm{s}}(t) - \alpha_{\mathrm{s}} \end{bmatrix} \right\|, \quad t \geq 0, \tag{13.45}$$

which implies that $\lim_{t\to\infty} \|\beta - z_{\sigma(t)}\| = 0$. Hence, $\lim_{t\to\infty} z_{\sigma(t)} = \beta$. Note that $z_\sigma \in \mathcal{N}(A_\sigma)$ for every $\sigma \in \mathcal{S}$. Now, it follows from $\mathcal{N}(A_{\sigma(t_i)}) \subseteq \bigcap_{l=0}^{i} \mathcal{N}(A_{\sigma(t_l)})$, $i \in \overline{\mathbb{Z}}_+$, that $\beta \in \bigcap_{i=0}^{\infty} \mathcal{N}(A_{\sigma(t_i)}) = \bigcap_{p\in\mathcal{P}} \mathcal{N}(A_p) = \mathcal{E}$. Hence, $x(t) \to \mathcal{E}$ as $t \to \infty$. Finally, it follows from Proposition 13.2 that (13.1) is semistable. □

Theorem 13.5. Consider the switched dynamical system (13.1). Assume that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric, positive-definite matrices such that, for every $p \in \mathcal{P}$ and $\sigma \in \mathcal{S}$, (13.4) and (13.31) hold, and there exists an infinite sequence of nonempty, bounded, nonoverlapping time-intervals $[t_{i_j}, t_{i_j+k_j})$, $i \in \mathbb{Z}_+$, $j \in \mathbb{Z}_+$, where t_k denotes switching time instants, such that the switching times t_k satisfy $t_{k+1} - t_k \geq \tau > 0$, $k \in \overline{\mathbb{Z}}_+$, $t_0 \triangleq 0$, with the property that across each such interval the pair $(A_{\sigma(t_{i_j+\ell})}, C_{\sigma(t_{i_j+\ell})})$ is semiobservable for every $\ell = 0, 1, \ldots, k_j - 1$ and an appropriately defined compact set of matrices $\{C_p : p \in \mathcal{P}\}$. Furthermore, assume that $\{A_p : p \in \mathcal{P}\}$ is compact. If $\mathcal{N}(A_{\sigma(t_i)}) \subseteq \bigcap_{l=0}^{i} \mathcal{N}(A_{\sigma(t_l)})$, $i \in \overline{\mathbb{Z}}_+$, then (13.1) is semistable.

Proof. Note that since the pair $(A_{\sigma(t_{i_j+\ell})}, C_{\sigma(t_{i_j+\ell})})$ is semiobservable for every $\ell = 0, 1, \ldots, k_j - 1$, it follows from Lemma 13.3 that $\hat{A}_{\sigma(t_{i_j+\ell})}$ in (13.32) is asymptotically stable. Now, the rest of the proof is similar to the proof of Theorem 13.4. $\square$

Chapter 14

Complexity, Robustness, Self-Organization, Swarms, and System Thermodynamics

14.1. Introduction

Due to technological advances in sensing, actuation, communication, and computation over the last several years, a considerable research effort has been devoted to the control of networks and control over networks. Network systems involve distributed decision-making for coordination of dynamic agents involving information flow enabling enhanced operational effectiveness via cooperative control in autonomous systems. These dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles (UAV's) and autonomous underwater vehicles (AUV's) for combat, surveillance, and reconnaissance; distributed reconfigurable sensor networks for managing power levels of wireless networks; air and ground transportation systems for air traffic control and payload transport and traffic management; swarms of air and space vehicle formations for command and control between heterogeneous air and space vehicles; and congestion control in communication networks for routing the flow of information through a network.

To enable the autonomous operation for these multiagent systems, the development of functional algorithms for agent coordination and control is needed. In particular, control algorithms need to address agent interactions, cooperative and non-cooperative control, task assignments, and resource allocations. To realize these tasks, appropriate sensory and cognitive capabilities such as adaptation, learning, decision-making, and agreement (or consensus) on the agent and multiagent levels are required. The common approach for addressing the autonomous operation of multiagent systems is using distributed control algorithms involving neighbor-to-neighbor interaction between agents wherein agents update their information state based on the information states of the neighboring agents. Since most multiagent network systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication networks, these systems are characterized by high-dimensional, large-scale interconnected dynamical systems. To develop distributed methods for control and coordination of autonomous multiagent systems, many researchers have looked to autonomous *swarm* systems appearing in nature for inspiration [152, 154, 176, 197, 207, 230].

Biology has shown that many species of animals such as insect swarms, ungulate flocks, fish schools, ant colonies, and bacterial colonies *self-organize* in nature [18, 46, 184, 196]. These biological aggregations give rise to remarkably complex global behaviors from simple local interactions between large numbers of relatively unintelligent agents without the need for centralized control. The spontaneous development (i.e., self-organization) of these autonomous biological systems and their spatio-temporal evolution to more complex states often appears without any external system interaction. In other words, structure morphing into coherent groups is internal to the system and results from local interactions among subsystem components that are independent of the physical nature of the individual components. These local interactions often comprise a simple set of rules that lead to remarkably complex global behaviors. *Complexity* here refers to the quality of a system wherein interacting subsystems self-organize to form hierarchical evolving structures exhibiting *emergent* system properties. Hence, a complex dynamical system is a system that is greater than the sum of its subsystems or parts. In addition, the spatially distributed sensing and actuation control architecture prevalent in such systems is inherently robust to individual subsystem (or agent) failures and unplanned behavior at the individual subsystem (or agent) level.

The connection between the local subsystem interactions and the globally complex system behavior is often elusive. Complex dynamical systems involving self-organizing components forming spatio-temporally evolving structures that exhibit a hierarchy of emergent system properties are not limited to biological aggregation systems. Such systems include, for example, nervous systems, immune systems, ecological systems, quantum particle systems, chemical reaction systems, economic systems, cellular systems, and galaxies, to cite but a few examples. These systems are known as *dissipative systems* [104,143] and consume energy and matter while maintaining their stable structure by dissipating entropy to the environment. For example, as in biology,[8] in the physical universe billions of stars and galaxies interact to form self-organizing dissipative nonequilibrium structures [143, 202]. The fundamental common phenomenon among these systems are that they evolve in accordance to the laws of (nonequilibrium) thermodynamics which are among the most firmly established laws of nature. System thermodynamics, in the sense of [104], involves open interconnected dynamical systems that exchange matter and energy with their environment in accordance with the first law (conservation of energy) and the second law (nonconservation of entropy) of thermodynamics. Self-organization can spontaneously occur in such systems by invoking the two fundamental axioms of the science of heat. Namely, *i*) if the energies in the connected subsystems of an interconnected system are equal, then energy exchange between these subsystems is not possible, and *ii*) energy flows from more energetic subsystems to less energetic subsystems. These axioms establish the existence of a system entropy function as well as *equipartition of energy* [104] in system thermodynamics and *information consensus* [126] in cooperative networks; an *emergent* behavior in thermodynamic systems as well as swarm systems. Hence, in complex interconnected dynamical systems, self-organization is not a property of the system's parts but rather emerges as a result of the nonlinear subsystem interactions.

In light of the above discussion, engineering swarm systems necessitates the development of relatively simple autonomous agents that are inherently distributed, self-organized, and truly scalable. Scalability follows from the fact that such systems do not involve centralized control and communication architectures. In addition, engineered swarming systems should

[8] All living systems are dissipative systems, the converse, however, is not necessarily true. Dissipative living systems involve pattern interactions by which life emerges. This nonlinear interaction between the subsystems making up a living system is characterized by *autopoiesis* (self-creation).

be inherently robust to individual agent failures, unplanned task assignment changes, and environmental changes. Mathematical models for large-scale swarms can involve Lagrangian and Eulerian models. In a Lagrangian model, each agent is modeled as a particle governed by a difference or differential equation, whereas an Eulerian model describes the local energy or information flux for a distribution of swarms with an advection-diffusion (conservation) equation. The two formulations can be connected by a Fokker-Plank approximation relating jump distance distributions of individual agents to terms in the advection-diffusion equation [184].

As discussed in Chapter 8, in many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In particular, it is important to develop information consensus protocols for networks of dynamic agents wherein a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus is the existence of a continuum of equilibria representing a state of equipartitioning or *consensus*. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well. For such systems possessing a continuum of equilibria, semistability [31,32], and not asymptotic stability, is the relevant notion of stability.

In this chapter, we develop distributed boundary control algorithms for addressing the consensus problem for an Eulerian swarm model. The proposed distributed boundary controller architectures are predicated on the recently developed notion of system thermodynamics [104] resulting in controller architectures involving the exchange of information between uniformly distributed swarms over an n-dimensional (not necessarily Euclidian) space that guarantee that the closed-loop system is consistent with basic thermodynamic principles. For our thermodynamically consistent model we further establish the existence of a unique continuously differentiable entropy functional for all equilibrium and nonequilibrium states of our system. Information consensus and semistability are shown using the well-known Sobolev embedding theorems and the notion of generalized (or weak) solutions. Finally, since the closed-loop system is guaranteed to satisfy basic thermodynamic principles, robustness to individual agent failures and unplanned individual agent behavior is automatically guaranteed.

14.2. Mathematical Preliminaries

In this chapter, we consider an Eulerian swarm model involving a nonlocal spatio-temporal distribution of swarm density. Specifically, consider the evolution equation for swarm aggregations defined over a compact connected set $\mathcal{V} \subset \mathbb{R}^n$ with a smooth boundary $\partial\mathcal{V}$ and volume $\mathrm{vol}\,\mathcal{V}$ characterized by the *conservation* equation [70, 104]

$$\frac{\partial u(x,t)}{\partial t} = -\nabla \cdot \phi(x, u(x,t), \nabla u(x,t)), \quad x \in \mathcal{V}, \quad t \geq t_0, \tag{14.1}$$

$$u(x,t_0) = u_{t_0}(x) \in \mathcal{X}, \quad x \in \mathcal{V}, \quad \phi(x, u(x,t), \nabla u(x,t)) \cdot \mathbf{n}(x) \geq 0, \quad x \in \partial\mathcal{V}, \quad t \geq t_0, \tag{14.2}$$

where $u : \mathcal{V} \times [0, \infty) \to \overline{\mathbb{R}}_+$ denotes the density distribution at the point $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathcal{V}$ and time instant $t \geq t_0$, $\phi : \mathcal{V} \times [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ denotes a continuously differentiable *flux* function, ∇ denotes the nabla operator, "$\cdot$" denotes the dot product in $\mathbb{R}^n$, $\mathbf{n}^{\mathrm{T}}(x)$ denotes the outward normal vector to the boundary $\partial\mathcal{V}$ at $x \in \partial\mathcal{V}$, and $\mathcal{X}$ denotes a space of two-times continuously differentiable scalar functions defined on $\mathcal{V}$. Here, we assume that $\mathcal{V} = \{x \in \mathbb{R}^n : f(x) \leq 0\}$ and $\partial\mathcal{V} = \{x \in \mathbb{R}^n : f(x) = 0\}$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a given continuously differentiable function, and consequently, the outward normal vector to the boundary $\partial\mathcal{V}$ at $x \in \partial\mathcal{V}$ is given by $\mathbf{n}^{\mathrm{T}}(x) = \nabla f(x)$.

Equations (14.1) and (14.2) involve an information (or energy) flow equation for a uniformly distributed continuous system. Specifically, note that for a smooth, bounded region $\mathcal{V} \subset \mathbb{R}^n$, the integral $\int_{\mathcal{V}} u(x,t)\mathrm{d}\mathcal{V}$ denotes the total information (or energy) amount within $\mathcal{V}$ at time t. Hence, the rate of information change within $\mathcal{V}$ is governed by the flux function $\phi : \mathcal{V} \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, which controls the rate of information transmission through the boundary $\partial\mathcal{V}$. Hence, for each time t,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} u(x,t)\mathrm{d}\mathcal{V} = -\int_{\partial\mathcal{V}} \phi(x,u(x,t),\nabla u(x,t))\cdot \mathbf{n}(x)\mathrm{d}\mathcal{S}_{\mathcal{V}}, \tag{14.3}$$

where $\mathrm{d}\mathcal{S}_{\mathcal{V}}$ denotes an infinitesimal surface element of the boundary of the set $\mathcal{V}$. Using the divergence theorem, it follows from (14.3) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} u(x,t)\mathrm{d}\mathcal{V} &= -\int_{\partial\mathcal{V}} \phi(x,u(x,t),\nabla u(x,t))\cdot \mathbf{n}(x)\mathrm{d}\mathcal{S}_{\mathcal{V}}\\ &= -\int_{\mathcal{V}} \nabla\cdot\phi(x,u(x,t),\nabla u(x,t))\mathrm{d}\mathcal{V}.\end{aligned} \tag{14.4}$$

Since the region $\mathcal{V}\subset\mathbb{R}^n$ is arbitrary, it follows that the conservation equation over a unit volume within the continuum $\mathcal{V}$ involving the rate of information density change within the continuum is given by (14.1) and (14.2). The physical interpretation of (14.1) and (14.2) is straightforward. In particular, if $u(x,t)$ is an information (or energy) density at point $x\in\mathcal{V}$ and time $t\geq t_0$, then the conservation equation (14.1) describes the time evolution of the information (or energy) density $u(x,t)$ over the region $\mathcal{V}$, while the boundary condition in (14.2) involving the dot product implies that the information (or energy) of the system (14.1) and (14.2) can either be stored or transmitted but not supplied through the boundary of $\mathcal{V}$ from the environment.

We denote the information (or energy) distribution over the set $\mathcal{V}$ at time $t\geq t_0$ by $u_t\in\mathcal{X}$ so that for each $t\geq t_0$ the set of mappings generated by $u_t(x)\equiv u(x,t)$ for every $x\in\mathcal{V}$ gives the *flow* of (14.1) and (14.2). We assume that the function $\phi(\cdot,\cdot,\cdot)$ is continuously differentiable so that (14.1) and (14.2) admits a unique solution $u(x,t)$, $x\in\mathcal{V}$, $t\geq t_0$, and $u(\cdot,t)\in\mathcal{X}$, $t\geq t_0$, is continuously dependent on the initial information (or energy) distribution $u_{t_0}(x)$, $x\in\mathcal{V}$. It is well known, however, that nonlinear partial differential equations need not have smooth differentiable solutions (*classical solutions*), and one has to use the notion of Schwartz distributions that provides a framework in which the information (or energy) density function $u(x,t)$ may be differentiated in a generalized sense infinitely often [70]. In this case, one has a well-defined notion of solutions that have jump discontinuities, which propagate as shock waves. Thus, one has to deal with *generalized* or *weak* solutions wherein uniqueness is lost. In this case, the *Clausius-Duhem* inequality is invoked for identifying the physically relevant (i.e., thermodynamically admissible) solution [63, 70].

If u_{t_0} is a two-times continuously differentiable function with compact support and its derivative is sufficiently small on $[t_0, \infty)$, then the classical solution to (14.1) and (14.2) can break down at a finite time. As a consequence of this, one may only hope to find generalized (or weak) solutions to (14.1) and (14.2) over the semi-infinite interval $[t_0, \infty)$, that is, $\mathcal{L}_\infty$ functions[9] $u(\cdot,\cdot)$ that satisfy (14.1) in the sense of distributions, which provides a framework in which $u(\cdot,\cdot)$ may be differentiated in a general sense infinitely often. It is important to note that we do *not* assume strict hyperbolicity of (14.1) and (14.2) since our interest in this chapter is to address *semistability*, and hence, (14.1) and (14.2) cannot be hyperbolic. Thus, many results on well-posedness of solutions of (14.1) and (14.2) developed in the literature are not applicable in this case. Furthermore, the linearization method also fails to provide any stability information due to nonhyperbolicity. Global well-posedness of smooth solutions of nonhyperbolic partial differential equations of the form (14.1) and (14.2) remains an open problem in mathematics [73]. Finally, the control aim here is to design a *boundary control* law so that the corresponding closed-loop system achieves semistability and *uniform information distribution* [104].

In this chapter, $\mathcal{L}_2$ denotes the space of square-integrable Lebesgue measurable functions on $\mathcal{V}$ and the $\mathcal{L}_2$ operator norm $\|\cdot\|_{\mathcal{L}_2}$ on $\mathcal{X}$ is used for the definitions of Lyapunov, semi-, and asymptotic stability. Furthermore, we introduce the Sobolev spaces

$$\mathcal{W}_2^0(\mathcal{V}) \triangleq \{u_t : \mathcal{V} \to \mathbb{R} : u_t \in \mathrm{C}^0(\mathcal{V}) \cap \mathcal{L}_2(\mathcal{V})\}_{\mathrm{co}} \subset \mathcal{L}_2(\mathcal{V}), \tag{14.5}$$

$$\mathcal{W}_2^1(\mathcal{V}) \triangleq \{u_t : \mathcal{V} \to \mathbb{R} : u_t \in \mathrm{C}^1(\mathcal{V}) \cap \mathcal{L}_2(\mathcal{V}), (\nabla u_t)^{\mathrm{T}} \in \mathcal{L}_2(\mathcal{V})\}_{\mathrm{co}}, \tag{14.6}$$

where $\mathrm{C}^r(\mathcal{V})$ denotes a function space defined on $\mathcal{V}$ with r-continuous derivatives and $\{\cdot\}_{\mathrm{co}}$

[9] $\mathcal{L}_\infty$ denotes the space of bounded Lebesgue measurable functions on $\mathcal{V}$ and provides the broadest framework for weak solutions. Alternatively, a natural function class for weak solutions is the space $\mathcal{BV}$ consisting of functions of bounded variation. Recall that a bounded measurable function $u(x,t)$ has locally bounded variation if its distributional derivatives are locally finite Radon measures.

denotes completion[10] of $\{\cdot\}$ in $\mathcal{L}_2$ in the sense of [233], with norms

$$\|u_t\|_{\mathcal{W}_2^0} \triangleq \|u_t\|_{\mathcal{L}_2} = \left[\int_{\mathcal{V}} u_t^2(x)\mathrm{d}\mathcal{V}\right]^{\frac{1}{2}}, \tag{14.7}$$

$$\|u_t\|_{\mathcal{W}_2^1} \triangleq \left[\|u_t\|_{\mathcal{W}_2^0}^2 + D(u_t, u_t)\right]^{\frac{1}{2}}, \tag{14.8}$$

defined on $\mathcal{W}_2^0(\mathcal{V})$ and $\mathcal{W}_2^1(\mathcal{V})$, respectively, where the gradient $\nabla u_t(x)$ in (14.8) is interpreted in the sense of a generalized gradient [233], and $D(u_t, u_t) \triangleq \int_{\mathcal{V}} \nabla u_t(x) \nabla^{\mathrm{T}} u_t(x)\mathrm{d}\mathcal{V}$ is the *Dirichlet integral* of u [77, p. 88]. Physically the Dirichlet integral term represents the potential energy in $\mathcal{V}$ of the *electrostatic field* $-\nabla u$. Note that since the solutions to (14.1) and (14.2) are assumed to be two-times continuously differentiable functions on a compact set $\mathcal{V}$ and ϕ is continuously differentiable, it follows that $u_t(x)$, $t \geq t_0$, belongs to $\mathcal{W}_2^0(\mathcal{V})$ and $\mathcal{W}_2^1(\mathcal{V})$.

14.3. A Thermodynamic Model for Large-Scale Swarms

The nonlinear conservation equation (14.1) and (14.2) can exhibit a full range of nonlinear behavior, including bifurcations, limit cycles, and even chaos. To ensure a thermodynamically consistent information (or energy) flow model involving a diffusive (parabolic) character additional assumptions are required. In this section, we develop a large-scale swarm model that is consistent with basic thermodynamic principles. First, however, we establish several key definitions and stability results for nonlinear infinite-dimensional systems. Here, the state space is assumed to be a Banach space with fully nonlinear dynamics.

Let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|_{\mathcal{B}}$. A *dynamical system* $\mathcal{G}$ on $\mathcal{B}$ is the triple $(\mathcal{B}, [t_0, \infty), s)$, where $s : [t_0, \infty) \times \mathcal{B} \to \mathcal{B}$ is such that the following axioms hold: *i*) (*Continuity*): $s(\cdot, \cdot)$ is jointly continuous, *ii*) (*Consistency*): $s(t_0, z_0) = z_0$ for all $t_0 \in \mathbb{R}$ and $z_0 \in \mathcal{B}$,

[10] The space $\{\cdot\}$ defined as part of (14.6) is not complete with respect to the norm generated by the inner product (14.8). This space can be completed by adding the limit points of all Cauchy sequences in $\{\cdot\}$. In this way, $\{\cdot\}$ is embedded in the larger normed space $\{\cdot\}_{\mathrm{co}}$, which is complete. Of course, it follows from the Riesz-Fischer theorem [210, p. 125] that $\mathcal{L}_2$ is complete with respect to the norm generated by the inner product (14.7).

and *iii*) (*Semigroup property*): $s(t+\tau, z_0) = s(\tau, s(t, z_0))$ for all $z_0 \in \mathcal{B}$ and $t, \tau \in [t_0, \infty)$. Given $t \in [0, \infty)$ we denote the *flow* $s(t, \cdot) : \mathcal{B} \to \mathcal{B}$ of $\mathcal{G}$ by $s_t(x_0)$ or s_t. Likewise, given $x \in \mathcal{B}$ we denote the *solution curve* or *trajectory* $s(\cdot, x) : [0, \infty) \to \mathcal{B}$ of $\mathcal{G}$ by $s^x(t)$ or s^x. The *positive limit set* of $x \in \mathcal{B}$ is the set $\omega(x)$ of points $z \in \mathcal{B}$ such that there exists an increasing sequence $\{t_i\}_{i=1}^{\infty}$ satisfying $s(t_i, x) \to z$ as $i \to \infty$. Finally, the image of $\mathcal{U} \subset \mathcal{B}$ under the flow s_t is defined by $s_t(\mathcal{U}) \triangleq \{y : y = s_t(x_0) \text{ for all } x_0 \in \mathcal{U}\}$.

An *equilibrium point* of $\mathcal{G}$ is a point $z \in \mathcal{B}$ such that $s(t, z) = s(t_0, z)$ for all $t \geq t_0$. A set $\mathcal{M} \subseteq \mathcal{B}$ is *positively invariant* if $s_t(\mathcal{M}) \subseteq \mathcal{M}$ for all $t \geq 0$. The set $\mathcal{M}$ is *negatively invariant* if, for every $z \in \mathcal{M}$ and every $t \geq 0$, there exists $x \in \mathcal{M}$ such that $s(t, x) = z$ and $s(\tau, x) \in \mathcal{M}$ for all $\tau \in [0, t]$. The set $\mathcal{M}$ is *invariant* if $s_t(\mathcal{M}) = \mathcal{M}$, $t \geq 0$. Note that a set is invariant if and only if it is positively and negatively invariant.

Definition 14.1. Let $\mathcal{G}$ be a dynamical system on a Banach space $\mathcal{B}$ with norm $\|\cdot\|_{\mathcal{B}}$ and let $\mathcal{D}$ be a positively invariant set with respect to $\mathcal{G}$. An equilibrium point $x \in \mathcal{D}$ of $\mathcal{G}$ is *Lyapunov stable* if for every relatively open subset $\mathcal{N}_\varepsilon$ of $\mathcal{D}$ containing x, there exists a relatively open subset $\mathcal{N}_\delta$ of $\mathcal{D}$ containing x such that $s_t(\mathcal{N}_\delta) \subseteq \mathcal{N}_\varepsilon$ for all $t \geq t_0$. An equilibrium point $x \in \mathcal{D}$ of $\mathcal{G}$ is *semistable* if it is Lyapunov stable and there exists a relatively open subset $\mathcal{U}$ of $\mathcal{D}$ containing x such that for all initial conditions in $\mathcal{U}$, the trajectory $s(\cdot, \cdot)$ of $\mathcal{G}$ converges to a Lyapunov stable equilibrium point, that is, $\lim_{t\to\infty} s(t, z) = y$, where $y \in \mathcal{D}$ is a Lyapunov stable equilibrium point of $\mathcal{G}$ and $z \in \mathcal{U}$. Finally, an equilibrium point $x \in \mathcal{D}$ of $\mathcal{G}$ is *asymptotically stable* if it is Lyapunov stable and there exists a relatively open subset $\mathcal{U}$ of $\mathcal{D}$ containing x such that $\lim_{t\to\infty} s(t, z) = x$ for all $z \in \mathcal{U}$.

The next result gives a sufficient condition to guarantee semistability of the equilibria of $\mathcal{G}$. For the statement of this result, let $\mathcal{B}$ and $\mathcal{C}$ be Banach spaces and recall that $\mathcal{B}$ is *compactly embedded* in $\mathcal{C}$ if $\mathcal{B} \subset \mathcal{C}$ and a unit ball in $\mathcal{B}$ belongs to a compact subset in $\mathcal{C}$.

Furthermore, define

$$\dot{V}(z) \triangleq \lim_{h\to 0^+} \frac{1}{h}[Vs(t_0+h,z) - V(z)], \quad z \in \mathcal{B}, \tag{14.9}$$

for a given continuous function $V : \mathcal{B} \to \mathbb{R}$ and every $z \in \mathcal{B}$ such that the limit in (14.9) exists.

Theorem 14.1. Let $\mathcal{B}$ and $\mathcal{C}$ be Banach spaces such that $\mathcal{B}$ is compactly embedded in $\mathcal{C}$, and let $\mathcal{G}$ be a dynamical system defined in $\mathcal{B}$ and $\mathcal{C}$. Assume there exist locally Lipschitz continuous functions $V_{\mathcal{B}} : \mathcal{B} \to \mathbb{R}$ and $V_{\mathcal{C}} : \mathcal{C} \to \mathbb{R}$ such that $V_{\mathcal{B}}(z) \geq 0$, $z \in \mathcal{B}_{\mathrm{c}}$, and $V_{\mathcal{C}}(z) \geq 0$, $z \in \mathcal{C}_{\mathrm{c}}$, where $\mathcal{B}_{\mathrm{c}} = \{z \in \mathcal{B} : V_{\mathcal{B}}(z) < \eta\}$ and $\mathcal{C}_{\mathrm{c}} = \{z \in \mathcal{C} : V_{\mathcal{C}}(z) < \eta\}$ for some $\eta > 0$ such that $\mathcal{B}_{\mathrm{c}} \subset \mathcal{C}_{\mathrm{c}}$. Furthermore, assume that $V_{\mathcal{B}}(s(t,z_0)) \leq V_{\mathcal{B}}(s(\tau,z_0))$ for all $t_0 \leq \tau \leq t$ and $z_0 \in \mathcal{B}_{\mathrm{c}}$, and $V_{\mathcal{B}}(s(t,z_0)) \leq V_{\mathcal{B}}(s(\tau,z_0))$ for all $t_0 \leq \tau \leq t$ and $z_0 \in \mathcal{C}_{\mathrm{c}}$. If $\mathcal{B}_{\mathrm{c}}$ is bounded and every point in the largest invariant subset $\mathcal{M}$ contained in $\mathcal{R}$ given by $\mathcal{R} \triangleq \{z \in \overline{\mathcal{C}_{\mathrm{c}}} : \dot{V}_{\mathcal{C}}(z) = 0\}$ is a Lyapunov stable equilibrium point of $\mathcal{G}$, then every equilibrium point in $\mathcal{M}$ is semistable.

Proof. First note that the assumptions on $V_{\mathcal{B}}$ imply that the trajectory $s(t,x)$ of $\mathcal{G}$ remains in $\mathcal{B}_{\mathrm{c}}$ for all $x \in \mathcal{B}_{\mathrm{c}}$ and $t \geq t_0$. Furthermore, since $\mathcal{B}$ is compactly embedded in $\mathcal{C}$, $s(t,x)$ is contained in a compact set of $\mathcal{C}_{\mathrm{c}}$ for all $x \in \mathcal{B}_{\mathrm{c}}$ and $t \geq t_0$. Now, it follows from Lemma 3 and Theorem 1 of [113] that, for every $x \in \mathcal{B}_{\mathrm{c}}$, the positive limit set $\omega(x)$ of x is nonempty and contained in the largest invariant subset $\mathcal{M}$ of $\mathcal{R}$. Since every point in $\mathcal{M}$ is a Lyapunov stable equilibrium point, it follows that every point in $\omega(x)$ is a Lyapunov stable equilibrium point.

Next, let $z \in \omega(x)$ and let $\mathcal{U}_{\varepsilon}$ be an open neighborhood of z. By Lyapunov stability of z, it follows that there exists a relatively open subset $\mathcal{U}_{\delta}$ containing z such that $s_t(\mathcal{U}_{\delta}) \subseteq \mathcal{U}_{\varepsilon}$ for every $t \geq t_0$. Since $z \in \omega(x)$, it follows that there exists $h \geq 0$ such that $s(h,x) \in \mathcal{U}_{\delta}$. Thus, $s(t+h,x) = s_t(s(h,x)) \in s_t(\mathcal{U}_{\delta}) \subseteq \mathcal{U}_{\varepsilon}$ for every $t > t_0$. Hence, since $\mathcal{U}_{\varepsilon}$ was chosen arbitrarily, it follows that $z = \lim_{t\to\infty} s(t,x)$. Now, it follows that $\lim_{i\to\infty} s(t_i,x) \to z$ for every divergent sequence $\{t_i\}$, and hence, $\omega(x) = \{z\}$. Finally, since $\lim_{t\to\infty} s(t,x) \in \mathcal{M}$ is Lyapunov stable for every $x \in \mathcal{B}_{\rm c}$, it follows from the definition of semistability that every equilibrium point in $\mathcal{M}$ is semistable. $\square$

The following assumptions are needed for the main results of this chapter. For the statement of these assumptions, $\phi : \mathcal{V}\times\overline{\mathbb{R}}_+\times\mathbb{R}^n \to \mathbb{R}^n$ denotes the system information (or energy) flow within the continuum $\mathcal{V}$, that is, $\phi(x, u(x,t), \nabla u(x,t)) = [\phi_1(x, u(x,t), \nabla u(x,t)), \ldots, \phi_n (x, u(x,t), \nabla u(x,t))]^{\rm T}$, where $\phi_i(\cdot,\cdot,\cdot)$ denotes the information (or energy) flow through a unit area per unit time in the x_i direction for all $i = 1,\ldots,n$, and $\nabla u(x,t) \triangleq [D_1u(x,t),\ldots,D_n(x, t)]$, $x \in \mathcal{D}$, $t \geq t_0$, denotes the gradient of $u(\cdot,t)$ with respect to the spatial variable x.

Assumption 1: For every $x \in \mathcal{V}$ and unit vector $\mathbf{u} \in \mathbb{R}^n$, $\phi(x, u_t(x), \nabla u_t(x)) \cdot \mathbf{u} = 0$ if and only if $\nabla u_t(x)\mathbf{u} = 0$.

Assumption 2: For every $x \in \mathcal{V}$ and unit vector $\mathbf{u} \in \mathbb{R}^n$, $\phi(x, u_t(x), \nabla u_t(x)) \cdot \mathbf{u} > 0$ if and only if $\nabla u_t(x)\mathbf{u} < 0$, and $\phi(x, u_t(x), \nabla u_t(x)) \cdot \mathbf{u} < 0$ if and only if $\nabla u_t(x)\mathbf{u} > 0$.

Note that Assumption 1 implies that $\phi_i(x, u_t(x), \nabla u_t(x)) = 0$ if and only if $D_iu_t(x) = 0$, $x \in \mathcal{V}$, $i = 1,\ldots,n$, while Assumption 2 implies that $\phi_i(x, u_t(x), \nabla u_t(x))D_iu_t(x) \leq 0$, $x \in \mathcal{V}$, $i = 1,\ldots,n$, which further implies that $\nabla u_t(x)\phi(x, u_t(x), \nabla u_t(x)) \leq 0$, $x \in \mathcal{V}$. The physical interpretation of Assumption 1 is that if the flux function ϕ in a certain direction is zero, then information or energy density change in this direction is not possible. This statement is reminiscent of the *zeroth law of thermodynamics*, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium. Assumption 2 implies that information or energy flows from information rich or more energetic regions to information poor or less energetic regions and is reminiscent of the *second law of thermodynamics*, which states that heat (energy) must flow in the direction of lower temperatures. For further details of these assumptions, see [104].

The following proposition shows that the solution $u(x,t), x \in \mathcal{V},\ t \geq t_0$, to (14.1) and

(14.2) is nonnegative for all nonnegative initial information density distributions $u_{t_0}(x) \geq 0$, $x \in \mathcal{V}$.

Proposition 14.1. Consider the dynamical system $\mathcal{G}$ given by (14.1) and (14.2). Assume that Assumptions 1 and 2 hold. Furthermore, assume that if $u(\hat{x}, \hat{t}) = 0$ for some $\hat{x} \in \partial\mathcal{V}$ and $\hat{t} \geq t_0$, then $\phi(\hat{x}, u(\hat{x}, \hat{t}), \nabla u(\hat{x}, \hat{t})) = 0$. Then the solution $u(x, t)$, $x \in \mathcal{V}$, $t \geq t_0$, to (14.1) and (14.2) is nonnegative for all nonnegative initial density distributions $u_{t_0}(x) \geq 0$, $x \in \mathcal{V}$.

Proof. Note that if $u(\hat{x}, \hat{t}) = 0$ for some $\hat{x}$ in the interior of $\mathcal{V}$ and $\hat{t} \geq t_0$, then it follows from Assumption 2 that $\phi(y, u(y, \hat{t}), \nabla u(y, \hat{t}))$ is directed towards the point $\hat{x}$ for all points y in a sufficiently small neighborhood of $\hat{x}$. This property along with (14.1) implies that $\frac{\partial u(\hat{x},\hat{t})}{\partial t} \geq 0$. Alternatively, if $u(\hat{x}, \hat{t}) = 0$ for some $\hat{x} \in \partial\mathcal{V}$ and $\hat{t} \geq t_0$, then it follows from (14.1) and Assumptions 1 and 2 that $\frac{\partial u(\hat{x},\hat{t})}{\partial t} \geq 0$. Thus, the solution to (14.1) and (14.2) is nonnegative for all nonnegative initial density distributions. $\square$

Next, we show that a Clausius-type inequality holds for the Eulerian swarm model $\mathcal{G}$ given by (14.1) and (14.2). For this result, note that it follows from Assumption 1 that for $\phi(x, u(x, t), \nabla u(x, t)) \cdot \mathbf{n}(x) \equiv 0$, the function $u(x, t) = \alpha$, $x \in \mathcal{V}$, $t \geq t_0$, $\alpha \geq 0$, is the solution to (14.1) and (14.2) with $u_{t_0}(x) = \alpha$, $x \in \mathcal{V}$. Thus, we define an equilibrium process for the system $\mathcal{G}$ as a process where the trajectory of $\mathcal{G}$ moves along the equilibrium manifold $\mathcal{M}_{\mathrm{e}} \triangleq \{u_t \in \mathcal{X} : u_t(x) = \alpha,\ x \in \mathcal{V},\ \alpha \geq 0\}$, that is, $u(x, t) = \alpha(t), x \in \mathcal{V}$, $t \geq t_0$, for some $\mathcal{L}_\infty$ function $\alpha : [0, \infty) \to \overline{\mathbb{R}}_+$. A nonequilibrium process is a process that does not lie on $\mathcal{M}_{\mathrm{e}}$. The next result establishes a Clausius-type inequality for equilibrium and nonequilibrium states of the infinite-dimensional dynamical system $\mathcal{G}$.

Proposition 14.2. Consider the dynamical system $\mathcal{G}$ given by (14.1) and (14.2), and assume that Assumptions 1 and 2 hold. Then, for every initial energy density distribution

$u_{t_0} \in \mathcal{X}$, $t_{\mathrm{f}} \geq t_0$, such that $u_{t_{\mathrm{f}}}(x) = u_{t_0}(x)$, $x \in \mathcal{V}$,

$$\int_{t_0}^{t_{\mathrm{f}}} \left[-\int_{\partial\mathcal{V}} \frac{\phi(x, u(x,t), \nabla u(x,t)) \cdot \mathbf{n}(x)}{c + u(x,t)} \mathrm{d}\mathcal{S}_{\mathcal{V}} \right] \mathrm{d}t \leq 0, \tag{14.10}$$

where $c > 0$ and $u(x,t)$, $x \in \mathcal{V}$, $t \geq t_0$, is the solution to (14.1) and (14.2). Furthermore,

$$\int_{t_0}^{t_{\mathrm{f}}} \left[-\int_{\partial\mathcal{V}} \frac{\phi(x, u(x,t), \nabla u(x,t)) \cdot \mathbf{n}(x)}{c + u(x,t)} \mathrm{d}\mathcal{S}_{\mathcal{V}} \right] \mathrm{d}t = 0 \tag{14.11}$$

if and only if there exists an $\mathcal{L}_\infty$ function $\alpha : [t_0, t_{\mathrm{f}}] \to \overline{\mathbb{R}}_+$ such that $u(x,t) = \alpha(t)$, $x \in \mathcal{V}$, $t \in [t_0, t_{\mathrm{f}}]$.

Proof. It follows from (14.1), the Green-Gauss theorem, and Assumption 2 that

$$\begin{aligned}
\int_{t_0}^{t_{\mathrm{f}}} & \left[-\int_{\partial\mathcal{V}} \frac{\phi(x, u(x,t), \nabla u(x,t)) \cdot \mathbf{n}(x)}{c + u(x,t)} \mathrm{d}\mathcal{S}_{\mathcal{V}} \right] \mathrm{d}t \\
&= \int_{t_0}^{t_{\mathrm{f}}} \int_{\mathcal{V}} \frac{\frac{\partial u(x,t)}{\partial t} + \nabla \cdot \phi(x, u(x,t), \nabla u(x,t))}{c + u(x,t)} \mathrm{d}\mathcal{V}\mathrm{d}t \\
&\quad - \int_{t_0}^{t_{\mathrm{f}}} \int_{\partial\mathcal{V}} \frac{\phi(x, u(x,t), \nabla u(x,t)) \cdot \mathbf{n}(x)}{c + u(x,t)} \mathrm{d}\mathcal{S}_{\mathcal{V}}\mathrm{d}t \\
&= \int_{\mathcal{V}} \log_e \left(\frac{c + u(x,t_{\mathrm{f}})}{c + u(x,t_0)} \right) \mathrm{d}\mathcal{V} \\
&\quad + \int_{t_0}^{t_{\mathrm{f}}} \int_{\partial\mathcal{V}} \frac{\phi(x, u(x,t), \nabla u(x,t)) \cdot \mathbf{n}(x)}{c + u(x,t)} \mathrm{d}\mathcal{S}_{\mathcal{V}}\mathrm{d}t \\
&\quad + \int_{t_0}^{t_{\mathrm{f}}} \int_{\mathcal{V}} \frac{\nabla u(x,t) \phi(x, u(x,t), \nabla u(x,t))}{(c + u(x,t))^2} \mathrm{d}\mathcal{V}\mathrm{d}t \\
&\quad - \int_{t_0}^{t_{\mathrm{f}}} \int_{\partial\mathcal{V}} \frac{\phi(x, u(x,t), \nabla u(x,t)) \cdot \mathbf{n}(x)}{c + u(x,t)} \mathrm{d}\mathcal{S}_{\mathcal{V}}\mathrm{d}t \\
&= \int_{t_0}^{t_{\mathrm{f}}} \int_{\mathcal{V}} \frac{\nabla u(x,t) \phi(x, u(x,t), \nabla u(x,t))}{(c + u(x,t))^2} \mathrm{d}\mathcal{V}\mathrm{d}t \\
&\leq 0,
\end{aligned} \tag{14.12}$$

which proves (14.10).

To show (14.11), note that it follows from (14.12), Assumption 1, and Assumption 2 that (14.11) holds if and only if $\nabla u(x,t) = 0$ for all $x \in \mathcal{V}$ and $t \in [t_0, t_{\mathrm{f}}]$ or, equivalently, there exists an $\mathcal{L}_\infty$ function $\alpha : [t_0, t_{\mathrm{f}}] \to \overline{\mathbb{R}}_+$ such that $u(x,t) = \alpha(t)$, $x \in \mathcal{V}$, $t \in [t_0, t_{\mathrm{f}}]$. □

Inequality (14.10) is a generalization of Clausius' inequality for reversible and irreversible thermodynamics as applied to Eulerian swarm models and restricts the manner in which the system looses information over cyclic motions. Next, we define an entropy functional for the continuum dynamical system $\mathcal{G}$.

Definition 14.2. For the dynamical system $\mathcal{G}$ given by (14.1) and (14.2), the functional $\mathcal{S} : \mathcal{X} \to \mathbb{R}$ satisfying

$$\mathcal{S}(u_{t_2}) \geq \mathcal{S}(u_{t_1}) + \int_{t_1}^{t_2} q(t)\mathrm{d}t \tag{14.13}$$

for all $t_2 \geq t_1 \geq t_0$, where

$$q(t) \triangleq -\int_{\partial\mathcal{V}} \frac{\phi(x, u(x,t), \nabla u(x,t)) \cdot \mathbf{n}(x)}{c + u(x,t)} \mathrm{d}\mathcal{S}_{\mathcal{V}} \tag{14.14}$$

and $c > 0$, is called the *entropy* functional of $\mathcal{G}$.

In the next theorem, we present a unique, continuously differentiable entropy functional for the dynamical system $\mathcal{G}$. This result holds for equilibrium and nonequilibrium processes.

Theorem 14.2. Consider the dynamical system $\mathcal{G}$ given by (14.1) and (14.2), and assume that Assumptions 1 and 2 hold. Then the functional $\mathcal{S} : \mathcal{X} \to \mathbb{R}$ given by

$$\mathcal{S}(u_t) = \int_{\mathcal{V}} \log_e(c + u_t(x))\mathrm{d}\mathcal{V} - \mathcal{V}_{\mathrm{vol}} \log_e c \tag{14.15}$$

is a unique (modulo a constant of integration), continuously differentiable entropy functional of $\mathcal{G}$. Furthermore, if $u_t \notin \mathcal{M}_{\mathrm{e}}$, $t \geq t_0$, where $u_t = u(x,t)$ denotes the solution to (14.1) and (14.2) and $\mathcal{M}_{\mathrm{e}} = \{u_t \in \mathcal{X} : \, u_t = \alpha, \, \alpha \geq 0\}$, then (14.15) satisfies

$$\mathcal{S}(u_{t_2}) > \mathcal{S}(u_{t_1}) + \int_{t_1}^{t_2} q(t)\mathrm{d}t. \tag{14.16}$$

Proof. It follows from the Green-Gauss theorem, Assumption 2, and (14.15) that

$$\dot{\mathcal{S}}(u_t) = \int_{\mathcal{V}} \frac{1}{c + u(x,t)} \frac{\partial u(x,t)}{\partial t} \, \mathrm{d}\mathcal{V}$$

$$
\begin{aligned}
&= \int_{\mathcal{V}} \frac{1}{c+u(x,t)} \left(-\nabla \cdot \phi(x,u(x,t),\nabla u(x,t))\right) \mathrm{d}\mathcal{V} \\
&= -\int_{\mathcal{V}} \frac{\nabla u(x,t)\phi(x,u(x,t),\nabla u(x,t))}{(c+u(x,t))^2} \mathrm{d}\mathcal{V} \\
&\quad - \int_{\partial\mathcal{V}} \frac{\phi(x,u(x,t),\nabla u(x,t)) \cdot \mathbf{n}(x)}{c+u(x,t)} \mathrm{d}\mathcal{S}_{\mathcal{V}} \\
&\geq q(t).
\end{aligned} \tag{14.17}
$$

Now, integrating (14.17) over $[t_1, t_2]$ yields (14.13). Furthermore, if $u_t \notin \mathcal{M}_{\mathrm{e}}$, $t \geq t_0$, then it follows from Assumption 1, Assumption 2, and (14.17) that (14.16) holds.

To show that (14.15) is a unique, continuously differentiable entropy function of $\mathcal{G}$, let $\mathcal{S}(u_t)$ be a continuously differentiable entropy functional of $\mathcal{G}$ so that $\mathcal{S}(u_t)$ satisfies (14.13) or, equivalently,

$$
\begin{aligned}
\dot{\mathcal{S}}(u_t) &\geq -\int_{\partial\mathcal{V}} \frac{\phi(x,u_t,\nabla u_t) \cdot \mathbf{n}(x)}{c+u_t} \mathrm{d}\mathcal{S}_{\mathcal{V}} \\
&= -\int_{\mathcal{V}} \nabla \cdot (\mu(u_t) S(x,t)) \mathrm{d}\mathcal{V} \\
&= -\mu(u_t) S(x,t), \quad t \geq t_0,
\end{aligned} \tag{14.18}
$$

where $\mu(u_t) \triangleq \frac{1}{c+u_t}$, $S(x,t) \triangleq \phi(x,u_t,\nabla u_t)$, u_t, $t \geq t_0$, denotes the solution to (14.1) and (14.2), and $\dot{\mathcal{S}}(u_t)$ denotes the time derivative of $\mathcal{S}(u_t)$ along the solution u_t, $t \geq t_0$. Hence, it follows from (14.18) that

$$
\mathcal{S}'(u_t)[-\nabla \cdot S(x,t)] \geq -\mu(u_t) S(x,t), \quad u_t \in \overline{\mathbb{R}}_+, \quad x \in \mathcal{V}, \quad t \geq t_0, \tag{14.19}
$$

that is,

$$
\mathcal{S}'(u_t)\left[-S(x,t) - \int_{\mathcal{V}} \nabla^2 S(x,t) \mathrm{d}\mathcal{V}\right] \geq -\mu(u_t) S(x,t), \quad u_t \in \overline{\mathbb{R}}_+, \quad x \in \mathcal{V}, \quad t \geq t_0, \tag{14.20}
$$

which implies that there exist continuous functions $\ell : \overline{\mathbb{R}}_+ \to \mathbb{R}^p$ and $\mathcal{W} : \overline{\mathbb{R}}_+ \to \mathbb{R}^{p \times q}$ such that

$$
\begin{aligned}
0 = {}& \mathcal{S}'(u_t)\left[-S(x,t) - \int_{\mathcal{V}} \nabla^2 S(x,t) \mathrm{d}\mathcal{V}\right] + \mu(u_t) S(x,t) \\
& -[\ell(u_t) + \mathcal{W}(u_t) S(x,t)]^{\mathrm{T}}[\ell(u_t) + \mathcal{W}(u_t) S(x,t)], \quad u_t \in \overline{\mathbb{R}}_+, \quad x \in \mathcal{V}, \quad t \geq t_0.
\end{aligned}
$$

Now, equating coefficients of equal powers (of S), it follows that $\mathcal{W}(u_t) \equiv 0$, $\mathcal{S}'(u_t) = \mu(u_t)$, $u_t \in \overline{\mathbb{R}}_+$, and

$$0 = \mathcal{S}'(u_t) \int_{\mathcal{V}} \nabla^2 S(x,t) \mathrm{d}\mathcal{V} + \ell^{\mathrm{T}}(u_t)\ell(u_t), \quad u_t \in \overline{\mathbb{R}}_+. \tag{14.21}$$

Hence, $\mathcal{S}(u_t) = \int_{\mathcal{V}} \log_e(c + u_t(x))\mathrm{d}\mathcal{V} - \mathcal{V}_{\mathrm{vol}} \log_e c$, $u_t \in \overline{\mathbb{R}}_+$. Thus, (14.15) is a unique, continuously differentiable entropy functional for $\mathcal{G}$. □

It follows from Theorem 14.2 that if no information flow is allowed into or out of $\mathcal{V}$ (i.e., the system is isolated), then $\mathcal{S}(u_{t_2}) \geq \mathcal{S}(u_{t_1})$, $t_2 \geq t_1$. This shows that for an adiabatically isolated system, the entropy of the final state is greater than or equal to the entropy of the initial state.

14.4. Boundary Semistable Control for Large-Scale Swarms

In this section, we develop a boundary controller that guarantees that the infinite-dimensional information flow model (14.1) and (14.2) has convergent flows to Lyapunov stable uniform equilibrium information density distributions determined by the system initial information density distribution. First, we show that if no information flow is allowed into or out of $\mathcal{V}$ (i.e., the boundary $\partial\mathcal{V}$ is insulated), then (14.1) and (14.2) is Lyapunov stable.

Theorem 14.3. Consider the dynamical system given by (14.1) and (14.2). Assume that Assumptions 1 and 2 hold. If

$$\phi(x, u(x,t), \nabla u(x,t)) \cdot \mathbf{n}(x) = 0, \quad x \in \partial\mathcal{V}, \quad t \geq t_0, \tag{14.22}$$

then $u(x,t) \equiv \alpha$, $\alpha \geq 0$, is Lyapunov stable.

Proof. It follows from Assumption 1 that $u(x,t) \equiv \alpha$, $\alpha \geq 0$, is an equilibrium state for (14.1) and (14.2). To show Lyapunov stability of the equilibrium state $u(x,t) \equiv \alpha$, consider the shifted Lyapunov functional candidate

$$V(u_t - \alpha) = \frac{1}{2}\int_{\mathcal{V}}(u_t(x) - \alpha)^2 \mathrm{d}\mathcal{V} = \frac{1}{2}\|u_t - \alpha\|_{\mathcal{L}_2}^2. \tag{14.23}$$

Now, it follows from the Green-Gauss theorem and Assumptions 1 and 2 that

$$\begin{aligned}
\dot{V}(u_t - \alpha) = {} & \int_{\mathcal{V}}(u(x,t) - \alpha)\frac{\partial u(x,t)}{\partial t}\mathrm{d}\mathcal{V} \\
= {} & -\int_{\mathcal{V}} u(x,t)\nabla\cdot\phi(x,u(x,t),\nabla u(x,t))\mathrm{d}\mathcal{V} \\
& +\alpha\int_{\mathcal{V}}\nabla\cdot\phi(x,u(x,t),\nabla u(x,t))\mathrm{d}\mathcal{V} \\
= {} & \int_{\mathcal{V}}\nabla u(x,t)\phi(x,u(x,t),\nabla u(x,t))\mathrm{d}\mathcal{V} \\
& -\int_{\partial\mathcal{V}} u(x,t)\phi(x,u(x,t),\nabla u(x,t))\cdot\mathbf{n}(x)\mathrm{d}\mathcal{S}_{\mathcal{V}} \\
& +\alpha\int_{\partial\mathcal{V}}\phi(x,u(x,t),\nabla u(x,t))\cdot\mathbf{n}(x)\mathrm{d}\mathcal{S}_{\mathcal{V}} \\
= {} & \int_{\mathcal{V}}\nabla u(x,t)\phi(x,u(x,t),\nabla u(x,t))\mathrm{d}\mathcal{V} \\
\leq {} & 0, \quad u_t \in \mathcal{W}_2^0(\mathcal{V}),
\end{aligned} \tag{14.24}$$

which establishes Lyapunov stability of the equilibrium state $u(x,t) \equiv \alpha$. □

Next, we show that the total $\mathcal{L}_2$ norm of the energy of (14.1) and (14.2) is nonincreasing.

Proposition 14.3. Consider the dynamical system given by (14.1) and (14.2). Assume that Assumptions 1 and 2 hold. If either $u(x,t) = 0$ for all $x \in \partial\mathcal{V}$ and $t \geq t_0$ or (14.22) holds, then $\|u_t\|_{\mathcal{W}_2^0} \leq \|u_\tau\|_{\mathcal{W}_2^0}$ for all $t_0 \leq \tau \leq t$.

Proof. Assume $u(x,t) = 0$ for all $x \in \partial\mathcal{V}$ and $t \geq t_0$, and consider the functional

$$V(u_t) = \|u_t\|_{\mathcal{W}_2^0}^2. \tag{14.25}$$

Now, it follows from the Green-Gauss theorem and Assumptions 1 and 2 that

$$\frac{1}{2}\dot{V}(u_t) = \int_{\mathcal{V}} u(x,t)\frac{\partial u(x,t)}{\partial t}\mathrm{d}\mathcal{V}$$

$$
\begin{aligned}
&= -\int_{\mathcal{V}} u(x,t)\nabla\cdot\phi(x,u(x,t),\nabla u(x,t))\mathrm{d}\mathcal{V}\\
&= \int_{\mathcal{V}} \nabla u(x,t)\phi(x,u(x,t),\nabla u(x,t))\mathrm{d}\mathcal{V}\\
&\quad -\int_{\partial\mathcal{V}} u(x,t)\phi(x,u(x,t),\nabla u(x,t))\cdot\mathbf{n}(x)\mathrm{d}\mathcal{S}_{\mathcal{V}}\\
&\leq 0, \quad u_t\in\mathcal{W}_2^0(\mathcal{V}),
\end{aligned}
\tag{14.26}
$$

which implies that $\|u_t\|_{\mathcal{W}_2^0}\leq\|u_\tau\|_{\mathcal{W}_2^0}$ for all $t_0\leq\tau\leq t$. Alternatively, if (14.22) holds, then

$$
\frac{1}{2}\dot{V}(u_t)=\int_{\mathcal{V}}\nabla u(x,t)\phi(x,u(x,t),\nabla u(x,t))\mathrm{d}\mathcal{V}\leq 0, \quad u_t\in\mathcal{W}_2^0(\mathcal{V}), \tag{14.27}
$$

which implies that $\|u_t\|_{\mathcal{W}_2^0}\leq\|u_\tau\|_{\mathcal{W}_2^0}$ for all $t_0\leq\tau\leq t$. □

Next, we present necessary and sufficient conditions for semistability of the swarm aggregation model (14.1) and (14.2).

Theorem 14.4. Consider the dynamical system given by (14.1) and (14.2). Assume that Assumptions 1 and 2 hold, and $D(u_t,u_t)\leq D(u_\tau,u_\tau)$ for all $t_0\leq\tau\leq t$. Then for every $\alpha\geq 0$, $u(x,t)\equiv\alpha$ is a semistable equilibrium state of (14.1) and (14.2) if and only if (14.22) holds. In this case, $u(x,t)\to\frac{1}{\mathrm{vol}\mathcal{V}}\int_{\mathcal{V}}u_{t_0}(x)\mathrm{d}\mathcal{V}$ as $t\to\infty$ for every initial condition $u_{t_0}\in\mathcal{W}_2^1(\mathcal{V})$ and every $x\in\mathcal{V}$; moreover, $\frac{1}{\mathrm{vol}\mathcal{V}}\int_{\mathcal{V}}u_{t_0}(x)\mathrm{d}\mathcal{V}$ is a semistable equilibrium state of (14.1) and (14.2).

Proof. Assume that (14.22) holds. Then it follows from Theorem 14.3 that $u(x,t)\equiv\alpha$, $\alpha\geq 0$, is Lyapunov stable. Next, to show semistability of this equilibrium state, consider the Lyapunov functionals (14.25) and

$$
\mathcal{E}(u_t)=\|u_t\|^2_{\mathcal{W}_2^1}, \quad u_t\in\mathcal{W}_2^1(\mathcal{V}). \tag{14.28}
$$

It follows from Proposition 14.3 that $V(u_t)$ is a nonincreasing functional of time for all $u_{t_0}\in\mathcal{W}_2^0(\mathcal{V})$. Furthermore, note that $\mathcal{E}(u_t)=V(u_t)+D(u_t,u_t)$. Hence, by assumption, $\mathcal{E}(u_t)$ is a nonincreasing functional of time for all $u_{t_0}\in\mathcal{W}_2^1(\mathcal{V})$. Next, since the functionals

$V(u_t)$ and $\mathcal{E}(u_t)$ are nonincreasing and bounded from below by zero, it follows that $V(u_t)$ and $\mathcal{E}(u_t)$ are bounded functionals for every $u_{t_0} \in \mathcal{W}_2^1(\mathcal{V})$. This implies that the positive orbit $\mathcal{O}_{u_{t_0}}^+ \triangleq \{u_t \in \mathcal{W}_2^1(\mathcal{V}) : u_t(x) = u(x,t), x \in \mathcal{V}, t \in [t_0, \infty)\}$ of (14.1) and (14.2) is bounded in $\mathcal{W}_2^1(\mathcal{V})$ for all $u_{t_0} \in \mathcal{W}_2^1(\mathcal{V})$. Furthermore, it follows from Sobolev's embedding theorem [223,233] that $\mathcal{W}_2^1(\mathcal{V})$ is compactly embedded in $\mathcal{W}_2^0(\mathcal{V})$, and hence, $\mathcal{O}_{u_{t_0}}^+$ is contained in a compact subset of $\mathcal{W}_2^0(\mathcal{V})$.

Next, define the sets $\mathcal{D}_{\mathcal{W}_2^1} = \{u_t \in \mathcal{W}_2^1(\mathcal{V}) : \mathcal{E}(u_t) < \eta\}$ and $\mathcal{D}_{\mathcal{W}_2^0} = \{u_t \in \mathcal{W}_2^0(\mathcal{V}) : V(u_t) < \eta\}$ for some arbitrary $\eta > 0$. Note that $\mathcal{D}_{\mathcal{W}_2^1}$ and $\mathcal{D}_{\mathcal{W}_2^0}$ are invariant sets with respect to (14.1) and (14.2). Moreover, it follows from the definition of $\mathcal{E}(u_t)$ and $V(u_t)$ that $\mathcal{D}_{\mathcal{W}_2^1}$ and $\mathcal{D}_{\mathcal{W}_2^0}$ are bounded sets in $\mathcal{W}_2^1(\mathcal{V})$ and $\mathcal{W}_2^0(\mathcal{V})$, respectively, and $\mathcal{D}_{\mathcal{W}_2^1} \subset \mathcal{D}_{\mathcal{W}_2^0}$. Next, let $\mathcal{R} \triangleq \{u_t \in \overline{\mathcal{D}}_{\mathcal{W}_2^0} : \dot{V}(u_t) = 0\} = \{u_t \in \overline{\mathcal{D}}_{\mathcal{W}_2^0} : \nabla u_t(x)\phi(x, u_t(x), \nabla u_t(x)) = 0, x \in \mathcal{V}\}$. Now, it follows from Assumption 1 that $\mathcal{R} = \{u_t \in \overline{\mathcal{D}}_{\mathcal{W}_2^0} : \nabla u_t(x) = 0, x \in \mathcal{V}\}$ or $\mathcal{R} = \{u_t \in \mathcal{W}_2^0(\mathcal{V}) : u_t(x) \equiv \sigma, 0 \le \sigma \le \sqrt{\frac{\eta}{\mathrm{vol}\mathcal{V}}}\}$, that is, $\mathcal{R}$ is the set of uniform density distributions, which are the equilibrium states of (14.1) and (14.2). Since the set $\mathcal{R}$ consists of only the equilibrium states of (14.1) and (14.2), it follows that the largest invariant set $\mathcal{M}$ contained in $\mathcal{R}$ is given by $\mathcal{M} = \mathcal{R}$. Hence, noting that $\mathcal{M}$ belongs to the set of generalized (weak) solutions of (14.1) and (14.2) defined on $\mathcal{R}$, it follows from Theorem 14.1 that $u(x,t) \equiv \alpha$ is a semistable equilibrium state of (14.1) and (14.2). Moreover, since $\eta > 0$ can be arbitrary large but finite and $\mathcal{E}(u_t)$ is radially unbounded, the previous statement holds for all $u_{t_0} \in \mathcal{W}_2^1(\mathcal{V})$. Next, note that since, by the divergence theorem,

$$\begin{aligned}\int_{\mathcal{V}} \frac{\partial u(x,t)}{\partial t} \mathrm{d}\mathcal{V} &= -\int_{\mathcal{V}} \nabla \cdot \phi(x, u(x,t), \nabla u(x,t)) \mathrm{d}\mathcal{V} \\ &= -\int_{\partial\mathcal{V}} \phi(x, u(x,t), \nabla u(x,t)) \cdot \mathbf{n}(x) \mathrm{d}\mathcal{S}_{\mathcal{V}} \\ &= 0, \end{aligned} \tag{14.29}$$

it follows that $\int_{\mathcal{V}} u(x,t)\mathrm{d}\mathcal{V} = \int_{\mathcal{V}} u_{t_0}(x)\mathrm{d}\mathcal{V}$, $t \ge t_0$, which implies that $u(x,t) \to \frac{1}{\mathrm{vol}\mathcal{V}} \int_{\mathcal{V}} u_{t_0}(x) \mathrm{d}\mathcal{V}$ as $t \to \infty$.

Conversely, assume that for every $\alpha \geq 0$, $u(x,t) \equiv \alpha$ is a semistable equilibrium state of (14.1) and (14.2). Suppose, *ad absurdum*, there exists at least one point $x_{\mathrm{p}} \in \partial\mathcal{V}$ such that $\phi(x_{\mathrm{p}}, u_t(x_{\mathrm{p}}, \nabla u_t(x_{\mathrm{p}}))) \cdot \mathbf{n}(x_{\mathrm{p}}) > 0$. Consider the Lyapunov functional (14.25) and note that the Lyapunov derivative of $V(u_t)$ is given by (14.26). Let $\mathcal{R} \triangleq \{u_t \in \overline{\mathcal{D}}_{\mathcal{W}_2^0} : \dot{V}(u_t) = 0\} = \{u_t \in \overline{\mathcal{D}}_{\mathcal{W}_2^0} : \nabla u_t(x)\phi(x, u_t(x), \nabla u_t(x)) = 0, x \in \mathcal{V}\} \cap \{u_t \in \overline{\mathcal{D}}_{\mathcal{W}_2^0} : u(x,t)\phi(x, u_t(x), \nabla u_t(x)) \cdot \mathbf{n}(x) = 0, x \in \partial\mathcal{V}\}$. Now, since Assumption 1 holds, it follows that $\mathcal{R} = \{u_t \in \overline{\mathcal{D}}_{\mathcal{W}_2^0} : \nabla u_t(x) = 0, x \in \mathcal{V}\} \cap \{u_t \in \overline{\mathcal{D}}_{\mathcal{W}_2^0} : u_t(x_{\mathrm{p}}) = 0, x_{\mathrm{p}} \in \partial\mathcal{V}\} = \{0\}$, and the largest invariant set $\mathcal{M}$ contained in $\mathcal{R}$ is given by $\mathcal{M} = \{0\}$. By assumption, $\mathcal{E}(u_t)$ is a nonincreasing functional of time for all $u_{t_0} \in \mathcal{W}_2^1(\mathcal{V})$, and since $\mathcal{E}(u_t)$ is bounded from below by zero, the positive orbit $\mathcal{O}_{u_{t_0}}^+$ of (14.1) and (14.2) is bounded in $\mathcal{W}_2^1(\mathcal{V})$. Hence, since $\mathcal{W}_2^1(\mathcal{V})$ is compactly embedded in $\mathcal{W}_2^0(\mathcal{V})$, it follows from Sobolev's embedding theorem [223, 233] that $\mathcal{O}_{u_{t_0}}^+$ is contained in a compact subset of $\mathcal{W}_2^0(\mathcal{V})$. Thus, it follows from Theorem 3 of [113] that for any initial density distribution $u_{t_0} \in \mathcal{D}_{\mathcal{W}_2^0}$, $u(x,t) \to \mathcal{M} = \{0\}$ as $t \to \infty$ with respect to the norm $\|\cdot\|_{\mathcal{W}_2^0}$, which shows asymptotic stability of the zero equilibrium state of (14.1) and (14.2). However, since asymptotic stability of (14.1) and (14.2) is equivalent to semistability of (14.1) and (14.2) if and only if the equilibrium state of (14.1) and (14.2) is zero, this contradicts the assumption that for every $\alpha \geq 0$, $u(x,t) \equiv \alpha$ is an equilibrium state of (14.1) and (14.2). Hence, (14.22) holds. $\square$

Theorem 14.4 shows that the swarm aggregation model (14.1) and (14.2) with Assumptions 1 and 2 has convergent flows to Lyapunov stable uniform equilibrium information density distributions determined by the system initial information density distribution. This phenomenon is known as *equipartition of energy* [104] in system thermodynamics and *information consensus* or *protocol agreement* [126] in cooperative network systems.

Corollary 14.1. Consider the dynamical system $\mathcal{G}$ given by (14.1) and (14.2). Assume that Assumptions 1 and 2 hold, and

$$\nabla^2 u_t(x) \nabla \cdot \phi(x, u_t(x), \nabla u_t(x)) \leq 0, \quad x \in \mathcal{V}, \quad u_t \in \mathcal{W}_2^1(\mathcal{V}), \tag{14.30}$$

where $\nabla^2 \triangleq \nabla \cdot \nabla$ denotes the Laplace operator. Then for every $\alpha \geq 0$, $u(x,t) \equiv \alpha$ is a semistable equilibrium state of (14.1) and (14.2) if and only if (14.22) holds. In this case, $u(x,t) \to \frac{1}{\mathrm{vol}\mathcal{V}} \int_{\mathcal{V}} u_{t_0}(x)\mathrm{d}\mathcal{V}$ as $t \to \infty$ for every initial condition $u_{t_0} \in \mathcal{W}_2^1(\mathcal{V})$ and every $x \in \mathcal{V}$; moreover, $\frac{1}{\mathrm{vol}\mathcal{V}} \int_{\mathcal{V}} u_{t_0}(x)\mathrm{d}\mathcal{V}$ is a semistable equilibrium state of (14.1) and (14.2).

Proof. The result is a direct consequence of Theorem 14.4 by showing that the Dirichlet integral $D(u_t, u_t)$ of u_t is nonincreasing. To see this, note that it follows from the Green-Gauss theorem and (14.22) that

$$\begin{aligned} \frac{1}{2}\dot{D}(u_t,u_t) &= \int_{\mathcal{V}} \nabla u(x,t) \frac{\partial}{\partial t}(\nabla u(x,t))^{\mathrm{T}}\mathrm{d}\mathcal{V} \\ &= \int_{\partial\mathcal{V}} \frac{\partial u(x,t)}{\partial t} D_{\mathbf{n}(x)} u(x,t)\mathrm{d}\mathcal{S}_{\mathcal{V}} \\ &\quad + \int_{\mathcal{V}} \nabla^2 u(x,t) \nabla \cdot \phi(x,u(x,t),\nabla u(x,t))\mathrm{d}\mathcal{V}, \end{aligned} \tag{14.31}$$

where $D_{\mathbf{n}(x)}u(x,t) \triangleq \nabla u(x,t)\mathbf{n}(x)$ denotes the directional derivative of $u(x,t)$ along $\mathbf{n}(x)$ at $x \in \partial\mathcal{V}$. Next, it follows from (14.22) and Assumption 1, with $\mathbf{u} = \mathbf{n}(x)$, that $D_{\mathbf{n}(x)}u(x,t) = 0$, $x \in \partial\mathcal{V}$. Hence, it follows from (14.30) and (14.31) that $\dot{D}(u_t,u_t) \leq 0$, $t \geq t_0$, for any $u_{t_0} \in \mathcal{W}_2^1(\mathcal{V})$. □

Condition (14.30) implies that for an information (or energy) density distribution $u_t(x)$, $x \in \mathcal{V}$, the information (or energy) flow $\phi(x, u_t(x), \nabla u_t(x))$ at $x \in \mathcal{V}$ is proportional to the information (or energy) density at this point. Note that for a linear information (or energy) flow model where $\phi(x,u_t(x),\nabla u_t(x)) = -k[\nabla u_t(x)]^{\mathrm{T}}$ and $k > 0$ is a conductivity constant, condition (14.30) is automatically satisfied since $\nabla^2 u_t(x) \nabla \cdot \phi(x,u_t(x),\nabla u_t(x)) = -k[\nabla^2 u_t(x)]^2 \leq 0$, $x \in \mathcal{V}$.

Equation (14.22) plays a critical role in (boundary) control design of (14.1) and (14.2). In particular, (14.22), along with Assumptions 1 and 2, give a criterion for guaranteeing semistability of (14.1) and (14.2). Next, we discuss boundary semistable control of (14.1) and (14.2) using (14.22). First, we consider *Dirichlet boundary control* [149]. The Dirichlet boundary control problem for (14.1) and (14.2) involves the control law given by (14.2) with

$$u(x,t) = U_{\mathrm{d}}(x,t), \quad x \in \partial\mathcal{V}, \quad t \geq t_0. \tag{14.32}$$

It follows from (14.22) and Assumption 1 that for the Dirichlet boundary control problem, the control input $U_{\mathrm{d}}(x,t)$ should be chosen to satisfy

$$\nabla f(x)\nabla^{\mathrm{T}} U_{\mathrm{d}}(x,t) = 0, \quad x \in \partial\mathcal{V}, \quad t \geq t_0. \tag{14.33}$$

Next, we consider *Neumann boundary control* [149] for (14.1) and (14.2). The Neumann boundary control problem for (14.1) and (14.2) involves the control law given by (14.2) with

$$\frac{\partial u(x,t)}{\partial \mathbf{n}} = U_{\mathrm{n}}(x,t), \quad x \in \partial\mathcal{V}, \quad t \geq t_0. \tag{14.34}$$

However, since $\frac{\partial u(x,t)}{\partial \mathbf{n}} = \nabla u_t(x)\cdot\mathbf{n}$, it follows from (14.22) and Assumption 1 that $U_{\mathrm{n}}(x,t) = 0$, $x \in \partial\mathcal{V}$, $t \geq t_0$, resulting in a trivial Neumann boundary controller.

Finally, we consider a linear form of (14.1) and (14.2). Specifically, consider the linear (heat) equation given by

$$\frac{\partial u(x,t)}{\partial t} = \nabla^2 u(x,t), \quad x \in \mathcal{V}, \quad t \geq t_0, \quad u(x,t_0) = u_{t_0}(x), \quad x \in \mathcal{V}, \tag{14.35}$$

where $u : \mathbb{R} \times [0,\infty) \to \overline{\mathbb{R}}_+$. It can be easily shown that Assumptions 1 and 2 hold, and (14.30) holds for (14.35). Now, using the Neumann boundary control law

$$\nabla u(x,t) \cdot \mathbf{n}(x) = 0, \quad x \in \partial\mathcal{V}, \quad t \geq t_0, \tag{14.36}$$

it follows that all the equilibrium points of (14.35) are given by $u(x,t) \equiv \alpha \in \mathbb{R}$ [70, p. 346]. Hence, it follows from Corollary 14.1 that the linear equation (14.35) achieves uniform information distributions over $\mathcal{V}$. The boundary condition (14.36) implies that there is no information (heat) flow into or out of $\mathcal{V}$, that is, the boundary $\partial\mathcal{V}$ is insulated.

Finally, we consider the Neumann boundary control law given by

$$U_{\mathrm{n}}(x,t) = -c(u(x,t) - u_{\mathrm{e}}), \quad x \in \partial\mathcal{V}, \quad t \geq t_0, \tag{14.37}$$

where $c > 0$ and $u_{\mathrm{e}} \geq 0$. This control law is also known as *Newton's law of cooling* in the literature [77, p. 155] and guarantees that, outside $\mathcal{V}$, the information (temperature) $u(x,t)$ is maintained at u_{e} and the rate of information (heat) flow across the boundary is proportional to $u - u_{\mathrm{e}}$.

Proposition 14.4. Consider the equation (14.35) with the boundary control (14.37). Then $u(x,t) \equiv u_{\mathrm{e}}$ is an asymptotically stable equilibrium state of (14.35) and (14.37).

Proof. Consider the Lyapunov functional candidate $V(u_t - u_{\mathrm{e}}) = \frac{1}{2}\int_{\mathcal{V}}(u_t(x) - u_{\mathrm{e}})^2 \mathrm{d}\mathcal{V} = \frac{1}{2}\|u_t - u_{\mathrm{e}}\|_{\mathcal{L}_2}^2$. Now, it follows from the Green-Gauss theorem that

$$\begin{aligned}
\dot{V}(u_t - u_{\mathrm{e}}) &= \int_{\mathcal{V}} (u(x,t) - u_{\mathrm{e}}) \frac{\partial u(x,t)}{\partial t} \mathrm{d}\mathcal{V} \\
&= \int_{\mathcal{V}} u(x,t)\nabla^2 u(x,t)\mathrm{d}\mathcal{V} - u_{\mathrm{e}} \int_{\mathcal{V}} \nabla^2 u(x,t)\mathrm{d}\mathcal{V} \\
&= -\int_{\mathcal{V}} \nabla u(x,t)\nabla^{\mathrm{T}} u(x,t)\mathrm{d}\mathcal{V} + \int_{\partial\mathcal{V}} u(x,t)\nabla u(x,t)\cdot \mathbf{n}(x)\mathrm{d}\mathcal{S}_{\mathcal{V}} \\
&\quad -u_{\mathrm{e}} \int_{\partial\mathcal{V}} \nabla u(x,t)\cdot \mathbf{n}(x)\mathrm{d}\mathcal{S}_{\mathcal{V}} \\
&= -D(u_t, u_t) + \int_{\partial\mathcal{V}} (u(x,t) - u_{\mathrm{e}})\nabla u(x,t)\cdot \mathbf{n}(x)\mathrm{d}\mathcal{S}_{\mathcal{V}} \\
&= -D(u_t, u_t) - c\int_{\partial\mathcal{V}} (u(x,t) - u_{\mathrm{e}})^2 \mathrm{d}\mathcal{S}_{\mathcal{V}} \\
&< 0, \quad u_t \in \mathcal{W}_2^0(\mathcal{V}), \quad u_t \neq u_{\mathrm{e}},
\end{aligned} \tag{14.38}$$

which establishes asymptotic stability of the equilibrium state $u(x,t) \equiv u_{\mathrm{e}}$. □

The control problem addressed by Proposition 14.4 can be viewed as a leader-follower coordination problem [135] for dynamical swarm systems.

14.5. Advection-Diffusion Model

The nonlinear partial differential equation (14.1) describes a general conservation equation which includes many important swarming models discussed in the literature. See, for example, [183]. In this section, we turn our attention to a specific form of (14.1) involving the *advection-diffusion* model [87,183] defined over a compact connected set $\mathcal{V} \subset \mathbb{R}^n$ with a smooth boundary $\partial\mathcal{V}$ and volume $\mathrm{vol}\,\mathcal{V}$ given by

$$\frac{\partial \rho(x,t)}{\partial t} = -\nabla \cdot (\rho(x,t)v(x,t)) + \nabla \cdot \left(B(x,t)\nabla^{\mathrm{T}}\rho(x,t)\right), \tag{14.39}$$

$$\rho(x,t_0) = \rho_{t_0}(x), \quad x \in \mathcal{V}, \quad t \geq t_0, \tag{14.40}$$

where $\rho : \mathcal{V} \times [0,\infty) \to \overline{\mathbb{R}}_+$ denotes the density distribution of mobile agents at the point $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathcal{V}$ and time instant $t \geq t_0$, $v : \mathcal{V} \times [0,\infty) \to \mathbb{R}^n$ is a density-dependent advection velocity, and $B : \mathcal{V} \times [0,\infty) \to \mathbb{R}^{n\times n}$ is a diffusion operator. Here, we consider the case where $v(x,t)$ is given by

$$v(x,t) = -k\nabla^{\mathrm{T}}\rho(x,t), \quad x \in \mathcal{V}, \quad t \geq t_0, \tag{14.41}$$

where $k \in \mathbb{R}$ and $B(x,t) = \lambda I_n \in \mathbb{R}^{n\times n}$ for all $x \in \mathcal{V}$ and $t \geq t_0$, where $\lambda \in \mathbb{R}$.

Theorem 14.5. Consider the dynamical system given by (14.39) and (14.40) with $B(x,t) \equiv \lambda I_n$. Assume that $v(x,t)$ satisfies (14.41). If $k, \lambda \geq 0$ are such that $k^2 + \lambda^2 \neq 0$, then for every $\alpha \in \overline{\mathbb{R}}_+$, $\rho(x,t) \equiv \alpha$ is a semistable equilibrium state of (14.39) and (14.40) if and only if $\nabla\rho(x,t) \cdot \mathbf{n}(x) = 0$, where $x \in \partial\mathcal{V}$ and $t \geq t_0$. In this case, $\rho(x,t) \to \frac{1}{\mathrm{vol}\mathcal{V}}\int_{\mathcal{V}} \rho_{t_0}(x)\mathrm{d}\mathcal{V}$ as $t \to \infty$ for every initial condition $\rho_{t_0} \in \mathcal{W}_2^1(\mathcal{V})$ and every $x \in \mathcal{V}$; moreover, $\frac{1}{\mathrm{vol}\mathcal{V}}\int_{\mathcal{V}} \rho_{t_0}(x)\mathrm{d}\mathcal{V}$ is a semistable equilibrium state of (14.39) and (14.40).

Proof. First, let $k \geq 0$ and $\lambda > 0$. In this case, $\phi(x,\rho(x,t),\nabla\rho(x,t)) = -(k\rho(x,t) + \lambda)\nabla^{\mathrm{T}}\rho(x,t)$, and hence, Assumptions 1 and 2 hold. Furthermore,

$$\begin{aligned}\nabla^2\rho_t(x)\nabla\cdot\phi(x,\rho_t(x),\nabla\rho_t(x)) &= \nabla^2\rho_t(x)\left[-k\nabla\rho_t(x)\nabla^{\mathrm{T}}\rho_t(x) - (k\rho_t(x)+\lambda)\nabla^2\rho_t(x)\right]\\ &= -k[\nabla^2\rho_t(x)]^2 - (k\rho_t(x)+\lambda)[\nabla^2\rho_t(x)]^2\\ &\leq 0, \quad x \in \mathcal{V},\end{aligned} \tag{14.42}$$

and hence, (14.30) holds. Now, the result is a direct consequence of Corollary 14.1.

Next, let $k > 0$ and $\lambda = 0$, and assume that $\rho(x,t)\nabla\rho(x,t) \cdot \mathbf{n}(x) = 0$ for $x \in \partial\mathcal{V}$ and $t \geq t_0$. To show Lyapunov stability of $\rho(x,t) \equiv \alpha$, consider the Lyapunov functional (14.23) with $u(x,t)$ replaced by $\rho(x,t)$. Now, it follows from the Green-Gauss theorem that

$$\begin{aligned}
\dot{V}(\rho_t - \alpha) &= \int_{\mathcal{V}} (\rho(x,t) - \alpha)\frac{\partial \rho(x,t)}{\partial t}\mathrm{d}\mathcal{V} \\
&= -\int_{\mathcal{V}} \rho(x,t)\nabla \cdot (\rho(x,t)v(x,t))\mathrm{d}\mathcal{V} + \alpha \int_{\mathcal{V}} \nabla \cdot (\rho(x,t)v(x,t))\mathrm{d}\mathcal{V} \\
&= \int_{\mathcal{V}} \nabla\rho(x,t)\rho(x,t)v(x,t)\mathrm{d}\mathcal{V} - \int_{\partial\mathcal{V}} \rho(x,t)\rho(x,t)v(x,t) \cdot \mathbf{n}(x)\mathrm{d}\mathcal{S}_{\mathcal{V}} \\
&\quad +\alpha \int_{\partial\mathcal{V}} \rho(x,t)v(x,t) \cdot \mathbf{n}(x)\mathrm{d}\mathcal{S}_{\mathcal{V}} \\
&= -\int_{\mathcal{V}} \rho(x,t)\nabla\rho(x,t)\nabla^{\mathrm{T}}\rho(x,t)\mathrm{d}\mathcal{V} \\
&\leq 0, \quad \rho_t \in \mathcal{W}_2^0(\mathcal{V}),
\end{aligned} \tag{14.43}$$

which proves Lyapunov stability of $\rho(x,t) \equiv \alpha$.

To show semistability of $\rho(x,t) \equiv \alpha$, consider the Lyapunov functionals (14.25) and (14.28). Now, it follows from (14.43), with $\alpha = 0$, that $V(\rho_t)$ is a nonincreasing functional of time for all $\rho_{t_0} \in \mathcal{W}_2^0(\mathcal{V})$. Furthermore, it follows from the Green-Gauss theorem that

$$\begin{aligned}
\frac{1}{2}\dot{D}(\rho_t, \rho_t) &= \int_{\mathcal{V}} \nabla\rho(x,t)\frac{\partial}{\partial t}(\nabla\rho(x,t))^{\mathrm{T}}\mathrm{d}\mathcal{V} \\
&= \int_{\partial\mathcal{V}} \frac{\partial\rho(x,t)}{\partial t} D_{\mathbf{n}(x)}\rho(x,t)\mathrm{d}\mathcal{S}_{\mathcal{V}} - k\int_{\mathcal{V}} [\nabla^2\rho(x,t)]^2\mathrm{d}\mathcal{V}.
\end{aligned} \tag{14.44}$$

Next, using similar arguments as in the proof of Corollary 14.1, it can be shown that $D(\rho_t, \rho_t)$ is a nonincreasing functional of time for all $\rho_{t_0} \in \mathcal{W}_2^1(\mathcal{V})$. Furthermore, note that $\mathcal{E}(\rho_t) = V(\rho_t) + D(\rho_t, \rho_t)$. Hence, $\mathcal{E}(\rho_t)$ is a nonincreasing functional of time for all $\rho_{t_0} \in \mathcal{W}_2^1(\mathcal{V})$. The rest proof follows as in the proof of Theorem 14.4.

The converse follows as in the proof of Theorem 14.4. □

14.6. Connections Between Eulerian and Lagrangian Models for Information Consensus

Information consensus for a Lagrangian network model involves the dynamical system

$$\dot{x}(t) = -Lx(t), \quad x(0) = x_0, \quad t \geq 0, \tag{14.45}$$

where $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ is the information state and $L \in \mathbb{R}^{n\times n}$ is the *Laplacian* of the underlying communication graph topology of the network [135]. Recall that the entries of a Laplacian matrix L of a directed graph are given by $L_{(i,i)} = \sum_{i=1,i\neq j}^{n} A_{(i,j)}$, $j = 1, \ldots, n$, and $L_{(i,j)} = -A_{(i,j)}$ for all $i \neq j$, where $A_{(i,j)}$, $i, j = 1, \ldots, n$, are the entries of the *weighted adjacency* matrix of the directed graph [206]. Consensus is achieved by a group of agents if, for all $x_i(0)$ and $i = 1, \ldots, n$, $\lim_{t\to\infty} x_i(t) \to \alpha$ as $t \to \infty$, where $x_i(t)$ denotes the ith component of $x(t)$ and $\alpha \in \mathbb{R}$.

Next, we compare our Eulerian framework for information consensus developed in this section with the Lagrangian framework for information consensus given by (14.45). Specifically, consider for simplicity the partial differential equation given by (14.35) and (14.36). In this case, (14.35) can be rewritten as

$$\frac{\partial}{\partial t}u(x,t) = -\mathfrak{L}u(x,t), \quad x \in \mathcal{V}, \quad t \geq t_0, \quad u(x,t_0) = u_{t_0}(x), \quad x \in \mathcal{V}, \tag{14.46}$$

where $\mathfrak{L} \triangleq -\nabla^2$ is the Laplacian operator so that (14.46) has the same form as (14.45). Condition (14.36) is a sufficient condition for guaranteeing a uniform information distribution of (14.35). Since $\mathfrak{L}$ is self-adjoint, consider (14.45) with $L = L^{\mathrm{T}}$ and note that, since L has zero row sums, 0 is an eigenvalue of L with an associated eigenvector $\mathbf{e} = [1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^n$. Next, by Proposition 6.1 of [104] (a lumped parameter version of) Assumptions 1 and 2 hold if and only if $\operatorname{rank} L = n - 1$. Now, information consensus for (14.45) is immediate by Theorem 6.1 of [104].

Definition 14.3. We say that λ is an *eigenvalue* of the operator $\mathfrak{L}$ on $\mathcal{V}$ subject to the Neumann boundary condition (14.36) if there exists a function w, not identically equal to zero, solving the boundary value problem

$$\mathfrak{L}w = \lambda w \text{ in } \mathcal{V}, \tag{14.47}$$

$$\frac{\partial w}{\partial \mathbf{n}} = 0 \text{ on } \partial\mathcal{V}. \tag{14.48}$$

Note that it follows from [70, p. 346] that the Neumann boundary value problem

$$\mathfrak{L}u = 0 \text{ in } \mathcal{V}, \tag{14.49}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\mathcal{V}, \tag{14.50}$$

has a smooth solution for $u \equiv C \in \mathbb{R}$. Hence, it follows from Definition 14.3 that 0 is an eigenvalue of the operator $\mathfrak{L}$ with an associated eigenfunction $w = C$. Thus, $\mathfrak{L}$ plays the same role as L. This provides an explicit connection between Lagrangian (discrete) and Eulerian (continuum) network consensus models.

Chapter 15

$\mathcal{H}_2$ Optimal Semistable Control for Linear Dynamical Systems: An LMI Approach

15.1. Introduction

As discussed in Chapters 8–14, dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles, autonomous underwater vehicles, distributed sensor networks, air and ground transportation systems, swarms of air and space vehicle formations, and congestion control in communication networks, to cite but a few examples. A unique feature of the closed-loop dynamics under any control algorithm in dynamical networks is the existence of a continuum of equilibria representing a desired state of convergence. Under such dynamics, the desired limiting state is not determined completely by the system dynamics, but depends on the initial system state as well [123, 124].

The dependence of the limiting state on the initial state is not limited to dynamical network systems, it is also seen in the dynamics of compartmental systems [134] which arise in chemical kinetics [24], and biomedical [132], environmental [182], economic [19], power [50], and thermodynamic systems [104]. In all such systems possessing a continuum of equilibria, semistability, and not asymptotic stability, is the relevant notion of stability.

Semistability was first introduced in [47] for linear systems, and applied to matrix second-order systems in [23]. Nonlinear extensions were considered in [32] and [31], which give several stability results for systems having a continuum of equilibria based on nontangency and arc length of trajectories, respectively. References [123,124] build on the results of [31,32] and give semistable stabilization results for nonlinear network dynamical systems. Optimal semistable stabilization, however, has never been considered in the literature.

In this chapter, we use linear matrix inequalities (LMIs) to develop $\mathcal{H}_2$ optimal semistable controllers for linear dynamical systems. Linear matrix inequalities provide a powerful design framework for linear control problems [36]. Since LMIs lead to convex or quasiconvex optimization problems, they can be solved very efficiently using interior-point algorithms. Unlike the standard $\mathcal{H}_2$ optimal control problem, a complicating feature of the $\mathcal{H}_2$ optimal semistable stabilization problem is that the closed-loop Lyapunov equation guaranteeing semistability can admit multiple solutions. An interesting feature of the proposed approach, however, is that a least squares solution over all possible semistabilizing solutions corresponds to the $\mathcal{H}_2$ optimal solution. It is shown that this least squares solution can be characterized by a linear matrix inequality minimization problem.

15.2. $\mathcal{H}_2$ Semistability Theory

In this section, we establish notation along with several key results on $\mathcal{H}_2$ semistability theory involving the notions of semistability, semicontrollability, and semiobservability. The notion we use in this chapter is fairly standard. Specifically, $\mathbb{R}$ (resp., $\mathbb{C}$) denotes the set of real (resp., complex) numbers, $\mathbb{R}^n$ (resp., $\mathbb{C}^n$) denotes the set of $n \times 1$ real (resp., complex) column vectors, $\mathbb{R}^{n\times m}$ (resp., $\mathbb{C}^{n\times m}$) denotes the set of $n \times m$ real (resp., complex) matrices, $(\cdot)^{\mathrm{T}}$ denotes transpose, $(\cdot)^*$ denotes complex conjugate transpose, $(\cdot)^{\#}$ denotes the group generalized inverse, and I_n or I denotes the $n \times n$ identity matrix. Furthermore, we write $\|\cdot\|$ for the Euclidean vector norm, $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm, $\mathcal{S}^{\perp}$ for the orthogonal complement of a set $\mathcal{S}$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ for the range space and the null space of a matrix A, $\mathrm{spec}(A)$ for the spectrum of the square matrix A, $\det A$ for the determinant of the square matrix A, $\mathrm{rank}\, A$ for the rank of a matrix A, $\mathrm{tr}(\cdot)$ for the trace operator, $\mathbb{E}$ for the expectation operator, and $A \geq 0$ (resp., $A > 0$) to denote the fact that the Hermitian matrix A is nonnegative (resp., positive) definite. Finally, we write $\mathcal{B}_{\varepsilon}(x)$, $x \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball with *radius* ε and *center* x, $\otimes$ for the Kronecker product, $\oplus$ for the Kronecker sum, and $\mathrm{vec}(\cdot)$ for the column stacking operator.

The following definition for semistability for a dynamical system is a restatement of Definition 8.1. For this definition, consider the nonlinear dynamical system given by

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{15.1}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \geq 0$, and $f : \mathcal{D} \to \mathbb{R}^q$ is locally Lipschitz continuous on $\mathcal{D}$.

Definition 15.1. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be positively invariant under (15.1). The equilibrium solution $x(t) \equiv x_{\mathrm{e}} \in \mathcal{D}$ of (15.1) is *Lyapunov stable* with respect to $\mathcal{D}$ if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_{\mathrm{e}}) \cap \mathcal{D}$, then $x(t) \in \mathcal{B}_\varepsilon(x_{\mathrm{e}}) \cap \mathcal{D}$, $t \geq 0$. The equilibrium solution $x(t) \equiv x_{\mathrm{e}} \in \mathcal{D}$ of (15.1) is *semistable* with respect to $\mathcal{D}$ if it is Lyapunov stable with respect to $\mathcal{D}$ and there exists $\delta > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_{\mathrm{e}}) \cap \mathcal{D}$, then $\lim_{t\to\infty} x(t)$ exists and corresponds to a Lyapunov stable equilibrium point in $\mathcal{D}$. Finally, the system (15.1) is said to be *semistable* with respect to $\mathcal{D}$ if every equilibrium point in $\mathcal{D}$ is semistable with respect to $\mathcal{D}$.

Note that if in (15.1) $f(x) = Ax$, where $A \in \mathbb{R}^{n\times n}$, then (15.1) is semistable if and only if A is *semistable*, that is, $\mathrm{spec}(A) \subset \{s \in \mathbb{C} : \mathrm{Re}\, s < 0\} \cup \{0\}$ and, if $0 \in \mathrm{spec}(A)$, then 0 is semisimple. In this case, it can be shown that for every $x_0 \in \mathbb{R}^n$, $\lim_{t\to\infty} x(t)$ exists or, equivalently, $\lim_{t\to\infty} e^{At}$ exists and is given by $\lim_{t\to\infty} e^{At} = I_n - AA^{\#}$ [22, p. 437-438].

Next, we present the notions of semicontrollability and semiobservability. For these definitions let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{l\times n}$, and consider the linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{15.2}$$

$$y(t) = Cx(t), \tag{15.3}$$

with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$, and output $y(t) \in \mathbb{R}^l$, where $t \geq 0$.

Definition 15.2. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$. The pair (A, B) is *semicontrollable* if

$$\left[\bigcap_{k=1}^{n} \mathcal{N}\left(B^{\mathrm{T}}(A^{k-1})^{\mathrm{T}}\right)\right]^{\perp} = \left[\mathcal{N}(A^{\mathrm{T}})\right]^{\perp}, \tag{15.4}$$

where $A^0 \triangleq I_n$.

Definition 15.3. Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$. The pair (A, C) is *semiobservable* if

$$\bigcap_{k=1}^{n} \mathcal{N}\left(CA^{k-1}\right) = \mathcal{N}(A). \tag{15.5}$$

Semicontrollability and semiobservability are extensions of controllability and observability. In particular, semicontrollability is an extension of null controllability to *equilibrium controllability*, whereas semiobservability is an extension of zero-state observability to *equilibrium observability*. It is important to note here that since Definition 15.2 and 15.3 are dual, dual results to the semiobservability results that we establish in this section also hold for semicontrollability.

Definition 15.4. Let $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{l\times n}$, and $K \in \mathbb{R}^{m\times n}$. The pair (A, C) is *semiobservable with respect to* K if

$$\mathcal{N}(K) \cap \left(\bigcap_{i=1}^{n} \mathcal{N}\left(CA^{i-1}\right)\right) = \mathcal{N}(K) \cap \mathcal{N}(A). \tag{15.6}$$

The following result shows that semiobservability is unchanged by full state feedback.

Proposition 15.1. Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, $K \in \mathbb{R}^{m\times n}$, and $R \in \mathbb{R}^{n\times n}$, where R is positive definite. If the pair (A, C) is semiobservable, then the pair $(A + BK, C^{\mathrm{T}}C + K^{\mathrm{T}}RK)$ is semiobservable with respect to K.

Proof. Note that $\mathcal{N}(C^{\mathrm{T}}C + K^{\mathrm{T}}RK) = \mathcal{N}(C) \cap \mathcal{N}(K)$. Hence,

$$\mathcal{N}(K) \cap \left(\bigcap_{i=1}^{n} \mathcal{N}((C^{\mathrm{T}}C + K^{\mathrm{T}}RK)(A + BK)^{i-1})\right)$$

$$
\begin{aligned}
&= \bigcap_{i=1}^{n} \mathcal{N}((C^{\mathrm{T}}C + K^{\mathrm{T}}RK)(A+BK)^{i-1}) \\
&= \mathcal{N}(K) \cap \left(\bigcap_{i=1}^{n} \mathcal{N}(CA^{i-1}) \right) \\
&= \mathcal{N}(K) \cap \mathcal{N}(A) \\
&= \mathcal{N}(K) \cap \mathcal{N}(A+BK),
\end{aligned}
\tag{15.7}
$$

which implies that the pair $(A+BK, C^{\mathrm{T}}C + K^{\mathrm{T}}RK)$ is semiobservable with respect to K.

$\square$

Next, we connect semistability with Lyapunov theory and semiobservability to arrive at a characterization of the $\mathcal{H}_2$ norm of semistable systems. For this result, we consider the linear dynamical system

$$
\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \geq 0, \tag{15.8}
$$

where $A \in \mathbb{R}^{n \times n}$, with output equation (5.39). Furthermore, for a given semistable system define the $\mathcal{H}_2$ norm of $G(s) \sim \left[\begin{array}{c|c} A & x_0 \\ \hline C & 0 \end{array} \right]$ by

$$
|||G|||_2 = \left[\int_0^\infty \|G(t)\|_{\mathrm{F}}^2 \mathrm{d}t \right]^{1/2} = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(\jmath\omega)\|_{\mathrm{F}}^2 \mathrm{d}\omega \right]^{1/2}. \tag{15.9}
$$

The following proposition presents necessary and sufficient conditions for well-posedness of the $\mathcal{H}_2$ norm of a semistable system.

Proposition 15.2. Consider the linear dynamical system (15.8) with output (15.3) and assume A is semistable. Then the following statements are equivalent:

i) For every $x_0 \in \mathbb{R}^n$, $|||G|||_2 < \infty$.

ii) $\int_0^\infty e^{A^{\mathrm{T}}t} R e^{At} \mathrm{d}t < \infty$, where $R = C^{\mathrm{T}}C$.

iii) $\mathcal{N}(A) \subset \mathcal{N}(C)$.

Proof. The equivalence of $i)$ and $ii)$ follows from the fact

$$|||G|||_2^2 = x_0^{\mathrm{T}} \int_0^\infty e^{A^{\mathrm{T}}t} R e^{At} \mathrm{d}t x_0. \tag{15.10}$$

To show $ii)$ implies $iii)$ note that since A is semistable it follows that either A is Hurwitz or there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that $A = S \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} S^{-1}$, where $J \in \mathbb{R}^{r\times r}$, $r = \operatorname{rank} A$, and J is Hurwitz. Now, if A is Hurwitz, then $iii)$ holds trivially since $\mathcal{N}(A) = \{0\} \subset \mathcal{N}(C)$.

Alternatively, if A is not Hurwitz, then

$$\mathcal{N}(A) = \left\{x \in \mathbb{R}^n : x = S[0_{1\times r}, y^{\mathrm{T}}]^{\mathrm{T}}, y \in \mathbb{R}^{n-r}\right\}. \tag{15.11}$$

Now,

$$\begin{aligned} \int_0^\infty e^{A^{\mathrm{T}}t} R e^{At} \mathrm{d}t &= S^{-\mathrm{T}} \int_0^\infty e^{\hat{J}t} \hat{R} e^{\hat{J}t} \mathrm{d}t S \\ &= S^{-\mathrm{T}} \int_0^\infty \begin{bmatrix} e^{J^{\mathrm{T}}t} \hat{R}_1 e^{Jt} & e^{J^{\mathrm{T}}t} \hat{R}_{12} \\ \hat{R}_{12}^{\mathrm{T}} e^{Jt} & \hat{R}_2 \end{bmatrix} \mathrm{d}t S, \end{aligned} \tag{15.12}$$

where

$$\hat{J} = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{R} = S^{\mathrm{T}} R S = \begin{bmatrix} \hat{R}_1 & \hat{R}_{12} \\ \hat{R}_{12}^{\mathrm{T}} & \hat{R}_2 \end{bmatrix}. \tag{15.13}$$

Next, it follows from (15.12) that

$$\int_0^\infty e^{A^{\mathrm{T}}t} R e^{At} \mathrm{d}t < \infty \tag{15.14}$$

if and only if $\hat{R}_2 = 0$ or, equivalently,

$$[0_{1\times r}, y^{\mathrm{T}}] \hat{R} [0_{1\times r}, y^{\mathrm{T}}]^{\mathrm{T}} = 0, \quad y \in \mathbb{R}^{n-r}, \tag{15.15}$$

which is further equivalent to $x^{\mathrm{T}} R x = 0$, $x \in \mathcal{N}(A)$. Hence, $\mathcal{N}(A) \subset \mathcal{N}(C)$.

Finally, the proof of $iii)$ implies $ii)$ is immediate by reversing the steps of the proof given above. $\square$

Theorem 15.1. Consider the linear dynamical system (15.8). Suppose there exist an $n \times n$ matrix $P \geq 0$ and an $m \times n$ matrix $C \in \mathbb{R}^{m\times n}$ such that (A, C) is semiobservable and

$$0 = A^{\mathrm{T}}P + PA + R, \tag{15.16}$$

where $R \triangleq C^{\mathrm{T}}C$. Then (15.8) is semistable with respect to $\mathbb{R}^n$. Furthermore, $|||G(s)|||_2^2 = (x_0 - x_{\mathrm{e}})^{\mathrm{T}}P(x_0 - x_{\mathrm{e}})$, where $x_{\mathrm{e}} \triangleq x_0 - AA^{\#}x_0$.

Proof. The first part of the result is a direct consequence of Proposition 4.1 of [29]. Now, since A is semistable, it follows from ix) of Proposition 11.7.2 of [22] that $\lim_{t\to\infty} e^{At} = I_q - AA^{\#}$. Next, noting that $Ax_{\mathrm{e}} = 0$, (15.8) can be equivalently written as

$$\dot{x}(t) = A(x(t) - x_{\mathrm{e}}), \quad x(0) = x_0, \quad t \geq 0. \tag{15.17}$$

Hence,

$$\int_0^t (x(s) - x_{\mathrm{e}})^{\mathrm{T}}R(x(s) - x_{\mathrm{e}})\mathrm{d}s = -(x(t) - x_{\mathrm{e}})^{\mathrm{T}}P(x(t) - x_{\mathrm{e}}) + (x_0 - x_{\mathrm{e}})^{\mathrm{T}}P(x_0 - x_{\mathrm{e}}). \tag{15.18}$$

Now, it follows from the semiobservability of (A, C) that $Rx_{\mathrm{e}} = 0$. Hence, letting $t \to \infty$ and noting that $x(t) \to x_{\mathrm{e}}$ as $t \to \infty$ it follows from (15.18) that

$$\int_0^\infty x^{\mathrm{T}}(t)Rx(t)\mathrm{d}t = (x_0 - x_{\mathrm{e}})^{\mathrm{T}}P(x_0 - x_{\mathrm{e}}). \tag{15.19}$$

Finally, defining the free response of (15.8) by $z(t) \triangleq Cx(t) = Ce^{At}x_0$, $t \geq 0$, and noting that $R = C^{\mathrm{T}}C$, it follows from Parseval's theorem that

$$(x_0 - x_{\mathrm{e}})^{\mathrm{T}}P(x_0 - x_{\mathrm{e}}) = \int_0^\infty z^{\mathrm{T}}(t)z(t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^\infty \|G(\jmath\omega)\|_{\mathrm{F}}^2\mathrm{d}\omega. \tag{15.20}$$

This completes the proof. □

Next, we give a necessary and sufficient condition for characterizing semistability using the Lyapunov equation (15.16). Before we state this result, the following lemmas are needed.

Lemma 15.1. Consider the linear dynamical system (15.8). If (15.8) is semistable, then, for every $n \times n$ nonnegative definite matrix R,

$$\int_0^\infty (x(t) - x_{\rm e})^{\rm T} R (x(t) - x_{\rm e}) {\rm d}t < \infty, \tag{15.21}$$

where $x_{\rm e} = (I_n - AA^{\#})x_0$.

Proof. Since A is semistable, it follows from the Jordan decomposition that there exists an invertible matrix $S \in \mathbb{C}^{n\times n}$ such that $A = S \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} S^{-1}$, where $J \in \mathbb{C}^{r\times r}$, $r = \operatorname{rank} A$, and J is asymptotically stable. Let $z(t) \triangleq S^{-1}x(t)$ and $z_{\rm e} \triangleq S^{-1}x_{\rm e}$, $t \geq 0$. Then (15.8) becomes

$$\dot{z}(t) = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} z(t), \quad z(0) = S^{-1}x_0, \quad t \geq 0, \tag{15.22}$$

which implies that $\lim_{t\to\infty} z_i(t) = 0$, $i = 1, \ldots, r$, and $z_j(t) = z_j(0)$, $j = r+1, \ldots, n$, that is, $z_{\rm e} = [0, \ldots, 0, z_{r+1}(0), \ldots, z_n(0)]^{\rm T}$. Now,

$$\begin{aligned} \int_0^\infty (x(t) - x_{\rm e})^{\rm T} R (x(t) - x_{\rm e}) {\rm d}t &= \int_0^\infty (z(t) - z_{\rm e})^* S^* R S (z(t) - z_{\rm e}) {\rm d}t \\ &= \int_0^\infty \hat{z}^*(t) S^* R S \hat{z}(t) {\rm d}t, \end{aligned} \tag{15.23}$$

where $\hat{z}(t) \triangleq [z_1(t), \ldots, z_r(t), 0, \ldots, 0]^{\rm T}$. Since

$$\dot{\hat{z}}(t) = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \hat{z}(t) \tag{15.24}$$

and J is asymptotically stable, it follows that

$$\int_0^\infty \hat{z}^*(t) S^* R S \hat{z}(t) {\rm d}t < \infty, \tag{15.25}$$

which proves the result. □

Lemma 15.2. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$. If A and B are semistable, then $A \oplus B$ is semistable.

Proof. Let $\lambda \in \operatorname{spec}(A)$ and $\mu \in \operatorname{spec}(B)$. Since A and B are both semistable, it follows that $\operatorname{Re}\lambda < 0$ or $\lambda = 0$ and $\operatorname{am}_A(0) = \operatorname{gm}_A(0)$, and $\operatorname{Re}\mu < 0$ or $\mu = 0$ and $\operatorname{am}_B(0) = \operatorname{gm}_B(0)$, where $\operatorname{am}_X(\lambda)$ and $\operatorname{gm}_X(\lambda)$ denote algebraic multiplicity of $\lambda \in \operatorname{spec}(X)$ and geometric multiplicity of $\lambda \in \operatorname{spec}(X)$, respectively. Now, it follows from the fact that $\lambda + \mu \in \operatorname{spec}(A \oplus B)$, that $\operatorname{spec}(A \oplus B) \subset \{z \in \mathbb{C} : \operatorname{Re} z < 0\} \cup \{0\}$. Next, it follows from Fact 7.5.2 of [22] that $\operatorname{gm}_A(0)\operatorname{gm}_B(0) \leq \operatorname{gm}_{A\oplus B}(0) \leq \operatorname{am}_{A\oplus B}(0) = \operatorname{am}_A(0)\operatorname{am}_B(0)$. Since $\operatorname{am}_A(0) = \operatorname{gm}_A(0)$ and $\operatorname{am}_B(0) = \operatorname{gm}_B(0)$, it follows that $\operatorname{gm}_{A\oplus B}(0) = \operatorname{am}_{A\oplus B}(0)$, and hence, $A \oplus B$ is semistable. $\square$

Lemma 15.3. Let $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, and assume A is semistable. Then $\int_0^\infty e^{At}x\mathrm{d}t$ exists if and only if $x \in \mathcal{R}(A)$. In this case, $\int_0^\infty e^{At}x\mathrm{d}t = -A^{\#}x$.

Proof. The proof is similar to the proofs of (vii) and (viii) of Lemma 2.2 of [26] and, hence, is omitted. $\square$

Lemma 15.4 [29]. Let $A \in \mathbb{R}^{n\times n}$. If there exist an $n \times n$ matrix $P \geq 0$ and an $m \times n$ matrix $C \in \mathbb{R}^{m\times n}$ such that (A, C) is semiobservable and (15.16) holds, then *i)* $\mathcal{N}(P) \subseteq \mathcal{N}(A) \subseteq \mathcal{N}(R)$ and *ii)* $\mathcal{N}(A) \cap \mathcal{R}(A) = \{0\}$.

Theorem 15.2. Consider the linear dynamical system (15.8). Then (15.8) is semistable if and only if for every semiobservable pair (A, C) there exists an $n \times n$ matrix $P \geq 0$ such that (15.16) holds. Furthermore, if (A, C) is semiobservable and P satisfies (15.16), then

$$P = \int_0^\infty e^{A^{\mathrm{T}}t} R e^{At} \mathrm{d}t + P_0 \tag{15.26}$$

for some $P_0 = P_0^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ satisfying

$$0 = A^{\mathrm{T}} P_0 + P_0 A \tag{15.27}$$

and

$$P_0 \geq -\int_0^\infty e^{A^{\mathrm{T}}t} R e^{At} \mathrm{d}t. \tag{15.28}$$

In addition, $\min_{P\in\mathcal{P}} \|P\|_{\mathrm{F}}$ has a unique solution P given by

$$P = \int_0^\infty e^{A^{\mathrm{T}}t} R e^{At} \mathrm{d}t, \tag{15.29}$$

where $\mathcal{P}$ denotes the set of all P satisfying (15.16). Finally, (15.8) is semistable if and only if for every semiobservable pair (A, C) there exists an $n \times n$ matrix $P > 0$ such that (15.16) holds.

Proof. Sufficiency for the first implication follows from Theorem 15.1. To show necessity, assume (15.8) is semistable. Then, $\lim_{t\to\infty} x(t) = x_{\mathrm{e}}$, where $x_{\mathrm{e}} = (I_n - AA^{\#})x_0$. For a semiobservable pair (A, C), let

$$P = \int_0^\infty (AA^{\#})^{\mathrm{T}} e^{A^{\mathrm{T}}t} R e^{At} AA^{\#} \mathrm{d}t. \tag{15.30}$$

Then, for $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} x_0^{\mathrm{T}} P x_0 &= \int_0^\infty x_0^{\mathrm{T}} (AA^{\#})^{\mathrm{T}} e^{A^{\mathrm{T}}t} R e^{At} AA^{\#} x_0 \mathrm{d}t \\ &= \int_0^\infty (x_0 - x_{\mathrm{e}})^{\mathrm{T}} e^{A^{\mathrm{T}}t} R e^{At} (x_0 - x_{\mathrm{e}}) \mathrm{d}t \\ &= \int_0^\infty (x(t) - x_{\mathrm{e}})^{\mathrm{T}} R (x(t) - x_{\mathrm{e}}) \mathrm{d}t, \end{aligned} \tag{15.31}$$

where we used the fact that $x(t) - x_{\mathrm{e}} = e^{At}(x_0 - x_{\mathrm{e}})$. It follows from Lemma 15.1 that P is well defined. Since $x_{\mathrm{e}} \in \mathcal{N}(A)$, it follows from (15.5) that $Rx_{\mathrm{e}} = 0$, and hence,

$$x_0^{\mathrm{T}} P x_0 = \int_0^\infty x^{\mathrm{T}}(t) R x(t) \mathrm{d}t = \int_0^\infty x_0^{\mathrm{T}} e^{A^{\mathrm{T}}t} R e^{At} x_0 \mathrm{d}t, \tag{15.32}$$

which implies that

$$P = \int_0^\infty e^{A^{\mathrm{T}}t} R e^{At} \mathrm{d}t. \tag{15.33}$$

Now, (15.16) is immediate using the fact that $Rx_{\mathrm{e}} = 0$.

Next, since A is semistable, it follows from the above result that there exists an $n \times n$ nonnegative-definite matrix P such that (15.16) holds or, equivalently, $(A \oplus A)^{\mathrm{T}}\mathrm{vec}\, P = -\mathrm{vec}\, R$. Hence, $\mathrm{vec}\, R \in \mathcal{R}((A \oplus A)^{\mathrm{T}})$ and

$$\mathcal{P} = \left\{P \in \mathbb{R}^{n\times n} : P = -\mathrm{vec}^{-1}\left(((A \oplus A)^{\mathrm{T}})^{\#}\mathrm{vec}\, R\right) + \mathrm{vec}^{-1}(z)\right\}$$

for some $z \in \mathcal{N}((A \oplus A)^{\mathrm{T}})$. Next, it follows from Lemma 15.2 that $A \oplus A$ is semistable, and hence, by Lemma 15.3,

$$\begin{aligned}
\mathrm{vec}^{-1}\left(((A \oplus A)^{\mathrm{T}})^{\#}\mathrm{vec}\, R\right) &= -\int_0^\infty \mathrm{vec}^{-1}\left(e^{(A\oplus A)^{\mathrm{T}}t}\mathrm{vec}\, R\right)\mathrm{d}t\\
&= -\int_0^\infty \mathrm{vec}^{-1}\left(e^{A^{\mathrm{T}}t} \otimes e^{A^{\mathrm{T}}t}\right)\mathrm{vec}\, R\mathrm{d}t\\
&= -\int_0^\infty e^{A^{\mathrm{T}}t}Re^{At}\mathrm{d}t, \qquad (15.34)
\end{aligned}$$

where in (15.34) we used the facts that $(X \otimes Y)^{\mathrm{T}} = X^{\mathrm{T}} \otimes Y^{\mathrm{T}}$, $e^{X\oplus Y} = e^X \otimes e^Y$, and $\mathrm{vec}(XYZ) = (Z^{\mathrm{T}} \otimes X)\mathrm{vec}\, Y$ [22, Chapter 7]. Hence,

$$P = \int_0^\infty e^{A^{\mathrm{T}}t}Re^{At}\mathrm{d}t + \mathrm{vec}^{-1}(z), \qquad (15.35)$$

where $\mathrm{vec}^{-1}(z)$ satisfies $\mathrm{vec}^{-1}(z) = (\mathrm{vec}^{-1}(z))^{\mathrm{T}}$, $A^{\mathrm{T}}\mathrm{vec}^{-1}(z)+\mathrm{vec}^{-1}(z)A = 0$, and $\mathrm{vec}^{-1}(z) \geq -\int_0^\infty e^{A^{\mathrm{T}}t}Re^{At}\mathrm{d}t$. If P is such that $\min_{P\in\mathcal{P}}\|P\|_{\mathrm{F}}$ holds, then it follows that P is the unique solution of a least squares minimization problem and is given by

$$P = -\mathrm{vec}^{-1}\left(((A \oplus A)^{\mathrm{T}})^{\#}\mathrm{vec}\, R\right) = \int_0^\infty e^{A^{\mathrm{T}}t}Re^{At}\mathrm{d}t. \qquad (15.36)$$

Finally, suppose (A, C) is semiobservable. Then it follows from the first part of the theorem that there exists an $n \times n$ matrix $P \geq 0$ such that (15.16) holds. Since, by Lemma 15.4, $\mathcal{N}(A) \cap \mathcal{R}(A) = \{0\}$, it follows from Lemma 4.14 of [19] that A is group invertible. Thus, let $L \triangleq I_n - AA^{\#}$ and note that $L^2 = L$. Hence, L is the unique $n \times n$ matrix satisfying $\mathcal{N}(L) = \mathcal{R}(A)$, $\mathcal{R}(L) = \mathcal{N}(A)$, and $Lx = x$ for all $x \in \mathcal{N}(A)$. Now, define

$$\hat{P} \triangleq P + L^{\mathrm{T}}L. \qquad (15.37)$$

Next, we show that $\hat{P}$ is positive definite. Consider the function $V(x) = x^{\mathrm{T}}\hat{P}x$, $x \in \mathbb{R}^n$. If $V(x) = 0$ for some $x \in \mathbb{R}^n$, then $Px = 0$ and $Lx = 0$. It follows from $i)$ of Lemma 15.4 that $x \in \mathcal{N}(A)$, and $Lx = 0$ implies that $x \in \mathcal{R}(A)$. Now, it follows from $ii)$ of Lemma 15.4 that $x = 0$. Hence, $\hat{P}$ is positive definite. Next, since $LA = A - AA^{\#}A = 0$, it follows that

$$A^{\mathrm{T}}\hat{P} + \hat{P}A + R = A^{\mathrm{T}}P + PA + R + A^{\mathrm{T}}L^{\mathrm{T}}L + L^{\mathrm{T}}LA = (LA)^{\mathrm{T}}L + L^{\mathrm{T}}LA = 0.$$

Conversely, if there exists $P > 0$ such that (15.16) holds, consider the function $U(x) = x^{\mathrm{T}}Px$, $x \in \mathbb{R}^n$. Then $\dot{U}(x) = -x^{\mathrm{T}}Rx \leq 0$ and $\dot{U}^{-1}(0) = \mathcal{N}(R)$. To obtain the largest invariant set $\mathcal{M}$ contained in $\mathcal{N}(R)$, consider a solution $x(t)$ of (15.8) such that $Cx(t) = 0$ for all $t \geq 0$. On $\mathcal{M}$, it follows that $C\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}x(t) = 0$ for all $t \geq 0$ and $k = 1, \ldots, n$, and hence, $CA^{k-1}x(t) = 0$ for all $t \geq 0$ and $k = 1, \ldots, n$. Now, it follows from (15.5) that $x(t) \in \mathcal{N}(A)$ for all $t \geq 0$. Thus, $\mathcal{M} \subseteq \mathcal{N}(A)$. Since $\mathcal{N}(A)$ consists of equilibrium points, it follows that $\mathcal{M} = \mathcal{N}(A)$. For $x_{\mathrm{e}} \in \mathcal{N}(A)$, Lyapunov stability of x_{e} now follows by considering the Lyapunov function $U(x - x_{\mathrm{e}})$. □

Next, we show that the unique solution P given by (15.16) and satisfying $\min_{P\in\mathcal{P}} \|P\|_{\mathrm{F}}$ can be characterized by a linear matrix inequality minimization problem.

Theorem 15.3. Consider the linear dynamical system (15.8) with output (15.3). Assume A is semistable and (A, C) is semiobservable. Let $P_{\min}$ be the solution to the linear matrix inequality minimization problem

$$\min \left\{ \operatorname{tr} PV : P \geq 0 \text{ and } A^{\mathrm{T}}P + PA + R \leq 0 \right\}, \tag{15.38}$$

where $V \in \mathbb{R}^{n\times n}$, $V \geq 0$. Then

$$\operatorname{tr} P_{\min}V = \operatorname{tr} \int_0^{\infty} e^{A^{\mathrm{T}}t}Re^{At}\mathrm{d}tV. \tag{15.39}$$

Proof. Let $\hat{P} = \int_0^{\infty} e^{A^{\mathrm{T}}t}Re^{At}\mathrm{d}t$ and let $P \geq 0$ be such that

$$A^{\mathrm{T}}P + PA + R \leq 0. \tag{15.40}$$

(Note that $A^{\mathrm{T}}\hat{P}+\hat{P}A+R=0$, which implies that a $P\geq 0$ satisfying (15.40) exists.) Now, let $W\in\mathbb{R}^{n\times n}$, $W\geq 0$, be such that

$$0=A^{\mathrm{T}}P+PA+R+W. \tag{15.41}$$

Next, since (A,C) is semiobservable it follows that if $x_{\mathrm{e}}\in\mathcal{N}(A)$, then $Rx_{\mathrm{e}}=0$, and hence, it follows from (15.41) that $Wx_{\mathrm{e}}=0$. Now, using identical arguments as in the proof of Theorem 15.2 it follows that

$$\begin{aligned} P &= \int_0^\infty e^{A^{\mathrm{T}}t}(R+W)e^{At}\mathrm{d}t \\ &\geq \int_0^\infty e^{A^{\mathrm{T}}t}Re^{At}\mathrm{d}t \\ &= \hat{P}. \end{aligned} \tag{15.42}$$

Finally, since $\hat{P}$ is an element of the feasible set of the optimization problem (15.38), $\operatorname{tr}P_{\min}V=\operatorname{tr}\hat{P}V$. □

Finally, we provide a dual result to Theorem 15.3 which is necessary for developing feedback controllers guaranteeing closed-loop semistability.

Theorem 15.4. Consider the linear dynamical system (15.8) with output (15.3). Assume A is semistable and let $V\in\mathbb{R}^{n\times n}$, $V\geq 0$, be such that (A,V) is semicontrollable. Let $Q_{\min}$ be the solution to the LMI minimization problem

$$\min\left\{\operatorname{tr}QR : Q\geq 0 \text{ and } AQ+QA^{\mathrm{T}}+V\leq 0\right\}. \tag{15.43}$$

Then

$$\operatorname{tr}Q_{\min}R=\operatorname{tr}\int_0^\infty e^{A^{\mathrm{T}}t}Re^{At}\mathrm{d}tV=\operatorname{tr}P_{\min}V, \tag{15.44}$$

where $P_{\min}$ is the solution to the LMI minimization problem given by (15.38).

Proof. The proof is a direct consequence of Theorem 15.3 by noting that (A, V) is semicontrollable if and only if (A^{T}, V) is semiobservable. Now, replacing A with A^{T} and R with V in Theorem 15.3 it follows that

$$\begin{aligned} \operatorname{tr} Q_{\min} R &= \operatorname{tr} \int_0^\infty e^{At} V e^{A^{\mathrm{T}} t} \mathrm{d}t R \\ &= \operatorname{tr} \int_0^\infty e^{A^{\mathrm{T}} t} R e^{At} \mathrm{d}t V \\ &= \operatorname{tr} P_{\min} V. \end{aligned} \tag{15.45}$$

This completes the proof. □

15.3. Optimal Semistable Stabilization

In this section, we consider the problem of optimal state feedback control for semistable stabilization of linear dynamical systems. Specifically, we consider the controlled linear system given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{15.46}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$, with the state feedback controller $u(t) = Kx(t)$, where $K \in \mathbb{R}^{m\times n}$, such that the closed-loop system given by

$$\dot{x}(t) = (A + BK)x(t), \quad x(0) = x_0, \quad t \geq 0, \tag{15.47}$$

is semistable and the performance criterion

$$J(K) \triangleq \int_0^\infty \left[(x(t) - x_{\mathrm{e}})^{\mathrm{T}} R_1 (x(t) - x_{\mathrm{e}}) + (u(t) - u_{\mathrm{e}})^{\mathrm{T}} R_2 (u(t) - u_{\mathrm{e}})\right] \mathrm{d}t \tag{15.48}$$

is minimized, where $R_1 \triangleq E_1^{\mathrm{T}} E_1$, $R_2 \triangleq E_2^{\mathrm{T}} E_2 > 0$, $R_{12} \triangleq E_1^{\mathrm{T}} E_2 = 0$, $u_{\mathrm{e}} = Kx_{\mathrm{e}}$, and $x_{\mathrm{e}} = \lim_{t\to\infty} x(t)$.

Note that it follows from Lemma 15.1 that if the closed-loop system is semistable, then $J(K)$ is well defined. To develop necessary conditions for the optimal semistable control problem, we assume that (A, B) is semicontrollable, (A, E_1) is semiobservable, and $x_{\mathrm{e}} \in \mathcal{N}(K)$.

In this case, it follows from Proposition 15.1 that $(A+BK, R_1+K^{\mathrm{T}}R_2K)$ is semiobservable with respect to K, and hence, $(R_1+K^{\mathrm{T}}R_2K)x_{\mathrm{e}}=0$. Thus,

$$\begin{aligned}
J(K) &= \int_0^\infty x_0^{\mathrm{T}} e^{\tilde{A}^{\mathrm{T}}t}(R_1+K^{\mathrm{T}}R_2K)e^{\tilde{A}t}x_0 \mathrm{d}t \\
&= \mathrm{tr} \int_0^\infty e^{\tilde{A}^{\mathrm{T}}t}(R_1+K^{\mathrm{T}}R_2K)e^{\tilde{A}t}x_0x_0^{\mathrm{T}} \mathrm{d}t \\
&= \mathrm{tr} P_{\mathrm{LS}}V, \qquad (15.49)
\end{aligned}$$

where we assume that the initial state x_0 is a random variable such that $\mathbb{E}[x_0]=0$ and $\mathbb{E}[x_0x_0^{\mathrm{T}}]=V$, $\tilde{A} \triangleq A+BK$, and $P_{\mathrm{LS}} \triangleq \int_0^\infty e^{\tilde{A}^{\mathrm{T}}t}(R_1+K^{\mathrm{T}}R_2K)e^{\tilde{A}t}\mathrm{d}t$ denotes the least squares solution to

$$0 = \tilde{A}^{\mathrm{T}}P + P\tilde{A} + \tilde{R}, \qquad (15.50)$$

where $\tilde{R} \triangleq R_1+K^{\mathrm{T}}R_2K$. Unlike the standard $\mathcal{H}_2$ optimal control problem, $P_{\mathrm{LS}} \geq 0$ is not a unique solution to (15.50).

The following theorem presents an LMI solution to the $\mathcal{H}_2$ optimal semistable control problem.

Theorem 15.5. Consider the linear dynamical system (15.46) and assume (A, E_1) is semiobservable and (A, V) is semicontrollable. Let $Q \in \mathbb{R}^{n\times n}$ and $X \in \mathbb{R}^{m\times n}$ be the solution to the LMI minimization problem

$$\min_{Q\in\mathbb{R}^{n\times n},\, X\in\mathbb{R}^{m\times n},\, W\in\mathbb{R}^{p\times p}} \mathrm{tr}\, W, \qquad (15.51)$$

subject to

$$\begin{bmatrix} Q & (E_1Q+E_2X)^{\mathrm{T}} \\ E_1Q+E_2X & W \end{bmatrix} > 0, \qquad (15.52)$$

$$AQ+BX+QA^{\mathrm{T}}+X^{\mathrm{T}}B^{\mathrm{T}}+V \leq 0. \qquad (15.53)$$

Then $K = XQ^{-1}$ is a semistabilizing controller for (15.46), that is, $A+BK$ is semistable. Furthermore, K minimizes the $\mathcal{H}_2$ performance criterion $J(K)$ given by (15.48).

Proof. Since $K = XQ^{-1}$ it follows from (15.53) that

$$(A + BK)Q + Q(A + BK)^{\mathrm{T}} + V \leq 0, \tag{15.54}$$

which, since (A, V) is semicontrollable, implies that $A + BK$ is semistable. Next, note that (15.52) holds if and only if

$$W > (E_1Q + E_2X)Q^{-1}(E_1Q + E_2X)^{\mathrm{T}}, \tag{15.55}$$

which implies that the minimization problem (15.51)–(15.53) is equivalent to

$$\min \operatorname{tr}(E_1Q + E_2X)Q^{-1}(E_1Q + E_2X)^{\mathrm{T}}, \tag{15.56}$$

subject to

$$AQ + BX + QA^{\mathrm{T}} + X^{\mathrm{T}}B^{\mathrm{T}} + V \leq 0, \tag{15.57}$$

$$Q > 0. \tag{15.58}$$

Hence, noting that (15.56)–(15.58) is equivalent to

$$\min \operatorname{tr} Q\tilde{R}, \tag{15.59}$$

subject to

$$\tilde{A}Q + Q\tilde{A}^{\mathrm{T}} + V \leq 0, \tag{15.60}$$

$$Q > 0, \tag{15.61}$$

the result follows as a direct consequence of Theorems 15.4 and 15.2. □

15.4. Optimal Fixed-Structure Control for Network Consensus

In this section, we use the optimal control framework developed in Section 15.3 to design optimal controllers for multiagent network dynamical systems. Specifically, we use undirected graphs to represent a dynamical network and present solutions to the consensus problem for networks with undirected graph *topologies* (or information flow) [187]. Specifically, let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted *directed graph* (or digraph) denoting the dynamical network (or dynamic graph) with the set of *nodes* (or vertices) $\mathcal{V} = \{1, \ldots, n\}$ involving a finite nonempty set denoting the agents, the set of *edges* $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow, and an *adjacency* matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ such that $\mathcal{A}_{(i,j)} = 1$, $i, j = 1, \ldots, n$, if $(j, i) \in \mathcal{E}$, and 0 otherwise. The edge $(i, j) \in \mathcal{E}$ denotes that agent $\mathcal{G}_j$ can obtain information from agent $\mathcal{G}_i$, but not necessarily vice versa. Moreover, we assume that $\mathcal{A}_{(i,i)} = 0$ for all $i \in \mathcal{V}$. A *graph* or *undirected graph* $\mathfrak{G}$ associated with the adjacency matrix $\mathcal{A} \in \mathbb{R}^{q\times q}$ is a directed graph for which the *arc set* is symmetric, that is, $\mathcal{A} = \mathcal{A}^{\mathrm{T}}$. A graph $\mathfrak{G}$ is *balanced* if $\sum_{j=1}^{n} \mathcal{A}_{(i,j)} = \sum_{j=1}^{n} \mathcal{A}_{(j,i)}$ for all $i = 1, \ldots, n$. Finally, we denote the *value* of the node i, $i = 1, \ldots, n$, at time t by $x_i(t) \in \mathbb{R}$. The consensus problem involves the design of a dynamic algorithm that guarantees information state equipartition, that is, $\lim_{t\to\infty} x_i(t) = \alpha \in \mathbb{R}$ for $i = 1, \ldots, n$.

As noted in Chapter 8, a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in a dynamical network is the existence of a continuum of equilibria representing a state of consensus. Under such dynamics, the limiting consensus state is not determined completely by the system dynamics, but on the initial system state as well. For such a system possessing a continuum of equilibria, semistability, and not asymptotic stability is the relevant notion of stability.

The information flow model is a network dynamical system involving the trajectories of the dynamical network characterized by the multiagent dynamical system $\mathcal{G}$ given by

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \quad i = 1, \ldots, q, \tag{15.62}$$

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \frac{1}{k_i} \mathcal{A}_{(i,j)}(x_j(t) - x_i(t)), \tag{15.63}$$

where $q \geq 2$, $x_i(t) \in \mathbb{R}$, $t \geq 0$, represents an information state, $u_i(t) \in \mathbb{R}$, $t \geq 0$, represents the control input, $k_i > 0$, $i = 1, \ldots, q$, and $\mathcal{A}_{(i,j)} \geq 0$, $i, j = 1, \ldots, q$, $i \neq j$.

Assumption 1: For the *connectivity matrix* $\mathcal{C} \in \mathbb{R}^{q\times q}$ associated with the multiagent dynamical system $\mathcal{G}$ defined by

$$\mathcal{C}_{(i,j)} = \begin{cases} 1, & \text{if } (j,i) \in \mathcal{E}, \\ 0, & \text{otherwise}, \end{cases} \quad i \neq j, \quad i,j = 1,\ldots,q, \tag{15.64}$$

and $\mathcal{C}_{(i,i)} = -\sum_{k=1,\,k\neq i}^{q} \mathcal{C}_{(i,k)}$, $i = j$, $i = 1,\ldots,q$, $\operatorname{rank} \mathcal{C} = q-1$ and $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$.

The negative of the connectivity matrix, that is, $-\mathcal{C}$, is known in the literature as the *Laplacian* of the graph $\mathfrak{G}$. Furthermore, note that $\mathcal{C}_{(i,j)} = \mathcal{A}_{(i,j)}$ for all $i,j = 1,\ldots,q$, $i \neq j$. In multiagent coordination [135, 187] and distributed network averaging [240] with a fixed communication topology, we require that $x_{\mathrm{e}} \in \operatorname{span}\{\mathbf{e}\}$, where $\mathbf{e} \in \mathbb{R}^q$ denotes the ones vector of order q, that is, $\mathbf{e} \triangleq [1,\ldots,1]^{\mathrm{T}}$. In this section, we consider the design of a fixed-structure consensus protocol for (15.62) and (15.63) such that the closed-loop system is semistable, that is, $\lim_{t\to\infty} x_i(t) = \alpha$, $i = 1,\ldots,q$, $\alpha \in \mathbb{R}$, and (15.48) is minimized.

Proposition 15.3. Consider the information flow model (15.62) and (15.63) and assume that Assumption 1 holds. Then $\alpha\mathbf{e}$, $\alpha \in \mathbb{R}$, is an equilibrium state of (15.62) and (15.63).

Proof. The proof is similar to the proof of Proposition 8.6 and, hence, is omitted. □

Proposition 15.4. Consider the information flow model (15.62) and (15.63) and assume that Assumption 1 holds. Then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a semistable equilibrium state of (15.62) and (15.63). Furthermore, $x(t) \to \alpha_*\mathbf{e}$ as $t \to \infty$, where $\alpha_* = \sum_{i=1}^{q} k_i x_i(0)/(\sum_{i=1}^{q} k_i)$, and $\alpha_*\mathbf{e}$ is a semistable equilibrium state.

Proof. First, note that if Assumption 1 holds for (15.62) and (15.63), then it follows from Proposition 15.3 that $\alpha\mathbf{e}$, $\alpha \in \mathbb{R}$, is an equilibrium state of (15.62) and (15.63). To show Lyapunov stability of the equilibrium state $\alpha\mathbf{e}$, consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}(x - \alpha\mathbf{e})^{\mathrm{T}} K (x - \alpha\mathbf{e}), \tag{15.65}$$

where $K \triangleq \mathrm{diag}[k_1, \ldots, k_q] \in \mathbb{R}^{q\times q}$. Now, using similar arguments as in the proof of Theorem 3.9 of [104] and noting that $\mathcal{A}^{\mathrm{T}} = \mathcal{A}$ and $k_i\phi_{ij}(x) = -k_j\phi_{ji}(x)$, $i, j = 1, \ldots, q$, $i \neq j$, it follows that

$$\begin{aligned}\dot{V}(x) &= (x - \alpha\mathbf{e})^{\mathrm{T}} K\dot{x} \\ &= \sum_{i=1}^{q} x_i \sum_{j=1, j\neq i}^{q} \mathcal{A}_{(i,j)}(x_j - x_i) \\ &= -\sum_{i=1}^{q}\sum_{j=i+1}^{q-1} \mathcal{A}_{(i,j)}(x_i - x_j)^2 \\ &\leq 0, \quad x \in \mathbb{R}^q, \end{aligned} \tag{15.66}$$

which establishes Lyapunov stability of the equilibrium state $\alpha\mathbf{e}$. Next, using similar arguments as in the proof of Theorem 3.9 of [104], it can be shown that the largest invariant set $\mathcal{M}$ contained in $\dot{V}^{-1}(0)$ is given by $\mathcal{M} = \{\alpha\mathbf{e}\}$, and hence, $\alpha\mathbf{e}$ is semistable.

Finally, note that $\mathbf{e}^{\mathrm{T}} K\dot{x}(t) = 0$ for all $t \geq 0$, and hence,

$$\mathbf{e}^{\mathrm{T}} Kx(0) = \lim_{t\to\infty} \mathbf{e}^{\mathrm{T}} Kx(t) = \alpha_*\mathbf{e}^{\mathrm{T}} K\mathbf{e}, \tag{15.67}$$

which completes the proof. □

Since, by Proposition 15.4, the closed-loop system given by (15.62) and (15.63) is semistable, the optimal fixed-structure control problem involves seeking $k_i > 0$, $i = 1, \ldots, q$, such that the cost functional

$$J(K) = \int_0^\infty [(x(t) - \alpha_*\mathbf{e})^{\mathrm{T}} R_1(x(t) - \alpha_*\mathbf{e}) + (u(t) - u_{\mathrm{e}})^{\mathrm{T}} R_2(u(t) - u_{\mathrm{e}})]\mathrm{d}t \tag{15.68}$$

is minimized, where $u_{\mathrm{e}} = \alpha_* K^{-1}\mathcal{A}\mathbf{e}$, $R_1 = E_1^{\mathrm{T}} E_2 \geq 0$, $R_2 = E_2^{\mathrm{T}} E_2 > 0$, and $E_1^{\mathrm{T}} E_2 = 0$.

The following theorem presents a bilinear matrix inequality (BMI) solution to the fixed-structure optimal semistable control problem for network consensus. For this result, define $\mathcal{L} \triangleq \{L \in \mathbb{R}^{q\times q} : L = \mathrm{diag}[\ell_1, \ldots, \ell_q] \in \mathbb{R}^{q\times q}, \ell_i > 0, i = 1, \ldots, q\}$.

Theorem 15.6. Consider the multiagent dynamical system (15.62) and (15.63) and assume $(\mathcal{A}, E_1)$ is semiobservable and $(\mathcal{A}, V)$ is semicontrollable. Let $Q \in \mathbb{R}^{q\times q}$ and $L \in \mathcal{L}$ be the solution to the BMI minimization problem

$$\min_{Q\in\mathbb{R}^{q\times q},\, L\in\mathcal{L},\, W\in\mathbb{R}^{p\times p}} \operatorname{tr} W, \tag{15.69}$$

subject to

$$\begin{bmatrix} Q & (E_1Q + E_2L\mathcal{A}Q)^{\mathrm{T}} \\ E_1Q + E_2L\mathcal{A}Q & W \end{bmatrix} > 0, \tag{15.70}$$

$$L\mathcal{A}Q + Q\mathcal{A}^{\mathrm{T}}L + V \leq 0. \tag{15.71}$$

Then $u = K^{-1}\mathcal{A}x$ is a semistabilizing controller for (15.62) and $x(t) \to \alpha_*\mathbf{e}$ as $t \to \infty$, where $K^{-1} = L$ and $\alpha_* = \sum_{i=1}^{q} k_i x_i(0)/(\sum_{i=1}^{q} k_i)$. Furthermore, K minimizes the $\mathcal{H}_2$ performance criterion $J(K)$ given by (15.68).

Proof. Convergence to the consensus state $\alpha_*\mathbf{e}$ is a direct consequence of Proposition 15.4. The optimality proof is similar to the proof of Theorem 15.5, and, hence, is omitted. □

Remark 15.1. Because of the diagonal structure on K, the optimization problem given in Theorem 15.6 is a bilinear matrix inequality. A suboptimal solution to this problem can be obtained by using a two-stage optimization process. Specifically, by fixing Q one can design the controller K. Then, with K fixed, Q can be obtained. This process continuous until convergence or an acceptable controller is found.

Chapter 16

$\mathcal{H}_2$ Optimal Semistable Stabilization for Linear Discrete-Time Dynamical Systems with Applications to Network Consensus

16.1. Introduction

In this chapter, we extend the results of Chapter 15 to discrete-time systems. As in the continuous-time case, a complicating feature of the discrete-time $\mathcal{H}_2$ optimal semistable stabilization problem is that the closed-loop Lyapunov equation guaranteeing semistability can admit multiple solutions. However, as in the continuous-time case, a least squares solution over all possible semistabilizing solutions corresponds to the $\mathcal{H}_2$ optimal solution. It is shown that this least squares solution can be characterized by a linear matrix inequality minimization problem.

16.2. Discrete-Time $\mathcal{H}_2$ Semistability Theory

In this section, we establish notation along with several key results on discrete-time $\mathcal{H}_2$ semistability theory involving the notions of semistability, semicontrollability, and semiobservability.

The following definition for semistability for a dynamical system is needed. For this definition, consider the nonlinear dynamical system given by

$$x(k+1) = f(x(k)), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{16.1}$$

where $x(k) \in \mathcal{D} \subseteq \mathbb{R}^n$, $k \in \overline{\mathbb{Z}}_+$, and $f : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

Definition 16.1. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be positively invariant under (16.1). The equilibrium solution $x(k) \equiv x_{\mathrm{e}} \in \mathcal{D}$ of (16.1) is *Lyapunov stable* with respect to $\mathcal{D}$ if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_\mathrm{e}) \cap \mathcal{D}$, then $x(k) \in \mathcal{B}_\varepsilon(x_\mathrm{e}) \cap \mathcal{D}$, $k \in \overline{\mathbb{Z}}_+$. The equilibrium solution $x(k) \equiv x_\mathrm{e} \in \mathcal{D}$ of (16.1) is *semistable* with respect to $\mathcal{D}$ if it is Lyapunov stable with respect to $\mathcal{D}$ and there exists $\delta > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_\mathrm{e}) \cap \mathcal{D}$, then $\lim_{k\to\infty} x(k)$ exists and corresponds to a Lyapunov stable equilibrium point in $\mathcal{D}$. Finally, the system (16.1) is said to be *semistable* with respect to $\mathcal{D}$ if every equilibrium point in $\mathcal{D}$ is semistable with respect to $\mathcal{D}$.

Proposition 16.1. Let $\mathcal{D}_\mathrm{c} \subset \mathbb{R}^n$ be a compact invariant set with respect to (16.1). Suppose there exists a continuous function $V : \mathcal{D}_\mathrm{c} \to \mathbb{R}$ such that $V(f(x)) - V(x) \leq 0$, $x \in \mathcal{D}_\mathrm{c}$. Let $\mathcal{R} \triangleq \{x \in \mathcal{D}_\mathrm{c} : V(f(x)) = V(x)\}$ and let $\mathcal{M}$ denote the largest invariant set contained in $\mathcal{R}$. If every element in $\mathcal{M}$ is a Lyapunov stable equilibrium point with respect to $\mathcal{D}_\mathrm{c}$, then (16.1) is semistable with respect to $\mathcal{D}_\mathrm{c}$.

Proof. Since every solution of (16.1) is bounded, it follows from the hypotheses on $V(\cdot)$that, for every $x \in \mathcal{D}_\mathrm{c}$, the positive limit set $\omega(x)$ of (16.1) is nonempty and contained in the largest invariant subset $\mathcal{M}$ of $\mathcal{R}$. Since every point in $\mathcal{M}$ is a Lyapunov stable equilibrium point, it follows that every point in $\omega(x)$ is a Lyapunov stable equilibrium point.

Next, let $z \in \omega(x)$ and let $\mathcal{U}_\varepsilon$ be an open neighborhood of z. By Lyapunov stability of z, it follows that there exists a relatively open subset $\mathcal{U}_\delta$ containing z such that $s_k(\mathcal{U}_\delta \subseteq \mathcal{U}_\varepsilon$ for every $k \geq k_0$. Since $z \in \omega(x)$, it follows that there exists $h \geq 0$ such that $s(h, x) \in \mathcal{U}_\delta$. Thus, $s(k+h, x) = s_k(s(h, x)) \in s_k(\mathcal{U}_\delta) \subseteq \mathcal{U}_\varepsilon$ for every $k > k_0$. Hence, since $\mathcal{U}_\varepsilon$ was chosen arbitrarily, it follows that $z = \lim_{k\to\infty} s(k, x)$. Now, it follows that $\lim_{i\to\infty} s(k_i, x) \to z$ for every divergent sequence $\{k_i\}$, and hence, $\omega(x) = \{z\}$. Finally, since $\lim_{k\to\infty} s(k, x) \in \mathcal{M}$ is Lyapunov stable for every $x \in \mathcal{D}_\mathrm{c}$, it follows from the definition of semistability that every equilibrium point in $\mathcal{M}$ is semistable. □

Note that if in (16.1) $f(x) = Ax$, where $A \in \mathbb{R}^{n\times n}$, then (16.1) is semistable with respect to $\mathbb{R}^n$ if and only if A is *semistable*, that is, $\mathrm{spec}(A) \subset \{s \in \mathbb{C} : |s| < 1\} \cup \{1\}$

and, if $1 \in \operatorname{spec}(A)$, then 1 is semisimple. In this case, it can be shown that for every $x_0 \in \mathbb{R}^n$, $\lim_{k\to\infty} x(k)$ exists or, equivalently, $\lim_{k\to\infty} A^k$ exists and is given by $\lim_{k\to\infty} A^k = I_n - (I_n - A)(I_n - A)^{\#}$ [22, 108].

Next, we present the notions of semicontrollability and semiobservability. For these definitions let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{l\times n}$, and consider the linear dynamical system

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{16.2}$$

$$y(k) = Cx(k), \tag{16.3}$$

with state $x(k) \in \mathbb{R}^n$, input $u(k) \in \mathbb{R}^m$, and output $y(k) \in \mathbb{R}^l$, where $k \in \overline{\mathbb{Z}}_+$.

Definition 16.2. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$. The pair (A, B) is *semicontrollable* if

$$\left[\bigcap_{i=1}^{n} \mathcal{N}\left(B^{\mathrm{T}}(A^{\mathrm{T}} - I_n)^{i-1}\right)\right]^{\perp} = \left[\mathcal{N}(A^{\mathrm{T}} - I_n)\right]^{\perp}, \tag{16.4}$$

where $(A^{\mathrm{T}} - I_n)^0 \triangleq I_n$.

Definition 16.3. Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$. The pair (A, C) is *semiobservable* if

$$\bigcap_{i=1}^{n} \mathcal{N}\left(C(A - I_n)^{i-1}\right) = \mathcal{N}(A - I_n). \tag{16.5}$$

As in the continuous-time case, semicontrollability and semiobservability are extensions of controllability and observability. In particular, semicontrollability is an extension of null controllability to *equilibrium controllability*, whereas semiobservability is an extension of zero-state observability to *equilibrium observability*. It is important to note here that since Definition 16.2 and 16.3 are dual, dual results to the semiobservability results that we establish in this section also hold for semicontrollability.

Definition 16.4. Let $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{l\times n}$, and $K \in \mathbb{R}^{m\times n}$. The pair (A, C) is *semiobservable with respect to K* if

$$\mathcal{N}(K) \cap \left(\bigcap_{i=1}^{n} \mathcal{N}\left(C(A - I_n)^{i-1}\right)\right) = \mathcal{N}(K) \cap \mathcal{N}(A - I_n). \tag{16.6}$$

The following result shows that semiobservability is unchanged by full state feedback.

Proposition 16.2. Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, $K \in \mathbb{R}^{m\times n}$, and $R \in \mathbb{R}^{n\times n}$, where R is positive definite. If the pair (A, C) is semiobservable, then the pair $(A + BK, C^{\mathrm{T}}C + K^{\mathrm{T}}RK)$ is semiobservable with respect to K.

Proof. Note that $\mathcal{N}(C^{\mathrm{T}}C + K^{\mathrm{T}}RK) = \mathcal{N}(C) \cap \mathcal{N}(K)$. Hence,

$$
\begin{aligned}
\mathcal{N}(K) \cap & \left(\bigcap_{i=1}^{n} \mathcal{N}((C^{\mathrm{T}}C + K^{\mathrm{T}}RK)(A - I_n + BK)^{i-1}) \right) \\
&= \bigcap_{i=1}^{n} \mathcal{N}((C^{\mathrm{T}}C + K^{\mathrm{T}}RK)(A - I_n + BK)^{i-1}) \\
&= \mathcal{N}(K) \cap \left(\bigcap_{i=1}^{n} \mathcal{N}(C(A - I_n)^{i-1}) \right) \\
&= \mathcal{N}(K) \cap \mathcal{N}(A - I_n) \\
&= \mathcal{N}(K) \cap \mathcal{N}(A - I_n + BK), \qquad (16.7)
\end{aligned}
$$

which implies that the pair $(A + BK, C^{\mathrm{T}}C + K^{\mathrm{T}}RK)$ is semiobservable with respect to K.

$\square$

Next, we connect semistability with Lyapunov theory and semiobservability to arrive at a characterization of the $\mathcal{H}_2$ norm of semistable systems. For this result, we consider the linear dynamical system

$$x(k+1) = Ax(k), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{16.8}$$

where $A \in \mathbb{R}^{n\times n}$, with output equation (5.39). Furthermore, for a given semistable system define the $\mathcal{H}_2$ norm of $G(z) \sim \left[\begin{array}{c|c} A & x_0 \\ \hline C & 0 \end{array}\right]$ by

$$|||G|||_2 = \left[\sum_{k=0}^{\infty} \|G(k)\|_{\mathrm{F}}^2 \right]^{1/2} = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \|G(e^{\jmath\theta})\|_{\mathrm{F}}^2 \mathrm{d}\theta \right]^{1/2}. \tag{16.9}$$

The following proposition presents necessary and sufficient conditions for well-posedness of the $\mathcal{H}_2$ norm of a semistable system.

Proposition 16.3. Consider the linear dynamical system (16.8) with output (16.3) and assume A is semistable. Then the following statements are equivalent:

i) For every $x_0 \in \mathbb{R}^n$, $|||G|||_2 < \infty$.

ii) $\sum_{k=0}^{\infty}(A^k)^{\mathrm{T}}RA^k < \infty$, where $R = C^{\mathrm{T}}C$.

iii) $\mathcal{N}(A - I_n) \subset \mathcal{N}(C)$.

Proof. The equivalence of *i)* and *ii)* follows from the fact

$$|||G|||_2^2 = \sum_{k=0}^{\infty} x_0^{\mathrm{T}}(A^k)^{\mathrm{T}}RA^k x_0. \tag{16.10}$$

To show *ii)* implies *iii)* note that since A is semistable it follows that either $\rho(A) < 1$ or there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that $A = S\begin{bmatrix} J & 0 \\ 0 & I_{n-r}\end{bmatrix}S^{-1}$, where $J \in \mathbb{R}^{r\times r}$, $r = \operatorname{rank} A$, and $\rho(J) < 1$. Now, if $\rho(A) < 1$, then *iii)* holds trivially since $\mathcal{N}(A - I_n) = \{0\} \subset \mathcal{N}(C)$.

Alternatively, if $1 \in \operatorname{spec}(A)$, then

$$\mathcal{N}(A - I_n) = \left\{x \in \mathbb{R}^n : x = S[0_{1\times r}, y^{\mathrm{T}}]^{\mathrm{T}}, y \in \mathbb{R}^{n-r}\right\}. \tag{16.11}$$

Now,

$$\begin{aligned}
\sum_{k=0}^{\infty}(A^k)^{\mathrm{T}}RA^k &= S^{-\mathrm{T}}\sum_{k=0}^{\infty}\begin{bmatrix} (J^k)^{\mathrm{T}} & 0 \\ 0 & I_{n-r}\end{bmatrix}\hat{R}\begin{bmatrix} J^k & 0 \\ 0 & I_{n-r}\end{bmatrix}S \\
&= S^{-\mathrm{T}}\sum_{k=0}^{\infty}\begin{bmatrix} (J^k)^{\mathrm{T}}\hat{R}_1J^k & (J^k)^{\mathrm{T}}\hat{R}_{12} \\ \hat{R}_{12}^{\mathrm{T}}J^k & \hat{R}_2\end{bmatrix}S,
\end{aligned} \tag{16.12}$$

where

$$\hat{R} = S^{\mathrm{T}}RS = \begin{bmatrix} \hat{R}_1 & \hat{R}_{12} \\ \hat{R}_{12}^{\mathrm{T}} & \hat{R}_2\end{bmatrix}. \tag{16.13}$$

Next, it follows from (16.12) that

$$\sum_{k=0}^{\infty}(A^k)^{\mathrm{T}}RA^k < \infty \tag{16.14}$$

if and only if $\hat{R}_2 = 0$ or, equivalently,

$$[0_{1\times r}, y^{\mathrm{T}}]\hat{R}[0_{1\times r}, y^{\mathrm{T}}]^{\mathrm{T}} = 0, \quad y \in \mathbb{R}^{n-r}, \tag{16.15}$$

which is further equivalent to $x^{\mathrm{T}}Rx = 0$, $x \in \mathcal{N}(A - I_n)$. Hence, $\mathcal{N}(A - I_n) \subset \mathcal{N}(C)$.

Finally, the proof of *iii*) implies *ii*) is immediate by reversing the steps of the proof given above. □

Lemma 16.1. Let $A \in \mathbb{R}^{n\times n}$. If there exist an $n \times n$ matrix $P \geq 0$ and an $l \times n$ matrix C such that (A, C) is semiobservable and

$$P = A^{\mathrm{T}}PA + R, \tag{16.16}$$

where $R \triangleq C^{\mathrm{T}}C$, then *i*) $\mathcal{N}(P) \subseteq \mathcal{N}(A - I_n) \subseteq \mathcal{N}(R)$ and *ii*) $\mathcal{N}(A - I_n) \cap \mathcal{R}(A - I_n) = \{0\}$.

Proof. *i*) If $(A - I_n)x = 0$, then (16.16) implies $x^{\mathrm{T}}Rx = x^{\mathrm{T}}(P - A^{\mathrm{T}}PA)x = 0$, and hence, $Rx = 0$. Thus, $\mathcal{N}(A - I_n) \subseteq \mathcal{N}(R)$. If $Px = 0$, then

$$0 \leq x^{\mathrm{T}}Rx = x^{\mathrm{T}}(P - A^{\mathrm{T}}PA)x = -x^{\mathrm{T}}A^{\mathrm{T}}PAx \leq 0, \tag{16.17}$$

and hence, $x^{\mathrm{T}}Rx = 0$ or, equivalently, $Rx = 0$. Thus, $\mathcal{N}(P) \subseteq \mathcal{N}(R)$.

Next, let $x \in \mathcal{N}(P) \subseteq \mathcal{N}(R)$. If $(A - I_n)^k x \in \mathcal{N}(P) \subseteq \mathcal{N}(R)$ for some $k \geq 0$, then

$$\begin{aligned} 0 &= x^{\mathrm{T}}(A^{\mathrm{T}} - I_n)^k R(A - I_n)^k x \\ &= x^{\mathrm{T}}(A^{\mathrm{T}} - I_n)^k (P - A^{\mathrm{T}}PA)(A - I_n)^k x \\ &= -x^{\mathrm{T}}(A^{\mathrm{T}} - I_n)^k A^{\mathrm{T}}PA(A - I_n)^k x \\ &= -x^{\mathrm{T}}(A^{\mathrm{T}} - I_n)^{k+1} P(A - I_n)^{k+1} x, \end{aligned} \tag{16.18}$$

and hence, $P(A - I_n)^{k+1}x = 0$, which implies that $(A - I_n)^{k+1}x \in \mathcal{N}(P) \subseteq \mathcal{N}(R)$. Since $(A - I_n)^k x \in \mathcal{N}(P) \subseteq \mathcal{N}(R)$ for $k = 0$, it follows by induction that x is contained in the null space of the left-hand side of (16.5). Equation (16.5) now implies that $x \in \mathcal{N}(A - I_n)$. Thus, $\mathcal{N}(P) \subseteq \mathcal{N}(A - I_n) \subseteq \mathcal{N}(R)$.

ii) Consider $x \in \mathcal{N}(A - I_n) \cap \mathcal{R}(A - I_n)$. Then $(A - I_n)x = 0$ and there exists $z \in \mathbb{R}^n$ such that $x = (A - I_n)z$. Now, it follows from *i*) that $Rx = R(A - I_n)z = 0$. Thus,

$$0 = z^{\mathrm{T}}Rx = z^{\mathrm{T}}(P - A^{\mathrm{T}}PA)x = -z^{\mathrm{T}}(A - I_n)^{\mathrm{T}}Px = -x^{\mathrm{T}}Px, \tag{16.19}$$

and hence, $Px = 0$. Finally,

$$z^{\mathrm{T}}Rz = z^{\mathrm{T}}(P - A^{\mathrm{T}}PA)z = z^{\mathrm{T}}Pz - (x+z)^{\mathrm{T}}P(x+z) = -x^{\mathrm{T}}Px - x^{\mathrm{T}}Pz - z^{\mathrm{T}}Px = 0,$$

and hence, $Rz = 0$. This implies that z is contained in the null space of the left-hand side of (16.5). Hence, by (16.5), $(A - I_n)z = x = 0$ as required. □

Theorem 16.1. Consider the linear dynamical system (16.8). Suppose there exist an $n \times n$ matrix $P \geq 0$ and a matrix $C \in \mathbb{R}^{l\times n}$ such that (A, C) is semiobservable and (16.16) holds. Then (16.8) is semistable with respect to $\mathbb{R}^n$. Furthermore, $|||G(z)|||_2^2 = (x_0 - x_{\mathrm{e}})^{\mathrm{T}}P(x_0 - x_{\mathrm{e}})$, where $x_{\mathrm{e}} \triangleq x_0 - (A - I_n)(A - I_n)^{\#}x_0$.

Proof. Since, by Lemma 16.1, $\mathcal{N}(A - I_n) \cap \mathcal{R}(A - I_n) = \{0\}$, it follows from Lemma 4.14 of [19] that $A - I_n$ is group invertible. Let $L \triangleq I_n - (A - I_n)(A - I_n)^{\#}$ and note that $L^2 = L$. Hence, L is the unique $n \times n$ matrix satisfying $\mathcal{N}(L) = \mathcal{R}(A - I_n)$, $\mathcal{R}(L) = \mathcal{N}(A - I_n)$, and $Lx = x$ for all $x \in \mathcal{N}(A - I_n)$.

Consider the nonnegative function

$$V(x) = x^{\mathrm{T}}Px + x^{\mathrm{T}}L^{\mathrm{T}}Lx. \tag{16.20}$$

If $V(x) = 0$ for some $x \in \mathbb{R}^n$, then $Px = 0$ and $Lx = 0$. It follows from *i*) of Lemma 16.1 that $x \in \mathcal{N}(A - I_n)$, while $Lx = 0$ implies $x \in \mathcal{R}(A - I_n)$. Now, it follows from *ii*) of Lemma 16.1 that $x = 0$. Hence, $V(\cdot)$ is positive definite. Next, since $L(A - I_n) = A - I_n - (A - I_n)(A - I_n)^{\#}(A - I_n) = 0$, it follows that

$$\begin{aligned}\Delta V(x) &= -x^{\mathrm{T}}Rx + x^{\mathrm{T}}(A - I_n)^{\mathrm{T}}L^{\mathrm{T}}L(A - I_n)x + x^{\mathrm{T}}(A - I_n)^{\mathrm{T}}L^{\mathrm{T}}Lx + x^{\mathrm{T}}L^{\mathrm{T}}L(A - I_n)x \\ &= -x^{\mathrm{T}}Rx \\ &\leq 0. \end{aligned} \tag{16.21}$$

Note that $\Delta V^{-1}(0) = \mathcal{N}(R)$.

To find the largest invariant subset $\mathcal{M}$ of $\mathcal{N}(R)$, consider a solution y of (16.8) such that $Cx(k) = 0$ for all $k \in \overline{\mathbb{Z}}_+$. Then, $Cx(k+1) - Cx(k) = 0$, that is, $C(A - I_n)x(k) = 0$. Similarly, $C(A - I_n)x(k+1) - C(A - I_n)x(k) = C(A - I_n)^2x(k) = 0$, and so on. This implies $C(A - I_n)^i x(k) = 0$ for all $k \in \overline{\mathbb{Z}}_+$ and $i = 1, 2, \ldots$. Equation (16.5) now implies that $x(k) \in \mathcal{N}(A - I_n)$ for all $k \in \overline{\mathbb{Z}}_+$. Thus, $\mathcal{M} \subseteq \mathcal{N}(A - I_n)$. However, $\mathcal{N}(A - I_n)$ consists of only equilibrium points and, hence, is invariant. Hence, $\mathcal{M} = \mathcal{N}(A - I_n)$.

Now, let $x_{\mathrm{e}} \in \mathcal{N}(A - I_n)$ be an equilibrium point of (16.8) and consider the function $U(x) = V(x - x_{\mathrm{e}})$, which is positive definite with respect to x_{e}. Then it follows that $\Delta U(x) = -(x - x_{\mathrm{e}})^{\mathrm{T}}R(x - x_{\mathrm{e}}) \leq 0$, $x \in \mathbb{R}^n$. Thus, it follows that x_{e} is Lyapunov stable, and hence, by Proposition 16.1, (16.8) is semistable.

Next, since A is semistable, it follows from $vi)$ of Proposition 11.9.2 of [22] that $\lim_{k\to\infty} A^k = I_n - (A - I_n)(A - I_n)^{\#}$. Now, noting that $Ax_{\mathrm{e}} = x_{\mathrm{e}}$, (16.8) can be equivalently written as

$$x(k+1) - x_{\mathrm{e}} = A(x(k) - x_{\mathrm{e}}), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+. \tag{16.22}$$

Hence,

$$\sum_{k=0}^{N}(x(k) - x_{\mathrm{e}})^{\mathrm{T}}R(x(k) - x_{\mathrm{e}}) = -(x(N) - x_{\mathrm{e}})^{\mathrm{T}}P(x(N) - x_{\mathrm{e}}) + (x_0 - x_{\mathrm{e}})^{\mathrm{T}}P(x_0 - x_{\mathrm{e}}). \tag{16.23}$$

Now, it follows from the semiobservability of (A, C) that $Rx_{\mathrm{e}} = 0$. Hence, letting $N \to \infty$ and noting that $x(k) \to x_{\mathrm{e}}$ as $t \to \infty$ it follows from (16.23) that

$$\sum_{k=0}^{\infty} x^{\mathrm{T}}(k) R x(k) = (x_0 - x_{\mathrm{e}})^{\mathrm{T}} P (x_0 - x_{\mathrm{e}}). \tag{16.24}$$

Finally, defining the free response of (16.8) by $z(k) \triangleq Cx(k) = CA^k x_0$, $k \in \overline{\mathbb{Z}}_+$, and noting that $R = C^{\mathrm{T}} C$, it follows from Parseval's theorem that

$$(x_0 - x_{\mathrm{e}})^{\mathrm{T}} P (x_0 - x_{\mathrm{e}}) = \sum_{k=0}^{\infty} z^{\mathrm{T}}(k) z(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \|G(e^{\jmath\theta})\|_{\mathrm{F}}^2 \mathrm{d}\theta. \tag{16.25}$$

This completes the proof. $\square$

Next, we give a necessary and sufficient condition for characterizing semistability using the Lyapunov equation (16.16). Before we state this result, the following lemmas are needed.

Lemma 16.2. Consider the linear dynamical system (16.8). If (16.8) is semistable, then, for every $n \times n$ nonnegative definite matrix R,

$$\sum_{k=0}^{\infty} (x(k) - x_{\mathrm{e}})^{\mathrm{T}} R (x(k) - x_{\mathrm{e}}) < \infty, \tag{16.26}$$

where $x_{\mathrm{e}} = [I_n - (A - I_n)(A - I_n)^{\#}] x_0$.

Proof. Since A is semistable, it follows from the Jordan decomposition that there exists an invertible matrix $S \in \mathbb{C}^{n \times n}$ such that $A = S \begin{bmatrix} J & 0 \\ 0 & I_{n-r} \end{bmatrix} S^{-1}$, where $J \in \mathbb{C}^{r \times r}$, $r = \operatorname{rank} A$, and $\rho(J) < 1$. Let $z(k) \triangleq S^{-1} x(k)$ and $z_{\mathrm{e}} \triangleq S^{-1} x_{\mathrm{e}}$, $k \in \overline{\mathbb{Z}}_+$. Then (16.8) becomes

$$z(k+1) = \begin{bmatrix} J & 0 \\ 0 & I_{n-r} \end{bmatrix} z(k), \quad z(0) = S^{-1} x_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{16.27}$$

which implies that $\lim_{k \to \infty} z_i(k) = 0$, $i = 1, \ldots, r$, and $z_j(k) = z_j(0)$, $j = r+1, \ldots, n$, that is, $z_{\mathrm{e}} = [0, \ldots, 0, z_{r+1}(0), \ldots, z_n(0)]^{\mathrm{T}}$. Now,

$$\begin{aligned} \sum_{k=0}^{\infty} (x(k) - x_{\mathrm{e}})^{\mathrm{T}} R (x(k) - x_{\mathrm{e}}) &= \sum_{k=0}^{\infty} (z(k) - z_{\mathrm{e}})^* S^* R S (z(k) - z_{\mathrm{e}}) \\ &= \sum_{k=0}^{\infty} \hat{z}^*(k) S^* R S \hat{z}(k), \end{aligned} \tag{16.28}$$

where $\hat{z}(k) \triangleq [z_1(k), \ldots, z_r(k), 0, \ldots, 0]^{\mathrm{T}}$. Since

$$\hat{z}(k+1) = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \hat{z}(k) \tag{16.29}$$

and $\rho(J) < 1$, it follows that

$$\sum_{k=0}^{\infty} \hat{z}^*(k) S^* R S \hat{z}(k) < \infty, \tag{16.30}$$

which proves the result. $\square$

Lemma 16.3. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$. If A and B are semistable, then $A \otimes B$ is semistable.

Proof. Let $\lambda \in \operatorname{spec}(A)$ and $\mu \in \operatorname{spec}(B)$. Since A and B are both semistable, it follows that $|\lambda| < 1$ or $\lambda = 1$ and $\mathrm{am}_A(1) = \mathrm{gm}_A(1)$, and $|\mu| < 1$ or $\mu = 1$ and $\mathrm{am}_B(1) = \mathrm{gm}_B(1)$, where $\mathrm{am}_X(\lambda)$ and $\mathrm{gm}_X(\lambda)$ denote algebraic multiplicity of $\lambda \in \operatorname{spec}(X)$ and geometric multiplicity of $\lambda \in \operatorname{spec}(X)$, respectively. Then it follows from the fact that $\lambda\mu \in \operatorname{spec}(A\otimes B)$, that $\operatorname{spec}(A \otimes B) \subset \{z \in \mathbb{C} : |z| < 1\} \cup \{1\}$. Next, it follows from Fact 7.4.12 of [22] that $\mathrm{gm}_A(1)\mathrm{gm}_B(1) \le \mathrm{gm}_{A\otimes B}(1) \le \mathrm{am}_{A\otimes B}(1) = \mathrm{am}_A(1)\mathrm{am}_B(1)$. Since $\mathrm{am}_A(1) = \mathrm{gm}_A(1)$ and $\mathrm{am}_B(1) = \mathrm{gm}_B(1)$, it follows that $\mathrm{gm}_{A\otimes B}(1) = \mathrm{am}_{A\otimes B}(1)$, and hence, $A \otimes B$ is semistable. $\square$

Lemma 16.4. Let $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, and assume A is semistable. Then $\sum_{k=0}^{\infty} A^k x$ exists if and only if $x \in \mathcal{R}(A - I_n)$. In this case, $\sum_{k=0}^{\infty} A^k x = -(A - I_n)^{\#} x$.

Proof. The proof is similar to the proofs of (viii) and (ix) of Lemma 5.2 of [108] and, hence, is omitted. $\square$

Theorem 16.2. Consider the linear dynamical system (16.8). Then (16.8) is semistable if and only if for every semiobservable pair (A, C) there exists an $n \times n$ matrix $P \ge 0$ such that (16.16) holds. Furthermore, if (A, C) is semiobservable and P satisfies (16.16), then

$$P = \sum_{k=0}^{\infty} (A^k)^{\mathrm{T}} R A^k + P_0 \tag{16.31}$$

for some $P_0 = P_0^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ satisfying

$$A^{\mathrm{T}} P_0 A = 0 \tag{16.32}$$

and

$$P_0 \geq -\sum_{k=0}^{\infty} (A^k)^{\mathrm{T}} R A^k. \tag{16.33}$$

In addition, $\min_{P\in\mathcal{P}} \|P\|_{\mathrm{F}}$ has a unique solution P given by

$$P = \sum_{k=0}^{\infty} (A^k)^{\mathrm{T}} R A^k, \tag{16.34}$$

where $\mathcal{P}$ denotes the set of all P satisfying (16.16). Finally, (16.8) is semistable if and only if for every semiobservable pair (A, C) there exists an $n \times n$ matrix $P > 0$ such that (16.16) holds.

Proof. Sufficiency for the first implication follows from Theorem 16.1. To show necessity, assume (16.8) is semistable. Then, $\lim_{k\to\infty} x(k) = x_{\mathrm{e}}$, where $x_{\mathrm{e}} = [I_n - (A - I_n)(A - I_n)^{\#}]x_0$. For a semiobservable pair (A, C), let

$$P = \sum_{k=0}^{\infty} ((I_n - A)(I_n - A)^{\#})^{\mathrm{T}} (A^k)^{\mathrm{T}} R A^k (I_n - A)(I_n - A)^{\#}. \tag{16.35}$$

Then, for $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} x_0^{\mathrm{T}} P x_0 &= \sum_{k=0}^{\infty} x_0^{\mathrm{T}} ((I_n - A)(I_n - A)^{\#})^{\mathrm{T}} (A^k)^{\mathrm{T}} R A^k (I_n - A)(I_n - A)^{\#} x_0 \\ &= \sum_{k=0}^{\infty} (x_0 - x_{\mathrm{e}})^{\mathrm{T}} (A^k)^{\mathrm{T}} R A^k (x_0 - x_{\mathrm{e}}) \\ &= \sum_{k=0}^{\infty} (x(k) - x_{\mathrm{e}})^{\mathrm{T}} R (x(k) - x_{\mathrm{e}}), \end{aligned} \tag{16.36}$$

where we used the fact that $x(k) - x_{\mathrm{e}} = A^k(x_0 - x_{\mathrm{e}})$. It follows from Lemma 16.2 that P is well defined. Since $x_{\mathrm{e}} \in \mathcal{N}(A - I_n)$, it follows from (16.5) that $Rx_{\mathrm{e}} = 0$, and hence,

$$x_0^{\mathrm{T}} P x_0 = \sum_{k=0}^{\infty} x^{\mathrm{T}}(k) R x(k) = \sum_{k=0}^{\infty} x_0^{\mathrm{T}} (A^k)^{\mathrm{T}} R A^k x_0, \quad x_0 \in \mathbb{R}^n, \tag{16.37}$$

which implies that

$$P = \sum_{k=0}^{\infty} (A^k)^{\mathrm{T}} R A^k. \tag{16.38}$$

Now, (16.16) is immediate using the fact that $Rx_{\mathrm{e}} = 0$.

Next, since A is semistable, it follows from the above result that there exists an $n \times n$ nonnegative-definite matrix P such that (16.16) holds or, equivalently, $\operatorname{vec} P = (A \otimes A)^{\mathrm{T}} \operatorname{vec} P + \operatorname{vec} R$, that is, $(I_{q^2} - (A \otimes A)^{\mathrm{T}}) \operatorname{vec} P = \operatorname{vec} R$. Hence, $\operatorname{vec} R \in \mathcal{R}(I_{q^2} - (A \otimes A)^{\mathrm{T}})$ and $\mathcal{P} = \{P \in \mathbb{R}^{n \times n} : P = \operatorname{vec}^{-1}((I_{q^2} - (A \otimes A)^{\mathrm{T}})^{\#} \operatorname{vec} R) + \operatorname{vec}^{-1}(z)\}$ for some $z \in \mathcal{N}(I_{q^2} - (A \otimes A)^{\mathrm{T}})$. Next, it follows from Lemma 16.3 that $A \otimes A$ is semistable, and hence, by Lemma 16.4,

$$\begin{aligned}
\operatorname{vec}^{-1}\big((I_{q^2} - (A \otimes A)^{\mathrm{T}})^{\#} \operatorname{vec} R\big) &= \sum_{k=0}^{\infty} \operatorname{vec}^{-1}(((A \otimes A)^{\mathrm{T}})^k \operatorname{vec} R) \\
&= \sum_{k=0}^{\infty} \operatorname{vec}^{-1}(((A^k)^{\mathrm{T}} \otimes (A^k)^{\mathrm{T}}) \operatorname{vec} R) \\
&= \sum_{k=0}^{\infty} (A^k)^{\mathrm{T}} R A^k,
\end{aligned} \tag{16.39}$$

where we used the facts that $(X \otimes Y)^{\mathrm{T}} = X^{\mathrm{T}} \otimes Y^{\mathrm{T}}$, $(X \otimes Y)(Z \otimes W) = XZ \otimes YW$, and $\operatorname{vec}(XYZ) = (Z^{\mathrm{T}} \otimes X) \operatorname{vec} Y$ [22, Chapter 7]. Hence,

$$P = \sum_{k=0}^{\infty} (A^k)^{\mathrm{T}} R A^k + \operatorname{vec}^{-1}(z), \tag{16.40}$$

where $\operatorname{vec}^{-1}(z)$ satisfies $\operatorname{vec}^{-1}(z) = (\operatorname{vec}^{-1}(z))^{\mathrm{T}}$, $A^{\mathrm{T}} \operatorname{vec}^{-1}(z) A = 0$, and $\operatorname{vec}^{-1}(z) \geq -\sum_{k=0}^{\infty} (A^k)^{\mathrm{T}} R A^k$. If P is such that $\min_{P \in \mathcal{P}} \|P\|_{\mathrm{F}}$ holds, then it follows that P is the unique solution of a least squares minimization problem and is given by

$$P = \operatorname{vec}^{-1}((I_{q^2} - (A \otimes A)^{\mathrm{T}})^{\#} \operatorname{vec} R) = \sum_{k=0}^{\infty} (A^k)^{\mathrm{T}} R A^k. \tag{16.41}$$

Finally, suppose (A, C) is semiobservable. Then it follows from the first part of the theorem that there exists an $n \times n$ matrix $P \geq 0$ such that (16.16) holds. Let $\hat{P} \triangleq P + L^{\mathrm{T}}L$, where $L = I_n - (A - I_n)(A - I_n)^{\#}$. Then using similar arguments as in the proof of Theorem 16.1, it can be shown that $\hat{P} > 0$ and satisfies (16.16). Conversely, if there exists $P > 0$ such that (16.16) holds, consider the function $V(x) = x^{\mathrm{T}}Px$. Using similar arguments as in the proof of Theorem 16.1, it can be shown that the largest invariant subset $\mathcal{M}$ of $\mathcal{N}(R)$ is given by $\mathcal{M} = \mathcal{N}(A - I_n)$. For $x_{\mathrm{e}} \in \mathcal{N}(A - I_n)$, Lyapunov stability of x_{e} now follows by considering the Lyapunov function $V(x - x_{\mathrm{e}})$. □

Next, we show that the unique solution P given by (16.16) and satisfying $\min_{P \in \mathcal{P}} \|P\|_{\mathrm{F}}$ can be characterized by a linear matrix inequality minimization problem.

Theorem 16.3. Consider the linear dynamical system (16.8) with output (16.3). Assume A is semistable and (A, C) is semiobservable. Let $P_{\min}$ be the solution to the linear matrix inequality minimization problem

$$\min \left\{ \operatorname{tr} PV : P \geq 0 \text{ and } A^{\mathrm{T}}PA + R - P \leq 0 \right\}, \tag{16.42}$$

where $V \in \mathbb{R}^{n \times n}$, $V \geq 0$. Then

$$\operatorname{tr} P_{\min} V = \operatorname{tr} \sum_{k=0}^{\infty} (A^k)^{\mathrm{T}} R A^k V. \tag{16.43}$$

Proof. Let $\hat{P} = \sum_{k=0}^{\infty} (A^k)^{\mathrm{T}} R A^k$ and let $P \geq 0$ be such that

$$A^{\mathrm{T}}PA + R - P \leq 0. \tag{16.44}$$

(Note that $A^{\mathrm{T}}\hat{P}A + R = \hat{P}$, which implies that a $P \geq 0$ satisfying (16.44) exists.) Now, let $W \in \mathbb{R}^{n \times n}$, $W \geq 0$, be such that

$$P = A^{\mathrm{T}}PA + R + W. \tag{16.45}$$

Next, since (A, C) is semiobservable it follows that if $x_{\rm e} \in \mathcal{N}(A - I_n)$, then $Rx_{\rm e} = 0$, and hence, it follows from (16.45) that $Wx_{\rm e} = 0$. Now, using identical arguments as in the proof of Theorem 16.2 it follows that

$$\begin{aligned} P &= \sum_{k=0}^{\infty} (A^k)^{\rm T}(R+W)A^k \\ &\geq \sum_{k=0}^{\infty} (A^k)^{\rm T} R A^k \\ &= \hat{P}. \end{aligned} \tag{16.46}$$

Finally, since $\hat{P}$ is an element of the feasible set of the optimization problem (16.42), $\operatorname{tr} P_{\min} V = \operatorname{tr} \hat{P} V$. □

Finally, we provide a dual result to Theorem 16.3 which is necessary for developing feedback controllers guaranteeing closed-loop semistability.

Theorem 16.4. Consider the linear dynamical system (16.8) with output (16.3). Assume A is semistable and let $V \in \mathbb{R}^{n \times n}$, $V \geq 0$, be such that (A, V) is semicontrollable. Let $Q_{\min}$ be the solution to the LMI minimization problem

$$\min \left\{ \operatorname{tr} QR : Q \geq 0 \text{ and } AQA^{\rm T} + V - Q \leq 0 \right\}. \tag{16.47}$$

Then

$$\operatorname{tr} Q_{\min} R = \operatorname{tr} \sum_{k=0}^{\infty} (A^k)^{\rm T} R A^k V = \operatorname{tr} P_{\min} V, \tag{16.48}$$

where $P_{\min}$ is the solution to the LMI minimization problem given by (16.42).

Proof. The proof is a direct consequence of Theorem 16.3 by noting that (A, V) is semicontrollable if and only if $(A^{\rm T}, V)$ is semiobservable. Now, replacing A with $A^{\rm T}$ and R with V in Theorem 16.3 it follows that

$$\operatorname{tr} Q_{\min} R = \operatorname{tr} \sum_{k=0}^{\infty} (A^k)^{\rm T} V A^k R$$

$$
\begin{aligned}
&= \operatorname{tr} \sum_{k=0}^{\infty} (A^k)^{\mathrm{T}} R A^k V \\
&= \operatorname{tr} P_{\min} V. \qquad (16.49)
\end{aligned}
$$

This completes the proof. □

16.3. Optimal Semistable Stabilization

In this section, we consider the problem of optimal state feedback control for semistable stabilization of linear dynamical systems. Specifically, we consider the discrete-time controlled linear system given by

$$
x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{16.50}
$$

where $x(k) \in \mathbb{R}^n$, $k \in \overline{\mathbb{Z}}_+$, is the state vector, $u(k) \in \mathbb{R}^m$, $k \in \overline{\mathbb{Z}}_+$, is the control input, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$, with the state feedback controller $u(k) = Kx(k)$, where $K \in \mathbb{R}^{m\times n}$ is such that the closed-loop system given by

$$
x(k+1) = (A + BK)x(k), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{16.51}
$$

is semistable and the performance criterion

$$
J(K) \triangleq \sum_{k=0}^{\infty} \left[(x(k) - x_{\mathrm{e}})^{\mathrm{T}} R_1 (x(k) - x_{\mathrm{e}}) + (u(k) - u_{\mathrm{e}})^{\mathrm{T}} R_2 (u(k) - u_{\mathrm{e}}) \right] \tag{16.52}
$$

is minimized, where $R_1 \triangleq E_1^{\mathrm{T}} E_1$, $R_2 \triangleq E_2^{\mathrm{T}} E_2 > 0$, $R_{12} \triangleq E_1^{\mathrm{T}} E_2 = 0$, $u_{\mathrm{e}} = Kx_{\mathrm{e}}$, and $x_{\mathrm{e}} = \lim_{k\to\infty} x(k)$.

Note that it follows from Lemma 16.2 that if the closed-loop system is semistable, then $J(K)$ is well defined. To develop necessary conditions for the optimal semistable control problem, we assume that (A, B) is semicontrollable, (A, E_1) is semiobservable, and $x_{\mathrm{e}} \in \mathcal{N}(K)$. In this case, it follows from Proposition 16.2 that $(A + BK, R_1 + K^{\mathrm{T}} R_2 K)$ is semiobservable with respect to K, and hence, $(R_1 + K^{\mathrm{T}} R_2 K)x_{\mathrm{e}} = 0$. Thus,

$$
J(K) = \sum_{k=0}^{\infty} x_0^{\mathrm{T}} (\tilde{A}^k)^{\mathrm{T}} (R_1 + K^{\mathrm{T}} R_2 K) \tilde{A}^k x_0
$$

$$
\begin{aligned}
&= \operatorname{tr} \sum_{k=0}^{\infty} (\tilde{A}^k)^{\mathrm{T}} \tilde{R} \tilde{A}^k V \\
&= \operatorname{tr} P_{\mathrm{LS}} V,
\end{aligned} \tag{16.53}
$$

where we assume that the initial state $x_0 \in \mathbb{R}^n$ is a random variable such that $\mathbb{E}[x_0] = 0$ and $\mathbb{E}[x_0 x_0^{\mathrm{T}}] = V$, $\tilde{A} \triangleq A + BK$, $\tilde{R} \triangleq R_1 + K^{\mathrm{T}} R_2 K$, and $P_{\mathrm{LS}} \triangleq \operatorname{tr} \sum_{k=0}^{\infty} (\tilde{A}^k)^{\mathrm{T}} \tilde{R} \tilde{A}^k$ denotes the least squares solution to

$$
P = \tilde{A}^{\mathrm{T}} P \tilde{A} + \tilde{R}. \tag{16.54}
$$

Unlike the standard $\mathcal{H}_2$ optimal control problem, $P_{\mathrm{LS}} \geq 0$ is not a unique solution to (16.54).

The following theorem presents an LMI solution to the $\mathcal{H}_2$ optimal semistable control problem.

Theorem 16.5. Consider the linear dynamical system (16.50) and assume (A, E_1) is semiobservable and (A, V) is semicontrollable. Let $Q \in \mathbb{R}^{n \times n}$ and $X \in \mathbb{R}^{m \times n}$ be the solution to the LMI minimization problem

$$
\min_{Q \in \mathbb{R}^{n \times n},\, X \in \mathbb{R}^{m \times n},\, W \in \mathbb{R}^{p \times p}} \operatorname{tr} W, \tag{16.55}
$$

subject to

$$
\begin{bmatrix} Q & (E_1 Q + E_2 X)^{\mathrm{T}} \\ E_1 Q + E_2 X & W \end{bmatrix} > 0, \tag{16.56}
$$

$$
\begin{bmatrix} V - Q & (AQ + BX)^{\mathrm{T}} \\ AQ + BX & -Q \end{bmatrix} \leq 0. \tag{16.57}
$$

Then $K = XQ^{-1}$ is a semistabilizing controller for (16.50), that is, $A + BK$ is semistable. Furthermore, K minimizes the $\mathcal{H}_2$ performance criterion $J(K)$ given by (16.52).

Proof. Since $K = XQ^{-1}$ it follows from (16.57) using Schur compliments that

$$
(A + BK) Q (A + BK)^{\mathrm{T}} + V - Q \leq 0, \tag{16.58}
$$

which, since (A, V) is semicontrollable, implies that $A + BK$ is semistable. Next, note that (16.56) holds if and only if

$$W > (E_1 Q + E_2 X) Q^{-1} (E_1 Q + E_2 X)^{\mathrm{T}}, \tag{16.59}$$

which implies that the minimization problem (16.55)–(16.57) is equivalent to

$$\min \operatorname{tr} (E_1 Q + E_2 X) Q^{-1} (E_1 Q + E_2 X)^{\mathrm{T}}, \tag{16.60}$$

subject to

$$AQA^{\mathrm{T}} + AX^{\mathrm{T}} B^{\mathrm{T}} + BXA^{\mathrm{T}} + BXQ^{-1} X^{\mathrm{T}} B^{\mathrm{T}} + V - Q \leq 0, \tag{16.61}$$

$$Q > 0. \tag{16.62}$$

Hence, noting that (16.60)–(16.62) is equivalent to

$$\min \operatorname{tr} Q\tilde{R}, \tag{16.63}$$

subject to

$$\tilde{A} Q \tilde{A}^{\mathrm{T}} + V - Q \leq 0, \tag{16.64}$$

$$Q > 0, \tag{16.65}$$

the result follows as a direct consequence of Theorems 16.4 and 16.2. □

16.4. Information Flow Models

In the remainder of this chapter, we use the optimal control framework developed in Section 16.3 to design optimal controllers for multiagent network dynamical systems. Specifically, we use undirected and directed graphs to represent a dynamical network and present solutions to the consensus problem for networks with both graph *topologies* (or information flow) [187]. Specifically, let $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted *directed graph* (or digraph) denoting the dynamical network (or dynamic graph) with the set of *nodes* (or vertices) $\mathcal{V} = \{1, \ldots, n\}$

involving a finite nonempty set denoting the agents, the set of *edges* $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow, and an *adjacency* matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$ such that $\mathcal{A}_{(i,j)} = 1$, $i, j = 1, \ldots, n$, if $(j, i) \in \mathcal{E}$, and 0 otherwise. The edge $(i, j) \in \mathcal{E}$ denotes that agent $\mathcal{G}_j$ can obtain information from agent $\mathcal{G}_i$, but not necessarily vice versa. Moreover, we assume that $\mathcal{A}_{(i,i)} = 0$ for all $i \in \mathcal{V}$. A *graph* or *undirected graph* $\mathfrak{G}$ associated with the adjacency matrix $\mathcal{A} \in \mathbb{R}^{q\times q}$ is a directed graph for which the *arc set* is symmetric, that is, $\mathcal{A} = \mathcal{A}^{\mathrm{T}}$. A graph $\mathfrak{G}$ is *balanced* if $\sum_{j=1}^{n} \mathcal{A}_{(i,j)} = \sum_{j=1}^{n} \mathcal{A}_{(j,i)}$ for all $i = 1, \ldots, n$. Finally, we denote the *value* of the node i, $i = 1, \ldots, n$, at time k by $x_i(k) \in \mathbb{R}$. The consensus problem involves the design of a dynamic algorithm that guarantees information state equipartition, that is, $\lim_{k\to\infty} x_i(k) = \alpha \in \mathbb{R}$ for $i = 1, \ldots, n$.

The information flow model is a network dynamical system involving the trajectories of the dynamical network characterized by the multiagent dynamical system $\mathcal{G}$ given by

$$x_i(k+1) = x_i(k) + \sum_{j=1, j\neq i}^{q} \phi_{ij}(x(k)), \quad x_i(0) = x_{i0}, \quad k \in \overline{\mathbb{Z}}_+, \quad i = 1, \ldots, q, \tag{16.66}$$

where $q \geq 2$, or, in vector form

$$x(k+1) = f(x(k)), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{16.67}$$

where $x(k) \triangleq [x_1(k), \ldots, x_q(k)]^{\mathrm{T}} \in \mathbb{R}^q$, $k \in \overline{\mathbb{Z}}_+$, represents the information state vector, $\phi_{ij} : \mathbb{R}^q \to \mathbb{R}$ is continuous, $i, j = 1, \ldots, q$, $i \neq j$, and represents the information flow from the jth agent to the ith agent, and $f = [f_1, \ldots, f_q]^{\mathrm{T}} : \mathbb{R}^q \to \mathbb{R}^q$ is such that $f_i(x) = x_i + \mathcal{I}_i(x)$, where for each $i \in \{1, \ldots, q\}$, $\mathcal{I}_i(x) \triangleq \sum_{j=1,\, j\neq i}^{q} \phi_{ij}(x)$. This nonlinear model is proposed in [104] and [108] and is called a *power balance equation*. Here, however, we address a slightly more general model in that $\phi_{ij}(\cdot)$ has no special structure and x need not be constrained to the nonnegative orthant.

Assumption 1: For the *connectivity matrix* $\mathcal{C} \in \mathbb{R}^{q\times q}$ associated with the multiagent dynamical system $\mathcal{G}$ defined by

$$\mathcal{C}_{(i,j)} = \begin{cases} 0, & \text{if } \phi_{ij}(x) \equiv 0, \\ 1, & \text{otherwise,} \end{cases} \quad i \neq j, \quad i, j = 1, \ldots, q, \tag{16.68}$$

and $\mathcal{C}_{(i,i)} = -\sum_{k=1,\, k\neq i}^{q} \mathcal{C}_{(i,k)}$, $i = j$, $i = 1, \ldots, q$, $\operatorname{rank} \mathcal{C} = q - 1$, and for $\mathcal{C}_{(i,j)} = 1$, $i \neq j$, $\phi_{ij}(x) = 0$ if and only if $x_i = x_j$.

Assumption 2: For $i, j = 1, \ldots, q$, $(x_i - x_j)\phi_{ij}(x) \leq 0$, $x \in \mathbb{R}^q$.

Assumption 3: For $i, j = 1, \ldots, q$, $|\phi_{ij}(x)| \leq \lambda_{ij}|x_i - x_j|$, $\lambda_{ij} > 0$, $x \in \mathbb{R}^q$.

The negative of the connectivity matrix, that is, $-\mathcal{C}$, is known in the literature as the *Laplacian* of the graph $\mathfrak{G}$. For further details on Assumptions 1-3, see [104] and [108] as well as Chapters 8–14.

16.5. Semistability of Information Flow Models

As noted in Chapter 8, a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in a dynamical network is the existence of a continuum of equilibria representing a state of consensus. Under such dynamics, the limiting consensus state is not determined completely by the system dynamics, but on the initial system state as well. For such a system possessing a continuum of equilibria, semistability, and not asymptotic stability is the relevant notion of stability. For the statement of the next result, let $\mathbf{e} \in \mathbb{R}^q$ denote the ones vector of order q, that is, $\mathbf{e} \triangleq [1, \ldots, 1]^{\mathrm{T}}$.

Proposition 16.4. Consider the information flow model (16.67) and assume that Assumptions 1 and 2 hold. Then $\mathcal{I}_i(x) = 0$ for all $i = 1, \ldots, q$ if and only if $x_1 = \cdots = x_q$. Furthermore, $\alpha\mathbf{e}$, $\alpha \in \mathbb{R}$, is an equilibrium state of (16.67).

Proof. The proof is similar to the proof of Proposition 8.6 and, hence, is omitted. □

The following lemmas involving graph-theoretic notions are needed for the main result of this section. For the statement of the next result, let $|\mathcal{V}|$ denote the cardinality of the set $\mathcal{V}$.

Lemma 16.5. Assume $\mathfrak{G}$ is an undirected strongly connected graph with n nodes and value $z_i \in \mathbb{R}$ for $i = 1, \ldots, n$. Furthermore, assume that for each node i, the set of nodes of its neighbors is given by $\mathcal{V}_{n_i} = \{i_1, \ldots, i_{n_i}\}$, where $n_i = |\mathcal{V}_{n_i}|$. If for each node i, $z_{i_1} = \cdots = z_{i_{n_i}}$ and, for some $m \in \{1, \ldots, n\}$ and some $m_j \in \mathcal{V}_{n_m}$, $z_m = z_{m_j}$, then $z_1 = \cdots = z_n$.

Proof. The result is trivial for the cases where $n = 2$ and $n = 3$. Consider the case where $n \geq 4$. Let m, $m \in \{1, \ldots, n\}$, be the node satisfying $z_m = z_{m_j}$ for some $m_j \in \mathcal{V}_{n_m}$. If $|\mathcal{V}_{n_m}| = 1$, then we consider the node m_j. Since $\mathfrak{G}$ is strongly connected and $n \geq 4$, it follows that $\mathcal{V}_{m_j} \neq \emptyset$. Hence, for every neighbor $s \in \mathcal{V}_{m_j}$, $z_s = z_{m_j} = z_m$. Choose a neighbor $s \in \mathcal{V}_{m_j}$ such that $|\mathcal{V}_s| \geq 2$ (this is possible since $\mathfrak{G}$ is strongly connected). Then, by connectivity, it follows that for every node $k \in \mathcal{V} \backslash \{s, m_j, m\}$, $z_k = z_{m_j} = z_m$ or $z_k = z_s = z_m$, and hence, the conclusion follows.

Otherwise, if $|\mathcal{V}_{n_m}| \geq 2$, then choose a neighbor $m_j \in \mathcal{V}_{n_m}$ such that $|\mathcal{V}_{m_j}| \geq 2$ (this is possible since $\mathfrak{G}$ is strongly connected). Then, by connectivity, it follows that for every node $k \in \mathcal{V} \backslash \{m, m_j\}$, $z_k = z_m$ or $z_k = z_{m_j}$, and hence, the conclusion follows. $\square$

For the next result, recall that a *cycle* of the graph $\mathfrak{G}$ is a connected graph where every vertex has exactly two neighbors [82] and an *odd cycle* of the graph $\mathfrak{G}$ is a cycle of $\mathfrak{G}$ with an odd number of edges [66, p. 14].

Lemma 16.6. Assume $\mathfrak{G}$ is an undirected strongly connected graph with n nodes and value $z_i \in \mathbb{R}$ for $i = 1, \ldots, n$. Furthermore, assume that for each node i, the set of nodes of its neighbors is given by $\mathcal{V}_{n_i} = \{i_1, \ldots, i_{n_i}\}$, where $n_i = |\mathcal{V}_{n_i}|$. If $\mathfrak{G}$ contains an odd cycle and for each i, $z_{i_1} = \cdots = z_{i_{n_i}}$, then $z_1 = \cdots = z_n$.

Proof. Since $\mathfrak{G}$ contains a cycle of length m, where $3 \leq m \leq n$ is odd, without loss of generality, let $1, \ldots, m$ be the nodes of the cycle. Then, by connectivity, $z_1 = z_3 = \cdots = z_m = z_2 = z_4 = \cdots = z_{m-1}$, which implies that there exists a node i such that $z_i = z_{i_m}$, where $i_m \in \mathcal{V}_{n_i}$. Thus, it follows from Lemma 16.5 that $z_1 = \cdots = z_n$. $\square$

Next, we present the main stability result of this section for information flow models. Note that although general stability results have been developed in [178] and [4], the conditions of those results are restrictive. Specifically, in [178] it is always required that for each $i \in \{1, \ldots, q\}$, the right hand side $f_i(x)$ of (16.67) is contained in the relative *interior* of the convex hull of x_i and its neighbors x_j. Although [4] extended the results of [178] to the case where the linear combination of x_i and its neighbors x_j is not necessarily convex, the results still need several technical assumptions. In the following result, we present improved results for semistability of (16.67). For this result, we define an *in-neighbor* of the ith agent to be those agents whose information can be received by the ith agent.

Theorem 16.6. Consider the information flow model (16.67) and assume that Assumptions 1–3 hold. For $i = 1, \ldots, q \geq 2$, let $n_i \geq 1$ be the number of neighbors of the ith agent in the case where $\mathfrak{G}$ is a graph and let $n_i \geq 1$ be the number of in-neighbors of the ith agent in the case where $\mathfrak{G}$ is a digraph. Then the following statements hold:

i) If $p_i\phi_{ij}(x) = -p_j\phi_{ji}(x)$ and $\lambda_{ij} < \frac{2p_j}{n_ip_j+n_jp_i}$ for all $i, j = 1, \ldots, q$, $i \neq j$, $p_i > 0$, then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a semistable equilibrium state of (16.67). Furthermore, $x(k) \to \alpha_*\mathbf{e}$ as $k \to \infty$, where $\alpha_* = \sum_{i=1}^{q} p_ix_i(0)/(\sum_{i=1}^{q} p_i)$.

ii) If $p_i\phi_{ij}(x) = -p_j\phi_{ji}(x)$, $\frac{n_i}{p_i} = \frac{n_j}{p_j}$, $\lambda_{ij} \leq \frac{2p_j}{n_ip_j+n_jp_i}$ for all $i, j = 1, \ldots, q$, $i \neq j$, $p_i > 0$, and $\lambda_{lm} < \frac{2p_m}{n_lp_l+n_mp_m}$ for some $l, m \in \{1, \ldots, q\}$ and $\mathcal{C}_{(l,m)} = 1$, $l \neq m$, then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a semistable equilibrium state of (16.67). Furthermore, $x(k) \to \alpha_*\mathbf{e}$ as $k \to \infty$.

iii) If $\mathfrak{G}$ contains an odd cycle, $p_i\phi_{ij}(x) = -p_j\phi_{ji}(x)$, $\frac{n_i}{p_i} = \frac{n_j}{p_j}$, and $\lambda_{ij} \leq \frac{2p_j}{n_ip_j+n_jp_i}$ for all $i, j = 1, \ldots, q$, $i \neq j$, $p_i > 0$, then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a semistable equilibrium state of (16.67). Furthermore, $x(k) \to \alpha_*\mathbf{e}$ as $k \to \infty$.

vi) Let $\phi_{ij}(x) = \phi_{ij}(x_i, x_j) = \frac{1}{p_i}\mathcal{A}_{(i,j)}(x_j - x_i)$ for all $i, j = 1, \ldots, q$, $i \neq j$. Assume that $\mathcal{C}^{\mathrm{T}}\mathbf{e} = 0$ and $p_i \geq n_i^+$, $i = 1, \ldots, q$. Furthermore, assume that $p_r > n_r^+$ for some $r \in \{1, \ldots, q\}$ such that $\mathcal{A}_{(r,j)} = 1$. Then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a semistable equilibrium state of (16.67). Furthermore, $x(k) \to \alpha_*\mathbf{e}$ as $k \to \infty$.

Proof. First note that it follows from Lemma 16.4 that $\alpha\mathbf{e} \in \mathbb{R}^q$, $\alpha \in \mathbb{R}$, is an equilibrium state of (16.67).

i) To show Lyapunov stability of the equilibrium state $\alpha\mathbf{e}$, consider the Lyapunov function candidate given by

$$V(x) = (x - \alpha\mathbf{e})^{\mathrm{T}} P (x - \alpha\mathbf{e}), \tag{16.69}$$

where $P \triangleq \mathrm{diag}[p_1, \ldots, p_q]$. Now, since $p_i\phi_{ij}(x) = -p_j\phi_{ji}(x)$, $x \in \mathbb{R}^q$, $i \neq j$, $i, j = 1, \ldots, q$, and $\mathbf{e}^{\mathrm{T}} P x(k+1) = \mathbf{e}^{\mathrm{T}} P x(k)$, $k \in \overline{\mathbb{Z}}_+$, it follows from Assumptions 2 and 3 that

$$\begin{aligned}
\Delta V(x(k)) &= 2\sum_{i=1}^{q} \sum_{j=1, j\neq i}^{q} x_i(k) p_i \phi_{ij}(x(k)) + \sum_{i=1}^{q} p_i \left[\sum_{j=1, j\neq i}^{q} \phi_{ij}(x(k)) \right]^2 \\
&= 2\sum_{i=1}^{q} \sum_{j\in\mathcal{K}_i} (x_i(k) - x_j(k)) p_i \phi_{ij}(x(k)) + \sum_{i=1}^{q} \frac{1}{p_i} \left[\sum_{j\in\mathcal{N}_i} p_i \phi_{ij}(x(k)) \right]^2 \\
&\leq 2\sum_{i=1}^{q} \sum_{j\in\mathcal{K}_i} (x_i(k) - x_j(k)) p_i \phi_{ij}(x(k)) + \sum_{i=1}^{q} \sum_{j\in\mathcal{N}_i} \frac{1}{p_i} n_i p_i^2 \phi_{ij}^2(x(k)) \\
&= 2\sum_{i=1}^{q} \sum_{j\in\mathcal{K}_i} (x_i(k) - x_j(k)) p_i \phi_{ij}(x(k)) + \sum_{i=1}^{q} \sum_{j\in\mathcal{K}_i} \left(\frac{n_i}{p_i} + \frac{n_j}{p_j} \right) p_i^2 \phi_{ij}^2(x(k)) \\
&= \sum_{i=1}^{q} \sum_{j\in\mathcal{K}_i} 2p_i \left[\left(\frac{n_i}{p_i} + \frac{n_j}{p_j} \right) \frac{p_i}{2} |\phi_{ij}(x(k))|^2 - |(x_i(k) - x_j(k))\phi_{ij}(x(k))| \right] \\
&\leq \sum_{i=1}^{q} \sum_{j\in\mathcal{K}_i} 2p_i \left[\left(\frac{n_i}{p_i} + \frac{n_j}{p_j} \right) \frac{p_i}{2} \lambda_{ij} - 1 \right] |(x_i(k) - x_j(k))\phi_{ij}(x(k))| \\
&\leq 0, \quad k \in \overline{\mathbb{Z}}_+,
\end{aligned} \tag{16.70}$$

where $\mathcal{K}_i \triangleq \mathcal{N}_i \backslash \bigcup_{l=1}^{i-1} \{l\}$ and $\mathcal{N}_i \triangleq \{j \in \{1, \ldots, q\} : \phi_{ij}(x) = 0 \text{ if and only if } x_i = x_j\}$, $i = 1, \ldots, q$, which establishes Lyapunov stability of the equilibrium state $\alpha\mathbf{e}$.

To show that $\alpha\mathbf{e}$ is semistable, note that

$$\Delta V(x(k)) \geq 2\sum_{i=1}^{q} \sum_{j\in\mathcal{K}_i} (x_i(k) - x_j(k)) p_i \phi_{ij}(x(k)), \quad k \in \overline{\mathbb{Z}}_+. \tag{16.71}$$

Next, we show that $\Delta V(x) = 0$ if and only if $(x_i - x_j)\phi_{ij}(x) = 0$, $i = 1, \ldots, q$, $j \in \mathcal{K}_i$. First, assume that $(x_i - x_j)\phi_{ij}(x) = 0$, $i = 1, \ldots, q$, $j \in \mathcal{K}_i$. Then it follows from (16.71) that $\Delta V(x) \geq 0$. However, it follows from (16.70) that $\Delta V(x) \leq 0$, and hence, $\Delta V(x) = 0$. Conversely, assume that $\Delta V(x) = 0$. In this case, note that

$$\Delta V(x) \leq \sum_{i=1}^{q} \sum_{j \in \mathcal{K}_i} 2p_i \left[\left(\frac{n_i}{p_i} + \frac{n_j}{p_j} \right) \frac{p_i}{2} \lambda_{ij} - 1 \right] |(x_i(t) - x_j(t))\phi_{ij}(x(t))| \leq 0, \quad (16.72)$$

and since $\left(\frac{n_i}{p_i} + \frac{n_j}{p_j} \right) \frac{p_i}{2} \lambda_{ij} - 1 < 0$, it follows that $(x_i - x_j)\phi_{ij}(x) = 0$, $i = 1, \ldots, q$, $j \in \mathcal{K}_i$.

Let $\mathcal{R} \triangleq \{x \in \mathbb{R}^q : \Delta V(x) = 0\} = \{x \in \mathbb{R}^q : (x_i - x_j)\phi_{ij}(x) = 0, i = 1, \ldots, q, j \in \mathcal{K}_i\}$. Now, by Assumption 1 the directed graph associated with the connectivity matrix $\mathcal{C}$ for the multiagent dynamical system (16.67) is strongly connected, which implies that $\mathcal{R} = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$. Since the set $\mathcal{R}$ consists of the equilibrium states of (16.67), it follows that the largest invariant set $\mathcal{M}$ contained in $\mathcal{R}$ is given by $\mathcal{M} = \mathcal{R}$. Hence, it follows from Proposition 16.1 that $\alpha\mathbf{e}$ is a semistable equilibrium state of (16.67). To show that $x(k) \to \alpha_* \mathbf{e}$ as $k \to \infty$, note that since $\mathbf{p}^{\mathrm{T}} x(k) = \mathbf{p}^{\mathrm{T}} x(0)$ and $x(k) \to \mathcal{M}$ as $k \to \infty$, where $\mathbf{p} \triangleq [p_1, \ldots, p_q]^{\mathrm{T}} \in \mathbb{R}^q$, it follows that $x(k) \to \alpha_* \mathbf{e}$ as $k \to \infty$.

ii) Using similar arguments as *i*), it can be shown that $\alpha\mathbf{e}$ is Lyapunov stable. To show semistability of $\alpha\mathbf{e}$, let $\mathcal{R} \triangleq \{x \in \mathbb{R}^q : \Delta V(x) = 0\}$, where $V(\cdot)$ is given by (16.69). In this case, it follows from (16.70) that

$$\mathcal{R} = (\mathcal{R}_1 \cup \mathcal{R}_2) \cap \mathcal{R}_3 = (\mathcal{R}_1 \cap \mathcal{R}_3) \cup (\mathcal{R}_2 \cap \mathcal{R}_3), \quad (16.73)$$

where $\mathcal{R}_1 \triangleq \{x \in \mathbb{R}^q : \phi_{ij}(x) = 0, i = 1, \ldots, q, j \in \mathcal{K}_i\}$, $\mathcal{R}_2 \triangleq \{x \in \mathbb{R}^q : \left(\frac{n_i}{p_i} + \frac{n_j}{p_j} \right) p_i \phi_{ij}(x) = 2(x_j - x_i), i = 1, \ldots, q, j \in \mathcal{K}_i\}$, and $\mathcal{R}_3 \triangleq \{x \in \mathbb{R}^q : \phi_{ij}(x) = \phi_{ik}(x), i = 1, \ldots, q, j \in \mathcal{N}_i, k \in \mathcal{N}_i \backslash \{j\}\}$. If $\phi_{ij}(x) = 0$, then $x_i = x_j$, $i = 1, \ldots, q$, $j \in \mathcal{K}_i$. Now, by Assumption 1 the directed graph associated with the connectivity matrix $\mathcal{C}$ for the multiagent dynamical system (16.67) is strongly connected, which implies that $x_1 = \cdots = x_q$. Hence, $\mathcal{R}_1 \cap \mathcal{R}_3 = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$.

Next, we consider the case where $\left(\frac{n_i}{p_i}+\frac{n_j}{p_j}\right)p_i\phi_{ij}(x) = 2(x_j - x_i)$ and $x \in \mathcal{R}_3$, $i = 1,\ldots,q$, $j \in \mathcal{K}_i$. Since $p_i\phi_{ij}(x) = -p_j\phi_{ji}(x)$, it follows that $\left(\frac{n_j}{p_j}+\frac{n_i}{p_i}\right)p_j\phi_{ji}(x) = 2(x_i - x_j)$, $i = 1,\ldots,q$, $j \in \mathcal{K}_i$. Hence, $\left(\frac{n_i}{p_i}+\frac{n_j}{p_j}\right)p_i\phi_{ij}(x) = 2(x_j - x_i)$, $i = 1,\ldots,q$, $j \in \mathcal{N}_i$. Since $\frac{n_i}{p_i} = \frac{n_j}{p_j}$, it follows that $\phi_{ij}(x) = \frac{1}{n_i}(x_j - x_i)$, $i = 1,\ldots,q$, $j \in \mathcal{N}_i$. Furthermore, since $\phi_{ij}(x) = \phi_{ik}(x)$, it follows that $x_j = x_k$, $i = 1,\ldots,q$ $j,k \in \mathcal{N}_i$, $j \neq k$. Note that since $p_i\phi_{ij}(x) = -p_j\phi_{ji}(x)$, $\mathfrak{G}$ is an undirected graph. Thus, $\mathcal{A} = \mathcal{A}^{\mathrm{T}}$, and hence, $\mathfrak{G}$ is strongly connected.

Now, it follows from (16.70) that for $x \in \mathcal{R}_2 \cap \mathcal{R}_3$, $(x_l - x_m)\phi_{lm}(x) = 0$, which implies that $x_l = x_m$. Hence, it follows from Lemma 16.5 that $x_1 = \cdots = x_q$, $\mathcal{R}_2 \cap \mathcal{R}_3 = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$. Therefore, $\mathcal{R} = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$. Now, since the set $\mathcal{R}$ consists of the equilibrium states of (16.67), it follows that the largest invariant set $\mathcal{M}$ contained in $\mathcal{R}$ is the set of equilibria of (16.67). Hence, it follows from Proposition 16.1 that $\alpha\mathbf{e}$ is a semistable equilibrium state of (16.67). To show that $x(k) \to \alpha_*\mathbf{e}$ as $k \to \infty$, note that since $\mathbf{p}^{\mathrm{T}}x(k) = \mathbf{p}^{\mathrm{T}}x(0)$ and $x(k) \to \mathcal{M}$ as $k \to \infty$, it follows that $x(k) \to \alpha_*\mathbf{e}$ as $k \to \infty$.

iii) Using similar arguments as *i*), it can be shown that $\alpha\mathbf{e}$ is Lyapunov stable. Furthermore, using similar arguments as *ii*), it follows that for $x \in \mathcal{R}_2 \cap \mathcal{R}_3$, $x_j = x_k$, $j,k \in \mathcal{N}_i$, $i = 1,\ldots,q$, $j \neq k$. Now, it follows from Lemma 16.6 that $x_1 = \cdots = x_q$. Hence, $\mathcal{R} = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$. The rest of the proof follows as the proof of *i*).

vi) Let $W \triangleq I_q + P^{-1}\mathcal{A}$. First, we show that W is irreducible. Note that W is a *stochastic matrix* [122, p. 526]. Furthermore, since

$$W - I_q = \begin{bmatrix} \frac{1}{p_1}\mathcal{A}_{(1,1)} & \frac{1}{p_1}\mathcal{A}_{(1,2)} & \cdots & \frac{1}{p_1}\mathcal{A}_{(1,q)} \\ \frac{1}{p_2}\mathcal{A}_{(2,1)} & \frac{1}{p_2}\mathcal{A}_{(2,2)} & \cdots & \frac{1}{p_2}\mathcal{A}_{(2,q)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_q}\mathcal{A}_{(q,1)} & \frac{1}{p_q}\mathcal{A}_{(q,2)} & \cdots & \frac{1}{p_q}\mathcal{A}_{(q,q)} \end{bmatrix}, \tag{16.74}$$

it follows that $\operatorname{rank}(W - I_q) = \operatorname{rank}\mathcal{C} = q - 1$. Since $\mathcal{C}^{\mathrm{T}}\mathbf{e} = 0$, it follows that $(W^{\mathrm{T}} - I_q)\mathbf{p} = 0$. Now, it follows from [19, p. 52] that W is irreducible.

Next, note that $|\lambda_i| \leq \|W\| = 1$, $i = 1, \ldots, q$, $\lambda_i \in \operatorname{spec}(W)$, and $\|W\|$ is an induced norm of W. Then, $\rho(W) = 1$. It follows from Theorem 1.4 of [19] that $\rho(W) = 1$ is a simple eigenvalue. Next, we show that W is a *primitive matrix* [122, p. 516]. Since $p_i \geq n_i^+$ for all $i \in \{1, \ldots, q\}$ and $p_r > n_r^+$ for some $r \in \{1, \ldots, q\}$, it follows that $\operatorname{tr} W = \sum_{i=1}^{q} 1 + \frac{1}{p_i}\mathcal{A}_{(i,i)} \geq 1 + \frac{1}{p_r}\mathcal{A}_{(r,r)} > 0$. Then it follows from Corollary 2.28 of [19] that W is primitive. Now, it follows from Theorem 2 of [97] that W is semistable, and hence, $\lim_{k\to\infty} W^k = I_q - (W - I_q)(W - I_q)^{\#}$. Next, it follows from *vi*) of Lemma 5.2 of [108] that $\mathcal{N}(W - I_q) = \mathcal{R}(I_q - (W - I_q)(W - I_q)^{\#})$. Since $(W - I_q)\mathbf{e} = 0$ and $\operatorname{rank}(W - I_q) = q - 1$, it follows that $\mathcal{N}(W - I_q) = \{\alpha\mathbf{e}\}$, where $\alpha \in \mathbb{R}$, and hence, $\mathcal{R}(I_q - (W - I_q)(W - I_q)^{\#}) = \{\alpha\mathbf{e}\}$, which implies that $\lim_{k\to\infty} x(k) = \lim_{k\to\infty} W^k x(0) = \alpha\mathbf{e}$. To show that $x(k) \to \alpha_*\mathbf{e}$ as $k \to \infty$, note that since $\mathbf{p}^{\mathrm{T}} x(k) = \mathbf{p}^{\mathrm{T}} x(0)$ and $x(k) \to \mathcal{M}$ as $k \to \infty$, it follows that $x(k) \to \alpha_*\mathbf{e}$ as $k \to \infty$. □

To illustrate some of the results of Theorem 16.6, consider the linear dynamical system

$$x_1(k+1) = \frac{1}{2}(x_2(k) + x_3(k)), \quad x_1(0) = x_{10}, \quad k \in \overline{\mathbb{Z}}_+, \tag{16.75}$$

$$x_2(k+1) = \frac{1}{2}(x_3(k) + x_1(k)), \quad x_2(0) = x_{20}, \tag{16.76}$$

$$x_3(k+1) = \frac{1}{2}(x_1(k) + x_2(k)), \quad x_3(0) = x_{30}. \tag{16.77}$$

Note that the system (16.75)–(16.77) is an information flow model of the form given by (16.67) and it follows from *iii*) of Theorem 16.6 that consensus and semistability of (16.75)–(16.77) are guaranteed. Figure 16.1 shows the trajectories of (16.75)–(16.77) versus time. Note that it is not easy to use the methods in [178] and [4] to prove semistability and consensus for (16.75)–(16.77). However, using Theorem 16.6 this is straightforward.

16.6. Optimal Fixed-Structure Control of Network Consensus

In multiagent coordination [135,187] and distributed network averaging [240] with a fixed communication topology, we require that $x_{\mathrm{e}} \in \operatorname{span}\{\mathbf{e}\}$. In this section, we consider the

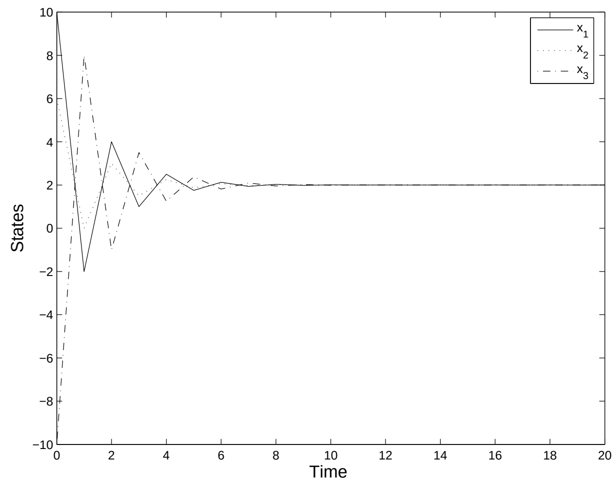

Figure 16.1: Trajectories versus time for (16.75)–(16.77)

design of a fixed-structure consensus protocol for (16.67) such that the closed-loop system is semistable, kernel $(f) = \mathrm{span}\{\mathbf{e}\}$, and (16.52) is minimized. Here, we consider the consensus protocol (16.67) given by

$$x_i(k+1) = u_i(k), \quad x_i(0) = x_{i0}, \quad k \in \overline{\mathbb{Z}}_+, \tag{16.78}$$

$$u_i(k) = x_i(k) + \sum_{j=1, j\neq i}^{q} \phi_{ij}(x(k)), \tag{16.79}$$

$$\phi_{ij}(x(k)) = \frac{1}{k_i}\mathcal{A}_{(i,j)}(x_j(k) - x_i(k)), \quad i, j = 1, \ldots, q, \quad i \neq j, \tag{16.80}$$

where $k_i > n_i^+$, $i = 1, \ldots, q$, $\mathcal{C}$ satisfies Assumption 1 and the conditions of Theorem 16.6. Note that for (16.78)–(16.80) Assumptions 2 and 3 are automatically satisfied. Since, by Theorem 16.6, the closed-loop system given by (16.67) is semistable, the optimal fixed-structure control problem involves seeking k_i, $k_i > n_i^+$, $i = 1, \ldots, q$, such that the cost functional

$$J(K) = \sum_{k=0}^{\infty} \left[(x(k) - \alpha^* \mathbf{e})^{\mathrm{T}} R_1 (x(k) - \alpha^* \mathbf{e}) + (u(k) - u_{\mathrm{e}})^{\mathrm{T}} R_2 (u(k) - u_{\mathrm{e}}) \right], \tag{16.81}$$

is minimized, where $u_{\mathrm{e}} = \alpha_* K^{-1}\mathcal{A}\mathbf{e}$, $R_1 = E_1^{\mathrm{T}} E_2 \geq 0$, $R_2 = E_2^{\mathrm{T}} E_2 > 0$, and $E_1^{\mathrm{T}} E_2 = 0$.

The following theorem presents a bilinear matrix inequality (BMI) solution to the fixed-structure optimal semistable control problem for network consensus. For this result, define $\mathcal{L} \triangleq \{L \in \mathbb{R}^{q\times q} : L = \mathrm{diag}[\ell_1, \ldots, \ell_q] \in \mathbb{R}^{q\times q}, \ell_i > n_i^+, i = 1, \ldots, q\}$.

Theorem 16.7. Consider the consensus protocol (16.78)–(16.80) and assume $(I_q+\mathcal{A}, E_1)$ is semiobservable and $(I_q+\mathcal{A}, V)$ is semicontrollable. Let $Q \in \mathbb{R}^{q\times q}$ and $L \in \mathcal{L}$ be the solution to the BMI minimization problem

$$\min_{Q\in\mathbb{R}^{q\times q},\, L\in\mathcal{L},\, W\in\mathbb{R}^{p\times p}} \operatorname{tr} W, \tag{16.82}$$

subject to

$$\begin{bmatrix} Q & (E_1Q + E_2Q + E_2L\mathcal{A}Q)^{\mathrm{T}} \\ E_1Q + E_2Q + E_2L\mathcal{A}Q & W \end{bmatrix} > 0, \tag{16.83}$$

$$\begin{bmatrix} V - Q & (E_1Q + E_2Q + E_2L\mathcal{A}Q)^{\mathrm{T}} \\ E_1Q + E_2Q + E_2L\mathcal{A}Q & -Q \end{bmatrix} \leq 0. \tag{16.84}$$

Then $u = (I_q + K^{-1}\mathcal{A})x$ is a semistabilizing controller for (16.78) and $x(k) \to \alpha_*\mathbf{e}$ as $k \to \infty$, where $K^{-1} = L$ and $\alpha_* = \sum_{i=1}^q k_i x_i(0)/(\sum_{i=1}^q k_i)$. Furthermore, K minimizes the $\mathcal{H}_2$ performance criterion $J(K)$ given by (16.81).

Proof. Convergence to the consensus state $\alpha_*\mathbf{e}$ is a direct consequence of Theorem 16.6. The optimality proof is similar to the proof of Theorem 16.5, and, hence, is omitted. $\square$

Remark 16.1. Due to the diagonal structure on K, the optimization problem given in Theorem 16.7 is a bilinear matrix inequality. A suboptimal solution to this problem can be obtained by using a two-stage optimization process. Specifically, by fixing Q one can design the controller K. Then, with K fixed, Q can be obtained. This process continues until convergence or an acceptable controller is found.

Chapter 17

Conclusions and Ongoing Research

17.1. Conclusions

In this dissertation we have extended the notion of dissipativity theory to vector dissipativity theory. Specifically, using vector storage functions and vector supply rates, dissipativity properties of aggregate large-scale, discrete-time dynamical systems are shown to be determined from the dissipativity properties of the individual subsystems and the nature of their interconnections. In particular, extended Kalman-Yakubovich-Popov conditions, in terms of the local subsystem dynamics and the subsystem interconnection constraints, characterizing vector dissipativeness via vector storage functions are derived. In addition, general stability criteria were given for feedback interconnections of discrete-time large-scale nonlinear dynamical systems in terms of vector storage functions serving as vector Lyapunov functions.

Motivated by energy flow modeling of large-scale interconnected systems, we also developed discrete-time nonlinear compartmental models that are consistent with thermodynamic principles. Specifically, using a discrete-time, large-scale systems perspective, we developed some of the key properties of thermodynamic systems involving conservation of energy and nonconservation of entropy and ectropy using dynamical systems theory. In addition, conditions were given under which steady-state energy and temperature distributions tend toward equipartition. Finally, the concept of entropy for a large-scale dynamical system is defined and shown to be consistent with the classical thermodynamic definition of entropy.

Next, we extended the notion of hybrid dissipativity theory to vector hybrid dissipativity theory. Specifically, using vector storage functions and hybrid supply rates, dissipativity properties of composite large-scale impulsive dynamical systems are shown to be determined from the dissipativity properties of the individual impulsive subsystems and the nature of their interconnections. Furthermore, extended Kalman-Yakubovich-Popov conditions, in terms of the local hybrid subsystem dynamics and the hybrid subsystem interconnection constraints, characterizing vector dissipativeness via vector storage functions are derived. In addition, general stability criteria were given for feedback interconnections of large-scale impulsive dynamical systems in terms of vector storage functions serving as vector Lyapunov functions.

Using the theory of impulsive dynamical systems, we have developed a general energy- and entropy-based hybrid control framework for lossless and dissipative dynamical systems. Specifically, two types of state-dependent hybrid controllers are developed and analyzed. In addition, unlike standard energy-based controllers for continuous-time systems, the proposed approach does not achieve stabilization via passivation. In addition, we have developed a general energy-based hybrid decentralized control framework for large-scale lossless dynamical systems. Specifically, using a subsystem decomposition for the large-scale system, two types of state-dependent hybrid controllers are developed and analyzed, and several examples are given to illustrate the enhanced ability of those controllers to remove energy from the open-loop system dynamics. In particular, we show that for our example the proposed energy-based hybrid decentralized controller provides finite-time stabilization resulting in superior performance to conventional decentralized control designs. Finally, we show that each decentralized controller corresponds to a maximum entropy controller.

Using the large-scale system framework developed in the first part of the dissertation, a vector Lyapunov function framework for addressing finite-time stability of nonlinear dynamical systems was developed. In addition, the newly developed notion of control vector Lyapunov functions was used to construct decentralized finite-time stabilizing controllers for large-scale dynamical systems with robustness guarantees against full modeling uncertainty. Finally, a family of continuous finite-time decentralized feedback stabilizers was developed for a class of large-scale homogeneous dynamical systems by exploiting connections between finite-time stability and geometric homogeneity.

Next, we unified the notions of semistability and finite-time stability for nonlinear dynamical systems having a continuum of equilibria. In particular, Lyapunov and converse Lyapunov theorems for semistability are established, as well as necessary and sufficient conditions for finite-time semistability of homogeneous systems are addressed. These results are used to develop a general framework for finite-time information consensus algorithms in dynamical networks. Specifically, nonlinear static and dynamic network protocols are designed that guarantee convergence to Lyapunov stable equilibria for a network of dynamic agents with undirected and directed information flows as well as fixed and switching topology. Our analysis relies on several tools from algebraic graph theory and system thermodynamics. In addition, we developed robust analysis results for control network consensus protocols involving higher-order perturbation terms. The proposed robust controllers use undirected and directed graphs to accommodate for a full range of possible information model uncertainty without limitations of bidirectional communication.

Extensions of the notions of semistability and finite-time semistability to nonlinear dynamical systems involving discontinuous time-invariant and time-varying vector fields are also developed. In particular, Lyapunov theorems for semistability, finite-time semistability, weak semistability, as well as uniform semistability are established. These results are used to develop a framework for information consensus algorithms in dynamical networks with switching topologies involving time-dependent and state-dependent communication links for addressing communication link failures, communication dropouts, and time-varying information exchange.

Finally, we presented a system thermodynamic framework for addressing consensus problems for Eulerian swarm models. In particular, necessary and sufficient conditions for information consensus and semistability are presented. In addition, connections between system thermodynamic models and Eulerian swarm models are developed using system entropy notions. In addition, we extended $\mathcal{H}_2$ theory to include semistable systems. Using this framework along with linear matrix inequalities we developed an $\mathcal{H}_2$ optimal semistable stabilization framework for linear dynamical systems.

17.2. Ongoing Research

There are many possible extensions of the results reported in this dissertation. First, the finite-time consensus protocol algorithms developed in Chapter 8 are limited to bidirectional communication. Extensions of this framework to the case where the network topology is a directed graph become more interesting since the communication graph between agents need not be bidirectional. In this case, it is difficult to find an appropriate Lyapunov function to prove semistability or test for nontangency of the vector field due to lack of information symmetry. Hence, we need to develop a new methodology for designing finite-time consensus protocols for dynamical networks with directed information flows. In addition, since the communication between agents is always limited due to capacity or security constraints, it is more natural and robust to use quantized feedback signals to design consensus protocols for dynamical networks. The challenging part of this extension is that quantization breaks symmetry of the network information.

In many applications such as the control of vehicular platoons, flow control, microelectromechanical systems (MEMS), smart structures, and systems described by partial differential equations with constant coefficients and distributed controls and measurements, the systems are always characterized by distributed parameter systems where the underlying dynamics are spatially invariant, and where the controls and measurements are spatially distributed. Such systems typically consist of an infinite collection of possibly heterogeneous linear control systems that are spatially interconnected via certain distant-dependent coupling functions over arbitrary graphs. This important class of networked dynamical systems is known as spatially invariant systems. It is no surprise that control of spatially invariant systems is gaining more and more attention, since an increasing interest has been arising in control of networks and control over networks due to technological advances in sensing, actuation, communication, and computation over last several years. A fruitful area of research is to extend our thermodynamic control framework to spatially invariant systems. This framework will be based on the recently developed system thermodynamics framework of continuum systems [104]. The main task is to develop a novel framework for addressing distributed control algorithms of spatially invariant systems. In addition, since spatially invariant systems are typically infinite dimensional, it is more natural to consider this control problem under a more general theory of dynamical systems such as ergodic theory. Specifically, the control problem for spatially invariant systems can be studied using ergodic theory. By using operator equations, one can design controllers for spatially invariant systems using system entropy notions.

Finally, we propose to merge system thermodynamics, communication system theory, and nonlinear dynamical system theory to develop a unified nonlinear stabilization framework with *a priori* achievable system performance guarantees. The fact that classical thermodynamics is a physical theory concerning systems in equilibrium, communication theory resorts to statistical (subjective or informational) probabilities, and control theory is based on a dynamical systems theory made it all but impossible to unify these theories, leaving these disciplines to stand in sharp contrast to one another in the half century of their coexistence. Yet all of the three theories involve fundamental limitations of performance giving rise to system entropy notions. Using the dynamical systems framework for nonequilibrium thermodynamics, we propose to harmoniously amalgamate thermodynamics, communication theory, and control theory under a single umbrella for quantifying limits of performance for nonlinear system stabilization. The starting point of this research is to place communication theory on a state-space footing using graph-theoretic notions. As in the case of thermodynamic entropy, this will allow us to develop an analytical description of an objective property of information entropy that can potentially offer a conceptual advantage over the subjective or informational expressions for information entropy proposed in the literature (e.g., Shannon entropy, von Neumann entropy, Kolmogorov-Sinai entropy). This can potentially allow us to quantify fundamental limitations for robustness and disturbance rejection of feedback systems with finite capacity input-output signal communication rates.

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